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MAXIMUM MODULUS IN A BIDISC OF ANALYTIC FUNCTIONS OF BOUNDED L-INDEX AND AN ANALOGUE OF HAYMAN'S THEOREM

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Abstract. We generalize some criteria of boundedness of **L**-index in joint variables for in a bidisc analytic functions. Our propositions give an estimate the maximum modulus on a skeleton in a bidisc and an estimate of $(p+1)$th partial derivative by lower order partial derivatives (analogue of Hayman's theorem).

Keywords: analytic function; bidisc; bounded **L**-index in joint variables; maximum modulus; partial derivative; Cauchy's integral formula

MSC 2010: 32A10, 32A17, 32A30, 30D60

1. Introduction

This paper continues our investigations from [5], where we introduced a definition of in a polydisc analytic function of bounded **L**-index in joint variables. Besides, we obtained a criterion of boundedness of **L**-index in joint variables which describes a local behaviour of partial derivatives and explored some interesting properties.

Our goal is to prove new analogues of the criteria of boundedness of **L**-index in joint variables. For entire functions, similar propositions were obtained by Bordulyak and Sheremeta [7] in the case of $\mathbf{L}(z) = (l_1(|z_1|), \dots, l_n(|z_n|))$ and by Bandura [1] in the case of $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, $z \in \mathbb{C}^n$.

Note that the corresponding theorems for entire functions of bounded l-index and of bounded L-index in direction are used to investigate index boundedness of entire solutions of ordinary and partial differential equations and infinite products (see bibliography in [14], [3]). Thus, those generalisations for in a polydisc analytic

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functions are necessary to explore **L**-index in joint variables of holomorphic solutions of PDE's, its systems and, for instance, multidimensional counterparts of Blaschke products.

Particularly, we prove an estimate of the maximum modulus on a greater bidisc by maximum modulus on a lesser bidisc (Theorems 4.1 and 4.2) and obtain an analogue of Hayman's theorem for in a bidisc analytic functions of bounded **L**-index in joint variables (Theorems 5.1 and 5.2).

2. Main definitions and notations

For simplicity, we consider two-dimensional complex space $\mathbb{C}^2$. This helps to distinguish main methods of investigation.

We need some standard notations. Denote $\mathbb{R}_+ = [0, \infty)$, $\mathbf{0} = (0, 0)$, $\mathbf{1} = (1, 1)$, $R = (r_1, r_2) \in \mathbb{R}^2_+$, $z = (z_1, z_2) \in \mathbb{C}^2$. For $A = (a_1, a_2) \in \mathbb{R}^2$, $B = (b_1, b_2) \in \mathbb{R}^2$ we will use formal notations without violation of the existence of these expressions

$$AB = (a_1b_1, a_2b_2), \quad \frac{A}{B} = \Big(\frac{a_1}{b_1}, \frac{a_2}{b_2}\Big), \ b_1 \neq 0, \ b_2 \neq 0, \quad A^B = a_1^{b_1} a_2^{b_2}, \quad b \in \mathbb{Z}^2_+,$$

the notation $A < B$ means that $a_j < b_j$, $j \in \{1, 2\}$; the relation $A \leqslant B$ is defined similarly. For $K = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$, denote $\|K\| = k_1 + \ldots + k_n$, $K! = k_1! \cdot \ldots \cdot k_n!$.

The polydisc $\{z = (z_1, z_2) \in \mathbb{C}^2 \colon \ |z_j - z_j^0| < r_j, \ j = 1, 2\}$ is denoted by $\mathbb{D}^2(z^0, R)$, its skeleton $\{z = (z_1, z_2) \in \mathbb{C}^2 \colon \ |z_j - z_j^0| = r_j, \ j = 1, 2\}$ is denoted by $\mathbb{T}^2(z^0, R)$, and the closed polydisc $\{z = (z_1, z_2) \in \mathbb{C}^2 \colon \ |z_j - z_j^0| \leqslant r_j, \ j = 1, 2\}$ is denoted by $\mathbb{D}^2[z^0, R]$, $\mathbb{D}^2 = \mathbb{D}^2(\mathbf{0}, \mathbf{1})$, $\mathbb{D} = \{z \in \mathbb{C} \colon \ |z| < 1\}$. For p, $q \in \mathbb{Z}_+$ and partial derivative of function $F(z)$ analytic in $\mathbb{D}^2$ we will use the notation

$$F^{(p,q)}(z) = F^{(p,q)}(z_1, z_2) := \frac{\partial^{p+q} F(z)}{\partial z_1^p \partial z_2^q}, \quad z = (z_1, z_2).$$

Let $\mathbf{L}(z) = (l_1(z), l_2(z))$, $z = (z_1, z_2)$, where $l_j(z) \colon \ \mathbb{D}^2 \to \mathbb{R}_+$ is a continuous function such that

$$(\forall z = (z_1, z_2) \in \mathbb{D}^2) \colon \ \ l_j(z) > \frac{\beta}{1 - |z_j|}, \ j \in \{1, 2\},$$

where $\beta > 1$ is a constant, $\boldsymbol{\beta} := (\beta, \beta)$. Strochyk, Sheremeta, Kushnir [15], [10], [14] imposed a similar condition for a function $l \colon \ \mathbb{D} \to \mathbb{R}_+$ and $l \colon \ G \to \mathbb{R}_+$, where G is an arbitrary domain in $\mathbb{C}$. We also used related condition by the study in the unit ball analytic functions of bounded **L**-index in direction [4].

An analytic function $F\colon \mathbb{D}^2 \to \mathbb{C}$ is called a function of *bounded* $\mathbf{L}$*-index* (*in joint variables*) if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z = (z_1, z_2) \in \mathbb{D}^2$ and for all $(p_1, p_2) \in \mathbb{Z}_+^2$,

$$\tag{2.1} \frac{1}{p_1!p_2!}\frac{|F^{(p_1,p_2)}(z)|}{l_1^{p_1}(z)l_2^{p_2}(z)} \leqslant \max\Big\{\frac{1}{k_1!k_2!}\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z)l_2^{k_2}(z)}\colon\ 0 \leqslant k_1 + k_2 \leqslant n_0\Big\}.$$

The least such integer n_0 is called the $\mathbf{L}$*-index in joint variables of the function* $F(z)$ and is denoted by $N(F, \mathbf{L}, \mathbb{D}^2) = n_0$. This is an analogue of the definition of entire function of bounded $\mathbf{L}$-index or bounded index ($\mathbf{L} \equiv 1$) in joint variables in $\mathbb{C}^2$ (see [2], [7], [6], [12], [13]) and the definition of in a domain analytic function of bounded index [8]. Note that the primary definition of in $\mathbb{C}$ entire function of bounded index was considered by Lepson in [11].

