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Safyan Ahmad; Shamsa Kanwal; Talat Firdous

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# COHEN-MACAULAY MODIFICATIONS OF THE VERTEX COVER IDEAL OF A GRAPH

Safyan Ahmad, Shamsa Kanwal, Lahore, Talat Firdous, Gujrat

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*Abstract.* We study when the modifications of the Cohen-Macaulay vertex cover ideal of a graph are Cohen-Macaulay.

*Keywords*: monomial ideal, minimal vertex cover, polarization of ideal, chordal graph

*MSC 2010*: 13A02, 13D02, 13D25, 13P10

## 1. Introduction

Let $K$ be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring in $n$ variables over $K$, and let $I$ be a squarefree Cohen-Macaulay monomial ideal in $S$. We will denote the unique minimal system of monomial generators of $I$ by $G(I)$. Let $G(I) = \{u_1, \ldots, u_m\}$, then we call a monomial ideal $J$ a modification of $I$, if $G(J) = \{v_1, \ldots, v_m\}$ and $\operatorname{supp}(u_i) = \operatorname{supp}(v_i)$ for all $i$. By support of a monomial $u$ we mean the set $\operatorname{supp}(u) = \{i\colon x_i \text{ divides } u\}$. A monomial ideal $J$ is called a *trivial modification* of $I$, if there exist nonnegative integers $a_1, \ldots, a_n$ such that $J$ is obtained from $I$ by the substitutions $x_i \mapsto {x_i}^{a_i}$ for all $i$. Obviously, if $J$ is a trivial modification of $I$, then $J$ is Cohen-Macaulay as $J = \varphi(I)S$ where $\varphi\colon S \to S$ is a flat $K$-algebra homomorphism with $\varphi(x_i) = {x_i}^{a_i}$ for all $i$.

Let $G$ be a simple connected graph on the vertex set $V(G) = \{v_1, \ldots, v_n\}$ with the edge set $E(G)$. The vertex cover ideal $I_G$ associated to $G$ is the ideal generated by all monomials of the form $\prod\limits_{x_i \in C} x_i$ for all minimal vertex covers $C$ of $G$. Recall that by a minimal vertex cover we mean a subset $C \subset V(G)$ such that every edge has at least one vertex in $C$ and no proper subset of $C$ has the same property,

$$I_G = \bigcap_{\{v_i, v_j\} \in E(G)} (x_i, x_j).$$

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Dually one defines the edge ideal

$$I(G) = (x_i x_j \colon \{v_i, v_j\} \in E(G))$$

Let $\Delta_G$ be the simplicial complex whose Stanley-Reisner ideal $I_{\Delta_G}$ coincides with $I(G)$. Then $I_G = I_{\Delta_G^\vee}$, where $\Delta_G^\vee$ is the Alexander dual of $\Delta_G$.

Recall that a graph $G$ is chordal if each cycle in $G$ of length greater than 3 has a chord and the complementary graph $\overline{G}$ of $G$ is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{\{v_i, v_j\} \colon \{v_i, v_j\} \notin E(G)\}$. By using the Alexander duality and results by Eagon-Reiner [4] as well as Fröberg [5] we immediately obtain the following statement.

**Proposition 1.1** ([1]). *The ideal $I_G$ is Cohen-Macaulay if and only if the complementary graph $\overline{G}$ is chordal.*

The purpose of this paper is to complement the results presented in the paper [1]; related questions have been studied in [2], [3] and [7].

Let us first review the concept of polarization. Given a monomial

$$u = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$$

we define the following monomial in a new set of variables:

$$u^p = \prod_{i=1}^{n} \prod_{j=1}^{a_i} x_{ij}.$$

Now let $I \subset S$ be an arbitrary monomial ideal with the minimal set of monomial generators $\{u_1, \dots, u_m\}$. Then we set

$$I^p = (u_1^p, \dots, u_m^p).$$

This ideal is called the *polarization* of $I$. If we choose an arbitrary set $\{v_1, \dots, v_r\}$ of monomial generators of $I$, then we have

$$I^p = I^p R = (v_1^p, \dots, v_r^p) R,$$

where $R$ is the polynomial ring over $K$ in the variables which are needed to polarize the monomials $v_i$. We will also need the following rule: Suppose $I = I_1 \cap I_2 \cap \dots \cap I_r$ where each $I_j$ is a monomial ideal, then

$$I^p = I^p R = (I_1^p R \cap I_2^p R \cap \dots \cap I_r^p R), \tag{1.1}$$

where $R$ is again the polynomial ring over $K$ in the variables which are needed to polarize all the monomials involved.

**Proposition 1.2** ([6], Corollary 1.6.3). *Let $I$ be a monomial ideal. The following condition are equivalent:*

- ▷ *$I$ is Cohen-Macaulay.*
- ▷ *$I^p$ is Cohen-Macaulay.*

We need some preparation to formulate the main results.