By $Q(\mathbb{D}^2)$ we denote the class of functions $\mathbf{L}$, which satisfy the condition

$$(\forall r_j \in [0, \beta],\ j \in \{1, 2\})\colon\ 0 < \lambda_{1,j}(R) \leqslant \lambda_{2,j}(R) < \infty,$$

where

$$\tag{2.2} \lambda_{1,j}(R) = \inf_{z^0\in\mathbb{D}^2} \inf\Big\{\frac{l_j(z)}{l_j(z^0)}\colon\ z \in \mathbb{D}^2[z^0, R/\mathbf{L}(z^0)]\Big\},$$

$$\tag{2.3} \lambda_{2,j}(R) = \sup_{z^0\in\mathbb{D}^2} \sup\Big\{\frac{l_j(z)}{l_j(z^0)}\colon\ z \in \mathbb{D}^2[z^0, R/\mathbf{L}(z^0)]\Big\}.$$

3. Auxiliary propositions

Denote $\mathcal{B} = (0, \beta]$ and $\mathcal{B}^2 = (0, \beta] \times (0, \beta]$, where $\times$ means the Cartesian product. We need three following theorems from [5].

Theorem 3.1 ([5]). *Let* $\mathbf{L} \in Q(\mathbb{D}^2)$*. A function* F *analytic in* $\mathbb{D}^2$ *has bounded* $\mathbf{L}$*-index in joint variables if and only if for each* $R \in \mathcal{B}^2$ *there exist* $n_0 \in \mathbb{Z}_+$, $p_0 > 0$ *such that for every* $z^0 = (z^0) \in \mathbb{D}^2$ *there exists* $(k_1^0, k_2^0) \in \mathbb{Z}_+^2$, $0 \leqslant k_1^0 + k_2^0 \leqslant n_0$*, and*

$$\tag{3.1} \max\Big\{\frac{1}{k_1!k_2!}\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z)l_2^{k_2}(z)}\colon\ k_1 + k_2 \leqslant n_0,\ (z_1, z_2) \in \mathbb{D}^2[z^0, R/\mathbf{L}(z^0)]\Big\} \leqslant \frac{p_0}{k_1^0!k_2^0!}\frac{|F^{(k_1^0,k_2^0)}(z^0)|}{l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}.$$

Theorem 3.2 ([5]). *Let $\mathbf{L}\in Q(\mathbb{D}^2)$. In order that a function F analytic in $\mathbb{D}^2$ be of bounded $\mathbf{L}$-index in joint variables it is necessary that for every $R\in\mathcal{B}^2$ exists $n_0\in\mathbb{Z}_+$ and exists $p\geqslant 1$ such that for all $z^0\in\mathbb{D}^2$ exists $k^0\in\mathbb{Z}^2_+$, $k_1^0+k_2^0\leqslant n_0$, and*

$$\max\{|F^{(k_1^0,k_2^0)}(z)|\colon\ z\in\mathbb{D}^2[z^0,R/\mathbf{L}(z^0)]\}\leqslant p|F^{(k_1^0,k_2^0)}(z^0)|, \tag{3.2}$$

and it is sufficient that for every $R\in\mathcal{B}^2$ exists $n_0\in\mathbb{Z}_+$ and exists $p\geqslant 1$ such that for all $z^0\in\mathbb{D}^2$ exist $k_1^0\leqslant n_0$, $k_2^0\leqslant n_0$ and

$$\max\{|F^{(k_1^0,0)}(z)|\colon\ z\in\mathbb{D}^2[z^0,R/\mathbf{L}(z^0)]\}\leqslant p|F^{(k_1^0,0)}(z^0)| \tag{3.3}$$

$$\max\{|F^{(0,k_2^0)}(z)|\colon\ z\in\mathbb{D}^2[z^0,R/\mathbf{L}(z^0)]\}\leqslant p|F^{(0,k_2^0)}(z^0)|. \tag{3.4}$$

Denote $\widetilde{\mathbf{L}}(z)=(\tilde{l}_1(z),\tilde{l}_2(z))$. $\mathbf{L}\asymp\widetilde{\mathbf{L}}$ means that exists $\Theta=(\theta_1,\theta_2)>\mathbf{0}$, such that

$$\forall\,(z_1,z_2)\in\mathbb{D}^2\colon\ \ \theta_1\widetilde{l}_j(z)\leqslant l_j(z)\leqslant\theta_2\widetilde{l}_j(z),\ j\in\{1,2\}.$$

Theorem 3.3 ([5]). *Let $\mathbf{L}\in Q(\mathbb{D}^2)$ and $\mathbf{L}\asymp\widetilde{\mathbf{L}}$. A function F analytic in $\mathbb{D}^2$ has bounded $\widetilde{\mathbf{L}}$-index in joint variables if and only if it has bounded $\mathbf{L}$-index.*

4. Estimate of maximum modulus on a bidisc

For an entire function $F(z)$ we put $M(R,z^0,F)=\max\{|F(z)|\colon\ z\in\mathbb{T}^2(z^0,R)\}$. Then $M(R,z^0,F)=\max\{|F(z)|\colon\ z\in\mathbb{D}^2[z^0,R]\}$, because the maximum modulus for an entire function in a closed polydisc is attained on its skeleton.

Theorem 4.1. *Let $\mathbf{L}\in Q(\mathbb{D}^2)$. If an analytic function F in $\mathbb{D}^2$ has bounded $\mathbf{L}$-index in joint variables, then for any $R',R''\in\mathbb{R}^2_+$, $\mathbf{0}<R'<R''\leqslant(\beta,\beta)$ there exists $p_1=p_1(R',R'')\geqslant 1$ such that for each $z^0\in\mathbb{D}^2$,*

$$M(R''/\mathbf{L}(z^0),z^0,F)\leqslant p_1M(R'/\mathbf{L}(z^0),z^0,F). \tag{4.1}$$

P r o o f. Let $N(F,L,\mathbb{D}^2)=N<\infty$. Suppose that inequality (4.1) does not hold, i.e. there exist R', R'', $\mathbf{0}<R'<R''$ such that for each $p_*\geqslant 1$ and for some $z^0=z^0(p_*)$,

$$M(R''/\mathbf{L}(z^0),z^0,F)>p_*M(R'/\mathbf{L}(z^0),z^0,F). \tag{4.2}$$

By Theorem 3.2 there exists a number $p_0=p_0(R'')\geqslant 1$ such that for every $z^0\in\mathbb{D}^2$ and for some $k^0\in\mathbb{Z}^2_+$, $k_1^0+k_2^0\leqslant N$ one has

$$M(R''/\mathbf{L}(z^0),z^0,F^{(k_1^0,k_2^0)})\leqslant p_0|F^{(k_1^0,k_2^0)}(z^0)|. \tag{4.3}$$

We put

$$b_1 = p_0 N! \Big(\frac{r_1'' r_2''}{r_1' r_2'}\Big)^N \lambda_{2,1}^N(R'')\lambda_{2,2}^N(R'') \sum_{j=1}^N \frac{(N-j)!}{(r_1'')^j},$$

$$b_2 = p_0 \lambda_{2,2}^N(R'') \sum_{j=1}^N \frac{(N-j)!}{(r_2'')^j} \max\{(r_1'')^{-N}, 1\},$$

$$p_* = p_0 (N!)^2 \Big(\frac{r_1'' r_2''}{r_1' r_2'}\Big)^N + b_1 + b_2 + 1.$$

Let $z^0 = z^0(p_*)$ be a point for which inequality (4.2) holds and k^0 be such that (4.3) holds. We choose z^* and $z^*_{(j_1,j_2)}$ such that

$$M(R'/\mathbf{L}(z^0), z^0, F) = |F(z^*)|, \quad M(R''/\mathbf{L}(z^0), z^0, F^{(j_1,j_2)}) = |F^{(j_1,j_2)}(z^*_{(j_1,j_2)})|$$

for every $j = (j_1, j_2) \in \mathbb{Z}_+^2$, $j_1 + j_2 \leqslant N$. We apply Cauchy's inequality

$$|F^{(j_1,j_2)}(z^0)| \leqslant j_1! j_2! \Big(\frac{l_1(z^0)}{r_1'}\Big)^{j_1} \Big(\frac{l_2(z^0)}{r_2'}\Big)^{j_2} |F(z^*)| \tag{4.4}$$

to estimate the difference

$$\begin{aligned} &|F^{(j_1,j_2)}(z^*_{j,1}, z^*_{j,2}) - F^{(j_1,j_2)}(z_1^0, z^*_{j,2})| \\ &= \Big|\int_{z_1^0}^{z^*_{j,1}} F^{(j_1+1,j_2)}(\zeta, z^*_{j,2})\,\mathrm{d}\zeta\Big| \\ &\leqslant \int_{z_1^0}^{z^*_{j,1}} \max\Big\{|F^{(j_1+1,j_2)}(\zeta, z^*_{j,2})| \colon\ |\zeta - z_1^0| = \frac{r_1''}{l_1(z^0)}\Big\}|\mathrm{d}\zeta| \\ &= |F^{(j_1+1,j_2)}(z^*_{(j_1+1,j_2)})| \frac{r_1''}{l_1(z^0)}. \end{aligned} \tag{4.5}$$

The point $(z_1^0, z^*_{j,2})$ belongs to $\mathbb{D}^2[z^0, R''/\mathbf{L}(z^0)]$. Therefore, for $k \in \{1,2\}$ we have $|z^*_{j,k} - z_k^0| = r_k''/l_k(z^0)$ and $l_k(z_1^0, z^*_{j,2}) \leqslant \lambda_{2,k}(R'')l_k(z^0)$. Putting $j = k^0$ in (4.4), by Theorem 3.1 we obtain

$$\begin{aligned} |F^{(j_1,j_2)}(z_1^0, z^*_{j,2})| &\leqslant j_1! j_2! p_0 |F^{(k^0)}(z^0)| \frac{l_1^{j_1}(z_1^0, z^*_{j,2}) l_2^{j_2}(z_1^0, z^*_{j,2})}{k_1^0! k_2^0! l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)} \\ &\leqslant \frac{j_1! j_2! \lambda_{2,1}^{j_1}(R'')\lambda_{2,2}^{j_2}(R'') l_1^{j_1}(z^0) l_2^{j_2}(z^0)}{k_1^0! k_2^0! l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)} p_0 k_1^0! k_2^0! \\ &\quad\times \Big(\frac{l_1(z^0)}{r_1'}\Big)^{k_1^0} \Big(\frac{l_2(z^0)}{r_2'}\Big)^{k_2^0} |F(z^*)| \\ &\leqslant j_1! j_2! \lambda_{2,1}^{j_1}(R'')\lambda_{2,2}^{j_2}(R'') p_0 \frac{l_1^{j_1}(z^0) l_2^{j_2}(z^0)}{(r_1')^{k_1^0}(r_2')^{k_2^0}} |F(z^*)|. \end{aligned} \tag{4.6}$$

From inequalities (4.5) and (4.6) it follows that

$$
\begin{aligned}
&|F^{(j_1+1,j_2)}(z^*_{(j_1+1,j_2)})| \\
&\quad \geqslant \frac{l_1(z^0)}{r''_1}(|F^{(j_1,j_2)}(z^*_{j,1}, z^*_{j,2})| - |F^{(j_1,j_2)}(z^0_1, z^*_{j,2})|) \\
&\quad \geqslant \frac{l_1(z^0)}{r''_1}\Big(|F^{(j_1,j_2)}(z^*_{j,1}, z^*_{j,2})| - j_1!j_2!\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')p_0\frac{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}{(r'_1)^{k_1^0}(r'_2)^{k_2^0}}|F(z^*)|\Big) \\
&\quad = \frac{l_1(z^0)}{r''_1}|F^{(j_1,j_2)}(z^*_{j,1}, z^*_{j,2})| - j_1!j_2!\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')p_0 l_1(z^0)\frac{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}{r''_1(r'_1)^{k_1^0}(r'_2)^{k_2^0}}|F(z^*)|.
\end{aligned}
$$

We choose $j = (j_1, j_2) = (k^0_1, k^0_2)$ and deduce

$$
\begin{aligned}
(4.7)\qquad &|F^{(k_1^0,k_2^0)}(z^*_{k^0})| \\
&\geqslant \frac{l_1(z^0)}{r''_1}|F^{(k_1^0-1,k_2^0)}(z^*_{(k_1^0-1,k_2^0)})| \\
&\quad - \frac{p_0(k^0_1-1)!k^0_2!l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{r''_1(r'_1)^{k_1^0}(r'_2)^{k_2^0}}\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')|F(z^*)| \\
&\geqslant \frac{l_1^2(z^0)}{(r''_1)^2}|F^{(k_1^0-2,k_2^0)}(z^*_{(k_1^0-2,k_2^0)})| \\
&\quad - \frac{p_0(k^0_1-2)!k^0_2!l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r''_1)^2(r'_1)^{k_1^0}(r'_2)^{k_2^0}}\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')|F(z^*)| \\
&\quad - \frac{p_0(k^0_1-1)!k^0_2!l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{r''_1(r'_1)^{k_1^0}(r'_2)^{k_2^0}}\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')|F(z^*)| \\
&\geqslant \dots \\
&\geqslant \frac{l_1^{k_1^0}(z^0)}{(r''_1)^{k_1^0}}|F^{(0,k_2^0)}(z^*_{0,k_2^0})| \\
&\quad - \frac{p_0k^0_2!l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r''_1)^{k_1^0}(r'_1)^{k_1^0}(r'_2)^{k_2^0}}\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')|F(z^*)| - \dots \\
&\quad - \frac{p_0(k^0_1-2)!k^0_2!l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r''_1)^2(r'_1)^{k_1^0}(r'_2)^{k_2^0}}\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')|F(z^*)| \\
&\quad - \frac{p_0(k^0_1-1)!k^0_2!l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{r''_1(r'_1)^{k_1^0}(r'_2)^{k_2^0}}\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')|F(z^*)| \\
&= \frac{l_1^{k_1^0}(z^0)}{(r''_1)^{k_1^0}}|F^{(0,k_2^0)}(z^*_{(0,k_2^0)})| \\
&\quad - \frac{p_0k^0_2!l_1^{k_1^0}(z^0)l_2^{k_2^0}(z^0)}{(r'_1)^{k_1^0}(r'_2)^{k_2^0}}\lambda^{j_1}_{2,1}(R'')\lambda^{j_2}_{2,2}(R'')|F(z^*)|\sum_{j_1=1}^{k_1^0}\frac{(k_1^0-j_1)!}{(r''_1)^{j_1}}
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant \frac{l_1^{k_1^0}(z^0)}{(r_1'')^{k_1^0}} \frac{l_2^{k_2^0}(z^0)}{(r_2'')^{k_2^0}} |F(z^*_{(0,0)})| \\
&\quad - \frac{p_0 k_2^0! l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}{(r_1')^{k_1^0}(r_2')^{k_2^0}} \lambda_{2,1}^{k_1^0}(R'')\lambda_{2,2}^{k_2^0}(R'')|F(z^*)| \sum_{j_1=1}^{k_1^0} \frac{(k_1^0-j_1)!}{(r_1'')^{j_1}} \\
&\quad - \frac{p_0 l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}{(r_1')^{k_1^0}(r_2')^{k_2^0}(r_1'')^{k_1^0}} \lambda_{2,2}^{k_2^0}(R'')|F(z^*)| \sum_{j_2=1}^{k_2^0} \frac{(k_2^0-j_2)!}{(r_2'')^{j_2}} \\
&= \frac{l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}} |F(z^*_{(0,0)})| - |F(z^*)|(\widetilde{b}_1 + \widetilde{b}_2),
\end{aligned}
$$

where

$$
\begin{aligned}
(4.8)\qquad \widetilde{b}_1 &= \frac{p_0 k_2^0! l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}{(r_1')^{k_1^0}(r_2')^{k_2^0}} \lambda_{2,1}^{k_1^0}(R'')\lambda_{2,2}^{k_2^0}(R'') \sum_{j_1=1}^{k_1^0} \frac{(k_1^0-j_1)!}{(r_1'')^{j_1}} \\
&= p_0 k_2^0! \frac{l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}} \frac{(r_1'')^{k_1^0}(r_2'')^{k_2^0}}{(r_1')^{k_1^0}(r_2')^{k_2^0}} \lambda_{2,1}^{k_1^0}(R'')\lambda_{2,2}^{k_2^0}(R'') \sum_{j_1=1}^{k_1^0} \frac{(k_1^0-j_1)!}{(r_1'')^{j_1}} \\
&\leqslant p_0 N! \frac{l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}} \Big(\frac{r_1'' r_2''}{r_1' r_2'}\Big)^N \lambda_{2,1}^{N}(R'')\lambda_{2,2}^{N}(R'') \sum_{j=1}^{N} \frac{(N-j)!}{(r_1'')^{j}} \\
&= \frac{l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}} b_1, \\
\widetilde{b}_2 &= \frac{p_0}{(r_1'')^{k_1^0}} \frac{l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}} \lambda_{2,2}^{k_2^0}(R'') \sum_{j_2=1}^{k_2^0} \frac{(k_2^0-j_2)!}{(r_2'')^{j_2}} \\
&\leqslant p_0 \frac{l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}} \lambda_{2,2}^{N}(R'') \sum_{j=1}^{N} \frac{(N-j)!}{(r_2'')^{j}} \max\Big\{\frac{1}{(r_1'')^N}, 1\Big\} \\
&= \frac{l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}} b_2.
\end{aligned}
$$

Inequality (4.7) implies that

$$
(4.9)\qquad |F^{(k_1^0,k_2^0)}(z^*_{(k_1^0,k_2^0)})| \geqslant \frac{l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}} |F(z^*)| \Big(\frac{|F(z^*_{(0,0)})|}{|F(z^*)|} - (b_1+b_2) \Big).
$$

In view of (4.2) we have that $|F(z^*_{(0,0)})|/|F(z^*)| \geqslant p_* > b_1+b_2$. Hence, applying (4.4) and (4.3) to (4.9), we deduce

$$
|F^{(k_1^0,k_2^0)}(z^*_{(k_1^0,k_2^0)})| \geqslant \frac{l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}} |F(z^*)| (p_* - (b_1+b_2))
$$

$$\geqslant \frac{l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}{(r_1'')^{k_1^0}(r_2'')^{k_2^0}}(p_* - (b_1+b_2)) \frac{|F^{(k_1^0,k_2^0)}(z^0)|(r_1')^{k_1^0}(r_2')^{k_2^0}}{k_1^0! k_2^0! l_1^{k_1^0}(z^0) l_2^{k_2^0}(z^0)}$$

$$\geqslant \left(\frac{r_1' r_2'}{r_1'' r_2''}\right)^N (p_* - (b_1+b_2)) \frac{|F^{(k_1^0,k_2^0)}(z^*_{(k_1^0,k_2^0)})|}{p_0(N!)^2}.$$

Therefore, $p_* \leqslant p_0(N!)^2(r_1''r_2''/r_1'r_2')^N + b_1 + b_2$, but it contradicts the choice $p_* = p_0(N!)^2(r_1''r_2''/r_1'r_2')^N + b_1 + b_2 + 1$. Thus, inequality (4.1) is valid. □

Theorem 4.2. *Let* $\mathbf{L} \in Q(\mathbb{D}^2)$, F *be a function analytic in* $\mathbb{D}^2$. *If there exist* $R', R'' \in \mathbb{R}^2_+$, $\mathbf{0} < R' < \mathbf{e} < R'' \leqslant (\beta, \beta)$ *and* $p_1 \geqslant 1$ *such that for each* $z^0 \in \mathbb{D}^2$ *inequality* (4.1) *holds, then function* F *has bounded* $\mathbf{L}$*-index in joint variables.*

P r o o f. Let $z^0 \in \mathbb{D}^2$ be an arbitrary point. We expand the function F in power series in $\mathbb{D}^2(z^0, R)$:

$$F(z) = \sum_{k \geqslant \mathbf{0}} b_k (z - z^0)^k = \sum_{k_1 \geqslant 0,\, k_2 \geqslant 0} b_{k_1,k_2}(z_1 - z_1^0)^{k_1}(z_2 - z_2^0)^{k_2}, \tag{4.10}$$

where $k = (k_1, k_2)$, $b_k = b_{k_1,k_2} = F^{(k_1,k_2)}(z_1^0, z_2^0)/k_1!\, k_2!$, $R = (r_1, r_2)$.

Let $\mu(R, z^0, F) = \max\{|b_k|R^k\colon\ k \geqslant \mathbf{0}\} = \max\{|b_{k_1,k_2}|r_1^{k_1} r_2^{k_2}\colon\ k_1 \geqslant 0,\ k_2 \geqslant 0\}$ be a maximal term of series (4.10) and $\nu(R) = \nu(R, z^0, F) = (\nu_1(R), \nu_2(R))$ be a set of indices such that

$$\mu(R, z^0, F) = |b_{\nu(R)}| R^{\nu(R)},$$
$$\|\nu(R)\| = \nu_1(R) + \nu_2(R) = \max\{k_1 + k_2\colon\ k_1 \geqslant 0,\ k_2 \geqslant 0,\ |b_k|R^k = \mu(R, z^0, F)\}.$$

We apply Cauchy's inequality:

$$\forall R = (r_1, r_2),\ 0 < r_j < 1,\ j \in \{1,2\}\colon\ \mu(R, z^0, F) \leqslant M(R, z^0, F).$$

For given R' and R'' such that $0 < r_j' < 1$, $1 < r_j'' < \beta$ we conclude

$$M(R'R, z^0, F) \leqslant \sum_{k \geqslant \mathbf{0}} |b_k|(R'R)^k \leqslant \sum_{k \geqslant \mathbf{0}} \mu(R, z^0, F)(R')^k$$
$$= \mu(R, z^0, F) \sum_{k \geqslant \mathbf{0}} (R')^k = \prod_{j=1}^{2} \frac{1}{1 - r_j'} \mu(R, z^0, F).$$

Besides,

$$\ln \mu(R, z^0, F) = \ln(|b_{\nu(R)}| R^{\nu(R)}) = \ln\left(|b_{\nu(R)}|(RR'')^{\nu(R)} \frac{1}{(R'')^{\nu(R)}}\right)$$
$$= \ln(|b_{\nu(R)}|(RR'')^{\nu(R)}) + \ln\left(\frac{1}{(R'')^{\nu(R)}}\right)$$
$$\leqslant \ln \mu(R''R, z^0, F) - \|\nu(R)\| \ln \min\{r_1'', r_2''\}.$$

This implies that

$$(4.11)\quad \|\nu(R)\| \leqslant \frac{1}{\ln\min\{r_1'', r_2''\}}(\ln\mu(R''R, z^0, F) - \ln\mu(R, z^0, F))$$
$$\leqslant \frac{1}{\ln\min\{r_1'', r_2''\}}$$
$$\times (\ln M(R''R, z^0, F) - \ln((1-r_1')(1-r_2')M(R'R, z^0, F)))$$
$$\leqslant \frac{1}{\ln\min\{r_1'', r_2''\}}(\ln M(R''R, z^0, F) - \ln M(R'R, z^0, F))$$
$$- \frac{1}{\ln\min\{r_1'', r_2''\}}\sum_{j=1}^{2}\ln(1-r_j')$$
$$= \frac{1}{\ln\min\{r_1'', r_2''\}}\ln\frac{M(R''R, z^0, F)}{M(R'R, z^0, F)} - \frac{1}{\ln\min\{r_1'', r_2''\}}\sum_{j=1}^{2}\ln(1-R_j).$$

Put $R = \mathbf{1}/\mathbf{L}(z^0)$. Now let $N(F, z^0, \mathbf{L})$ be an **L**-index of function F in joint variables at point z^0, i.e. it is the least integer for which inequality (2.1) holds at point z^0. Clearly,

$$(4.12)\qquad N(F, z^0, \mathbf{L}) \leqslant \nu(1/\mathbf{L}(z^0), z^0, F) = \nu(R, z^0, F).$$

But

$$(4.13)\qquad M(R''/\mathbf{L}(z^0), z^0, F) \leqslant p_1(R', R'')M(R'/\mathbf{L}(z^0), z^0, F).$$

Therefore, from (4.11), (4.12), (4.13) we obtain that for all $z^0 \in \mathbb{D}^2$:

$$N(F, z^0, \mathbf{L}) \leqslant -\frac{1}{\ln\min\{r_1'', r_2''\}}\sum_{j=1}^{2}\ln(1-r_j') + \frac{\ln p_1(R', R'')}{\ln\min\{r_1'', r_2''\}}.$$

This means that F has bounded **L**-index in joint variables. □

5. An analogue of Hayman's theorem for in a bidisc analytic function of bounded L-index in joint variables

Theorem 5.1 is an analogue of known Hayman's theorem, which was established for entire functions of one complex variable (see [9]).

Theorem 5.1. *Let $\mathbf{L} \in Q(\mathbb{D}^2)$. A function F analytic in $\mathbb{D}^2$ has bounded* **L***-index in joint variables if and only if there exist $p \in \mathbb{Z}_+$ and $c \in \mathbb{R}_+$ such that for each*

$z \in \mathbb{D}^2$ *the inequality*

$$(5.1)\quad \max\left\{\frac{|F^{(j_1,j_2)}(z)|}{l_1^{j_1}(z)l_2^{j_2}(z)}\colon\ j_1+j_2=p+1\right\} \leqslant c\max\left\{\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z)l_2^{k_2}(z)}\colon\ k_1+k_2\leqslant p\right\}$$

holds.

Proof. Let $N=N(F,\mathbf{L},\mathbb{D}^2)<\infty$. The necessity we obtain immediately from the definition of the boundedness of $\mathbf{L}$-index in joint variables with $p=N$ and $c=((N+1)!)^2$. We prove the sufficiency. If $F\equiv 0$, then theorem is obvious. Thus, we suppose that $F\not\equiv 0$. Let (5.1) hold, $z^0\in\mathbb{D}^2$, $z\in\mathbb{T}^2(z^0,\boldsymbol{\beta}/\mathbf{L}(z^0))$. For all $j=(j_1,j_2)\in\mathbb{Z}_+^2$, $j_1+j_2\leqslant p+1$ we have

$$\begin{aligned}
\frac{|F^{(j_1,j_2)}(z)|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)} &\leqslant \frac{|F^{(j_1,j_2)}(z)|l_1^{j_1}(z)l_2^{j_2}(z)}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)l_1^{j_1}(z)l_2^{j_2}(z)} \leqslant \lambda_{1,1}^{j_1}(\beta)\lambda_{1,2}^{j_2}(\beta)\frac{|F^{(j_1,j_2)}(z)|}{l_1^{j_1}(z)l_2^{j_2}(z)}\\
&\leqslant \lambda_{1,1}^{j_1}(\beta)\lambda_{1,2}^{j_2}(\beta)c\max\left\{\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z)l_2^{k_2}(z)}\colon\ k_1+k_2\leqslant p\right\}\\
&= \lambda_{2,1}^{j_1}(\beta)\lambda_{2,2}^{j_2}(\beta)c\max\left\{\frac{l_1^{k_1}(z^0)l_2^{k_2}(z^0)|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z^0)l_2^{k_2}(z^0)l_1^{k_1}(z)l_2^{k_2}(z)}\colon\ k_1+k_2\leqslant p\right\}\\
&\leqslant \lambda_{2,1}^{j_1}(\beta)\lambda_{2,2}^{j_2}(\beta)c\max\left\{\frac{1}{\lambda_{1,1}^{k_1}(\beta)\lambda_{1,2}^{k_2}(\beta)}\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z^0)l_2^{k_2}(z^0)}\colon\ k_1+k_2\leqslant p\right\}\\
&\leqslant \max\{\lambda_{2,1}^{j_1}(\beta)\lambda_{2,2}^{j_2}(\beta)\colon\ j_1+j_2\leqslant p+1\}\\
&\quad\times c\max\left\{\frac{1}{\lambda_{1,1}^{k_1}(\beta)\lambda_{1,2}^{k_2}(\beta)}\colon\ k_1+k_2\leqslant p\right\}\\
&\quad\times \max\left\{\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z^0)l_2^{k_2}(z^0)}\colon\ k_1+k_2\leqslant p\right\} = BG(z),
\end{aligned}$$

where

$$B=c\max\{\lambda_{2,1}^{j_1}(\beta)\lambda_{2,2}^{j_2}(\beta)\colon\ j_1+j_2\leqslant p+1\}\max\{\lambda_{1,1}^{-k_1}(\beta)\lambda_{1,2}^{-k_2}(\beta)\colon\ k_1+k_2\leqslant p\},$$

$$G(z)=\max\left\{\frac{|F^{(k_1,k_2)}(z)|}{l_1^{k_1}(z^0)l_2^{k_2}(z^0)}\colon\ k_1+k_2\leqslant p\right\}.$$

We choose $z^{(1)}=(z_1^{(1)},z_2^{(1)})\in\mathbb{T}^2(z^0,\mathbf{1}/(2\beta\mathbf{L}(z^0)))$ and $z^{(2)}=(z_1^{(2)},z_2^{(2)})\in\mathbb{T}^2(z^0,\boldsymbol{\beta}/\mathbf{L}(z^0))$ such that $F(z^{(1)})\neq 0$ and

$$(5.2)\qquad |F(z^{(2)})|=\max\{|F(z)|\colon\ z\in\mathbb{T}^2(z^0,\boldsymbol{\beta}/\mathbf{L}(z^0))\}\neq 0.$$

These points exist, otherwise if $F(z)\equiv 0$ on skeleton $\mathbb{T}^2(z^0,\mathbf{1}/(2\boldsymbol{\beta}\mathbf{L}(z^0)))$ or $\mathbb{T}^2(z^0,\boldsymbol{\beta}/\mathbf{L}(z^0))$, then by the uniqueness theorem, $F\equiv 0$ in $\mathbb{D}^2$. We connect the points $z^{(1)}$ and $z^{(2)}$ with a plane

$$\alpha\colon\ z_2 = k_2 z_1 + c_2, \quad \frac{z_2 - z_2^{(1)}}{z_2^{(2)} - z_2^{(1)}} = \frac{z_1 - z_1^{(1)}}{z_1^{(2)} - z_1^{(1)}},$$
$$k_2 = \frac{z_2^{(2)} - z_2^{(1)}}{z_1^{(2)} - z_1^{(1)}}, \quad c_2 = \frac{z_2^{(1)} z_1^{(2)} - z_1^{(1)} z_2^{(2)}}{z_1^{(2)} - z_1^{(1)}}.$$

Let $\widetilde{G}(z_1) = G(z)|_\alpha$ be a restriction of function G onto α. All functions $F^{(k_1,k_2)}|_\alpha$ are analytic functions of variable z_1 and $\widetilde{G}(z_1^{(1)}) = G(z^{(1)})|_\alpha \neq 0$, because $F(z^{(1)}) \neq 0$. That's why zeros of function $\widetilde{G}(z_1)$ are isolated as zeros of a function of one variable. Therefore we can choose piecewise analytic curve γ onto α:

$$z = z(t) = (z_1(t), k_2 z_1(t) + c_2), \quad t \in [0,1],$$

which connects points $z^{(1)}$, $z^{(2)}$ and such that $G(z(t)) \neq 0$ and $\int_0^1 |z_1'(t)|\,\mathrm{d}t \leqslant 2\beta/l_1(z^0)$. For a construction of the curve we connect $z^{(1)}$ and $z^{(2)}$ by a line $z_1^*(t) = (z_1^{(2)} - z_1^{(1)})t + z_1^{(1)}$, $t \in [0,1]$. The curve γ can cross points z_1 at which the function $\widetilde{G}(z_1) = 0$. The number of such points $m = m(z^{(1)}, z^{(2)})$ is finite. Let $(z_{1,k}^*)$ be a sequence of these points in ascending order of the value $|z_1^{(1)} - z_{1,k}^*|$, $k \in \{1, 2, \ldots, m\}$. We choose

$$r < \min_{1 \leqslant k \leqslant m-1} \Big\{ |z_{1,k}^* - z_{1,k+1}^*|, |z_{1,1}^* - z_1^{(1)}|, |z_{1,m}^* - z_1^{(2)}|, \frac{2\beta^2 - 1}{2\pi\beta l_1(z^0)} \Big\}.$$

Now we construct circles with the centre at points $z_{1,k}^*$ and corresponding radii $r_k' < r2^{-k}$ such that $\widetilde{G}(z_1) \neq 0$ for all z_1 on the circles. It is possible, because $F \not\equiv 0$. Every such circle is divided into two semicircles by the line $z_1^*(t)$. The required piecewise analytic curve consists of arcs of the constructed semicircles and segments of line $z_1^*(t)$, which connect the arcs in series between themselves or with points $z_1^{(1)}, z_1^{(2)}$. The length of $z_1(t)$ in $\mathbb{C}$ is less than

$$\frac{\beta}{l_1(z^0)} + \frac{1}{2\beta l_1(z^0)} + \pi r \leqslant \frac{2\beta}{l_1(z^0)}.$$

Then

$$\int_0^1 |z_2'(t)|\,\mathrm{d}t = |k_2| \int_0^1 |z_1'(t)|\,\mathrm{d}t \leqslant \frac{|z_2^{(2)} - z_2^{(1)}|}{|z_1^{(2)} - z_1^{(1)}|} \frac{2\beta}{l_1(z^0)}$$
$$\leqslant \frac{2\beta^2 + 1}{2\beta l_2(z^0)} \frac{2\beta l_1(z^0)}{2\beta^2 - 1} \frac{2\beta}{l_1(z^0)} = \frac{2\beta(2\beta^2 + 1)}{(2\beta^2 - 1) l_2(z^0)}.$$

Hence,

$$(5.3)\quad \int_0^1 \sum_{i=1}^{2} l_i(z^0)|z_i'(t)|\,\mathrm{d}t = \int_0^1 l_1(z^0)|z_1'(t)|\,\mathrm{d}t + \int_0^1 l_2(z^0)|z_2'(t)|\,\mathrm{d}t$$

$$\leqslant l_1(z^0)\frac{2\beta}{l_1(z^0)} + l_2(z^0)\frac{2\beta(2\beta^2+1)}{(2\beta^2-1)l_2(z^0)} = \frac{8\beta^3}{2\beta^2-1} = S.$$

Since the function $z = z(t)$ is piecewise analytic on $[0,1]$, for arbitrary $k \in \mathbb{Z}_+^2$, $j \in \mathbb{Z}_+^2$, $\|k\| \leqslant p$, $\|j\| \leqslant p$, $k \neq j$ either

$$(5.4)\qquad \frac{|F^{(k_1,k_2)}(z_1(t),z_2(t))|}{l_1^{k_1}(z^0)l_2^{k_2}(z^0)} \equiv \frac{|F^{(j_1,j_2)}(z_1(t),z_2(t))|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}$$

or the equality

$$(5.5)\qquad \frac{|F^{(k_1,k_2)}(z_1(t),z_2(t))|}{l_1^{k_1}(z^0)l_2^{k_2}(z^0)} = \frac{|F^{(j_1,j_2)}(z_1(t),z_2(t))|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}$$

holds only for a finite set of points $t_k \in [0;1]$. Then for function $G(z(t))$ as maximum of such expressions $|F^{(j_1,j_2)}(z_1(t),z_2(t))|/(l_1^{j_1}(z^0)l_2^{j_2}(z^0))$ by all $\|j\| \leqslant p$ two cases are possible:

Case 1. In some interval of analyticity of the curve γ, function $G(z(t))$ identically equals simultaneously to some derivatives, that is, (5.4) holds. It means that $G(z(t)) \equiv |F^{(j_1,j_2)}(z_1(t),z_2(t))|/(l_1^{j_1}(z^0)l_2^{j_2}(z^0))$ for some $\|j\| \leqslant p$. Clearly, function $F^{(j_1,j_2)}(z_1(t),z_2(t))$ is analytic. Then $|F^{(j_1,j_2)}(z_1(t),z_2(t))|$ is continuously differentiable function on the interval of analyticity except points where this partial derivative equals zero $|F^{(j_1,j_2)}(z_1(t),z_2(t))| = 0$. However, such points don't exist, because in the opposite case $G(z(t)) = 0$. But it contradicts the construction of the curve γ.

Case 2. In some interval of analyticity of the curve γ, function $G(z(t))$ equals simultaneously to some derivatives at a finite number of points t_k, that is, (5.5) holds. Then points t_k divide the interval of analyticity into a finite number of segments, in which $G(z(t))$ equals to one of the partial derivatives, i.e. $G(z(t)) \equiv |F^{(j_1,j_2)}(z_1(t),z_2(t))|/(l_1^{j_1}(z^0)l_2^{j_2}(z^0))$ for some j, $\|j\| \leqslant p$. As above, in each of these segments, functions $|F^{(j_1,j_2)}(z_1(t),z_2(t))|$ and $G(z(t))$ are continuously differentiable except points t_k.

Taking into account (3.1) and using the inequality

$$\frac{\mathrm{d}}{\mathrm{d}x}|\varphi(x)| \leqslant \left|\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x)\right|,$$

which holds for complex-valued functions of real argument outside a countable set of points, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}G(z(t))\\
&\quad\leqslant \max\biggl\{\frac{1}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}\Bigl|\frac{\mathrm{d}}{\mathrm{d}t}F^{(j_1,j_2)}(z_1(t),z_2(t))\Bigr|\colon\ j_1+j_2\leqslant p\biggr\}\\
&\quad\leqslant \max_{j_1+j_2\leqslant p}\left\{\frac{|F^{(j_1+1,j_2)}(z_1(t),z_2(t))|\cdot|z_1'(t)|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}+\frac{|F^{(j_1,j_2+1)}(z_1(t),z_2(t))|\cdot|z_2'(t)|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}\right\}\\
&\quad= \max_{j_1+j_2\leqslant p}\left\{|F^{(j_1+1,j_2)}(z(t))|\frac{|z_1'(t)|l_1(z^0)}{l_1^{j_1+1}(z^0)l_2^{j_2}(z^0)}+|F^{(j_1,j_2+1)}(z(t))|\frac{|z_2'(t)|l_2(z^0)}{l_1^{j_1}(z^0)l_2^{j_2+1}(z^0)}\right\}\\
&\quad\leqslant (|z_1'(t)|l_1(z^0)+|z_2'(t)|l_2(z^0))\max\biggl\{\frac{|F^{(j_1,j_2)}(z_1(t),z_2(t))|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}\colon\ j_1+j_2\leqslant p+1\biggr\}\\
&\quad\leqslant \biggl(\sum_{i=1}^{2}l_i(z^0)|z_i'(t)|\biggr)BG(z(t)).
\end{aligned}$$

Therefore, (5.3) yields

$$\Bigl|\ln\frac{G(z^{(2)})}{G(z^{(1)})}\Bigr|=\biggl|\int_0^1\frac{1}{G(z(t))}\frac{\mathrm{d}}{\mathrm{d}t}G(z(t))\biggr|\leqslant B\int_0^1\sum_{i=1}^{2}l_i(z^0)|z_i'(t)|\,\mathrm{d}t\leqslant BS.$$

Using (5.2), we deduce

$$\max\{|F(z)|\colon\ z\in\mathbb{T}^2(z^0,\boldsymbol{\beta}/\mathbf{L}(z^0))\}=|F(z^{(2)})|\leqslant G(z^{(2)})\leqslant G(z^{(1)})\exp(BS).$$

Since $z^{(1)}\in\mathbb{T}^2(z^0,\mathbf{1}/(2\beta\mathbf{L}(z^0)))$, the Cauchy inequality

$$\frac{|F^{(j)}(z^{(1)})|}{l_1^{j_1}(z^0)l_2^{j_2}(z^0)}\leqslant j_1!j_2!(2\beta)^{j_1+j_2}M(\mathbf{1}/(2\boldsymbol{\beta}\mathbf{L}(z^0)),z^0,F)$$

holds for every $j\in\mathbb{Z}_+^2$. Therefore, for $j_1+j_2\leqslant p$ we have

$$G(z^{(1)})\leqslant (p!)^2(2\beta)^{2p}M(\mathbf{1}/(2\boldsymbol{\beta}\mathbf{L}(z^0)),z^0,F)$$

and

$$\begin{aligned}
&\max\{|F(z)|\colon\ z\in\mathbb{T}^2(z^0,\boldsymbol{\beta}/\mathbf{L}(z^0))\}\\
&\qquad\qquad\leqslant \mathrm{e}^{BS}(p!)^2(2\beta)^{2p}\max\{|F(z)|\colon\ z\in\mathbb{T}^2(z^0,\mathbf{1}/(2\boldsymbol{\beta}\mathbf{L}(z^0)))\}.
\end{aligned}$$

Hence, by Theorem 4.2, F is a function of bounded **L**-index in joint variables. □

Theorem 5.2. *Let $\beta > 1$, $\mathbf{L} \in Q(\mathbb{D}^2)$. A function F analytic in $\mathbb{D}^2$ has bounded $\mathbf{L}$-index in joint variables if and only if there exist $c \in (0;\infty)$ and $N \in \mathbb{N}$ such that for each $z \in \mathbb{D}^2$ the inequality*

$$\sum_{k_1+k_2=0}^{N} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)} \geqslant c \sum_{k_1+k_2=N+1}^{\infty} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)} \tag{5.6}$$

holds.

Proof. Let $1/\beta < \theta_j < 1$, $j \in \{1,2\}$. If function F has bounded $\mathbf{L}$-index in joint variables, then by Theorem 3.3 F has bounded $\widetilde{\mathbf{L}}$-index in joint variables, where $\widetilde{\mathbf{L}} = (\widetilde{l}_1(z), \widetilde{l}_2(z))$, $\widetilde{l}_j(z) = \theta_j l_j(z)$, $j \in \{1,2\}$. Therefore,

$$\begin{aligned}
&\max\left\{\frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)} \colon\ 0 \leqslant k_1+k_2 \leqslant N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)\right\}\\
&\quad= \max\left\{\frac{\theta_1^{k_1}\theta_2^{k_2}|F^{(k_1,k_2)}(z)|}{k_1!k_2!\widetilde{l}_1^{k_1}(z)\widetilde{l}_2^{k_2}(z)} \colon\ 0 \leqslant k_1+k_2 \leqslant N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)\right\}\\
&\quad\geqslant (\theta_1\theta_2)^{N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)} \max\left\{\frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!\widetilde{l}_1^{k_1}(z)\widetilde{l}_2^{k_2}(z)} \colon\ 0 \leqslant k_1+k_2 \leqslant N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)\right\}\\
&\quad\geqslant (\theta_1\theta_2)^{N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)} \frac{|F^{(j_1,j_2)}(z)|}{j_1!j_2!\widetilde{l}_1^{j_1}(z)\widetilde{l}_2^{j_2}(z)}\\
&\quad= \theta_1^{N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)-j_1}\theta_2^{N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)-j_2} \frac{|F^{(j_1,j_2)}(z)|}{j_1!j_2!l_1^{j_1}(z)l_2^{j_2}(z)}
\end{aligned}$$

for all $j_1 \geqslant 0$, $j_2 \geqslant 0$ and

$$\begin{aligned}
&\sum_{j_1+j_2=N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)+1}^{\infty} \frac{|F^{(j_1,j_2)}(z)|}{j_1!j_2!l_1^{j_1}(z)l_2^{j_2}(z)}\\
&\qquad\leqslant \max_{0\leqslant k_1+k_2\leqslant N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)}\\
&\qquad\quad\times \sum_{j_1+j_2=N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)+1}^{\infty} \theta_1^{j_1-N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)}\theta_2^{j_2-N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)}\\
&\qquad= \frac{\theta_1\theta_2}{(1-\theta_1)(1-\theta_2)} \max\left\{\frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)} \colon\ 0 \leqslant k_1+k_2 \leqslant N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)\right\}\\
&\qquad\leqslant \frac{\theta_1\theta_2}{(1-\theta_1)(1-\theta_2)} \sum_{k_1+k_2=0}^{N(F,\widetilde{\mathbf{L}},\mathbb{D}^2)} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)}.
\end{aligned}$$

Hence, we obtain (5.6) with $N = N(F, \widetilde{\mathbf{L}}, \mathbb{D}^2)$ and $c = \theta_1\theta_2/((1-\theta_1)(1-\theta_2))$. On the contrary, inequality (5.6) implies

$$\max\left\{\frac{|F^{(j_1,j_2)}(z)|}{j_1!j_2!l_1^{j_1}(z)l_2^{j_2}(z)} \colon j_1+j_2=N+1\right\} \leqslant \sum_{k_1+k_2=N+1}^{\infty} \frac{F^{(k_1,k_2)}(z)}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)}$$
$$\leqslant \frac{1}{c}\sum_{k_1+k_2=0}^{N} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)}$$
$$\leqslant \frac{(N+1)N}{2c} \max_{0\leqslant k_1+k_2\leqslant N} \frac{|F^{(k_1,k_2)}(z)|}{k_1!k_2!l_1^{k_1}(z)l_2^{k_2}(z)}$$

and by Theorem 5.1, F is of bounded **L**-index in joint variables. □

References

[1] *A. Bandura*: New criteria of boundedness of **L**-index in joint variables for entire functions. Mat. Visn. Nauk. Tov. Im. Shevchenka *13* (2016), 58–67. (In Ukrainian.) zbl
[2] *A. I. Bandura, M. T. Bordulyak, O. B. Skaskiv*: Sufficient conditions of boundedness of L-index in joint variables. Mat. Stud. *45* (2016), 12–26. zbl MR doi
[3] *A. I. Bandura, O. B. Skaskiv*: Entire Functions of Several Variables of Bounded Index. Chyslo, Lviv, 2015. zbl
[4] *A. I. Bandura, O. B. Skaskiv*: Analytic in the unit ball functions of bounded L-index in direction. Avaible at `https://arxiv.org/abs/1501.04166`.
[5] *A. I. Bandura, N. V. Petrechko, O. B. Skaskiv*: Analytic functions in a polydisc of bounded **L**-index in joint variables. Mat. Stud. *46* (2016), 72–80. zbl MR doi
[6] *M. T. Bordulyak*: The space of entire functions in $\mathbb{C}^n$ of bounded L-index. Mat. Stud. *4* (1995), 53–58. zbl MR
[7] *M. T. Bordulyak, M. M. Sheremeta*: Boundedness of the L-index of an entire function of several variables. Dopov./Dokl. Akad. Nauk Ukraïni *9* (1993), 10–13. (In Ukrainian.) MR
[8] *J. Gopala Krishna, S. M. Shah*: Functions of bounded indices in one and several complex variables. Math. Essays dedicated to A. J. Macintyre. Ohio Univ. Press, Athens, Ohio, 1970, pp. 223–235. zbl MR
[9] *W. K. Hayman*: Differential inequalities and local valency. Pac. J. Math. *44* (1973), 117–137. zbl MR doi
[10] *V. O. Kushnir, M. M. Sheremeta*: Analytic functions of bounded l-index. Mat. Stud. *12* (1999), 59–66. zbl MR
[11] *B. Lepson*: Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index. Entire Funct. and Relat. Parts of Anal., La Jolla, Calif. 1966. Proc. Sympos. Pure Math. 11, AMS, Providence, Rhode Island, 1968, pp. 298–307. zbl MR
[12] *F. Nuray, R. F. Patterson*: Multivalence of bivariate functions of bounded index. Matematiche *70* (2015), 225–233. zbl MR doi
[13] *M. Salmassi*: Functions of bounded indices in several variables. Indian J. Math. *31* (1989), 249–257. zbl MR
[14] *M. Sheremeta*: Analytic Functions of Bounded Index. Mathematical Studies Monograph Series 6. VNTL Publishers, Lviv, 1999. zbl MR

[15] *S. N. Strochyk, M. M. Sheremeta*: Analytic in the unit disc functions of bounded index. Dopov./Dokl. Akad. Nauk Ukraïni *1* (1993), 19–22. (In Ukrainian.) zbl MR

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