## 2. Modifications of first type

Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$ and let $I_G$ be the vertex cover ideal of $G$,

$$I_G = \bigcap_{\{v_i,v_j\}\in E(G)} (x_i, x_j).$$

In this section we consider those modifications where we take powers of both variables in the prime ideal in $I_G$ corresponding to some edge, in the primary decomposition of $I_G$, i.e. of the form

$$J = (x_{a_i}^m, x_{b_i}^m) \cap \Bigg( \bigcap_{\{a_j,b_j\}\in E(G),\, i\neq j} (x_{a_j}, x_{b_j}) \Bigg).$$

**Definition 2.1.** For $\{a,b\} \in E(G)$ and $m \in \mathbb{Z}$, $m > 1$, we define a new graph $\overline{G_{m,ab}}$ with vertex set $V(\overline{G_{m,ab}}) = V(G) \cup \{a_{11},\dots,a_{1m-1},b_{11},\dots,b_{1m-1}\}$, where $a_{1i}, b_{1i} \notin V(G)$ for all $i = 1,\dots,m-1$ and edge set

$$E(\overline{G_{m,ab}}) = E(\overline{G}) \cup \Bigg( \bigcup_{i=1}^{m-1} \{\{a_{1i},v\},\{u,b_{1i}\}\colon\ u,v\in V(G),\ u\neq a,\ v\neq b\} \Bigg) \cup \Bigg( \bigcup_{i\neq j,\, i,j=1}^{m-1} \{a_{1i},a_{1j}\} \Bigg) \cup \Bigg( \bigcup_{i\neq j,\, i,j=1}^{m-1} \{b_{1i},b_{1j}\} \Bigg)$$

With the above notation, we have the following lemma.

**Lemma 2.2.** *Let $G$ be a graph with $|V(G)| \geqslant 4$ and $\{a,b\}\in E(G)$. If there exist $c,d \in V(G)\setminus\{a,b\}$ with $\{c,d\}\in E(G)$ then $\overline{G_{m,ab}}$ contains minimal cycles $\{a_{1i},c,b_{1i},d\}$ for all $i=1,\dots,m-1$.*

Proof. Suppose there exist $c, d \in V(G) \setminus \{a, b\}$ with $\{c, d\} \in E(G)$. By the definition of $\overline{G_{m,ab}}$, we know that

$$E(\overline{G_{m,ab}}) = E(\overline{G}) \cup \left(\bigcup_{i=1}^{m-1} \{\{a_{1i}, v\}, \{u, b_{1i}\}\colon\ u, v \in V(G),\ u \neq a,\ v \neq b\}\right)$$
$$\cup \left(\bigcup_{i\neq j,\, i,j=1}^{m-1} \{a_{1i}, a_{1j}\}\right) \cup \left(\bigcup_{i\neq j,\, i,j=1}^{m-1} \{b_{1i}, b_{1j}\}\right).$$

As $\{c, d\} \in E(G)$, so $\{c, d\} \notin E(\overline{G_{m,ab}})$. Also by the definition of $\overline{G_{m,ab}}$ it is clear that $\{a_{1i}, b_{1j}\} \notin E(\overline{G_{m,ab}})$ for all $i, j = 1, \ldots, m-1$ and $\{a_{1i}, c\}, \{a_{1i}, d\}, \{c, b_{1i}\}, \{d, b_{1i}\} \in E(\overline{G_{m,ab}})$ for all $i = 1, \ldots, m-1$. Using all these facts, we have minimal cycles $\{a_{1i}, c, b_{1i}, d\}$ in $\overline{G_{m,ab}}$ for all $i = 1, \ldots, m-1$. □

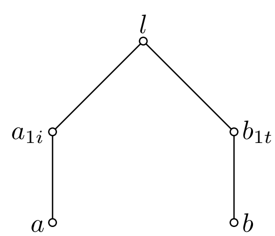
Another observation regarding $\overline{G_{m,ab}}$ is recorded as the following lemma.

**Lemma 2.3.** *Let $G$ be a graph with $|V(G)| \geqslant 4$. If $\overline{G}$ is chordal then $\overline{G_{m,ab}}$ has no minimal cycle of length not less than 5.*

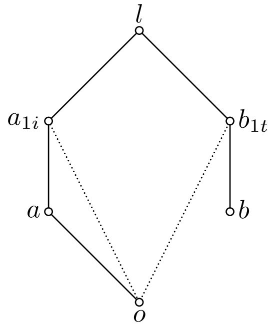
Proof. Since $\overline{G}$ is chordal, all its minimal cycles have length 3. Suppose that $\overline{G_{m,ab}}$ contains a minimal cycle $C$ of length not less than 5; as $\overline{G}$ is chordal it follows that $V(C) \not\subset V(\overline{G})$. Thus there exists $v \in V(C)$ such that $v \notin V(\overline{G})$ and by the definition of $\overline{G_{m,ab}}$, $v \in \{a_{11}, \ldots, a_{1m-1}, b_{11}, \ldots, b_{1m-1}\}$.

If $v = a_{1i}$ for some $i = 1, \ldots, m-1$, we know from the definition that $a_{1i}$ is adjacent to every vertex in $\overline{G_{m,ab}}$ except $b, b_{11}, \ldots, b_{1m-1}$. Thus $C$ must contain the edge formed by $b$ and $b_{1t}$ for some $t \in \{1, \ldots, m-1\}$; note that $b_{1k} \notin V(C)$ for $k \neq t$ because $\{b, b_{1k}\}, \{b_{1k}, b_{1j}\} \in E(\overline{G_{m,ab}})$ for all $k \neq j; k, j = 1, \ldots, m-1$.

Similar reasoning shows that $C$ also contains $a$ and so $\{a, a_{1i}\}, \{b, b_{1t}\} \in E(C)$. Now for any $l \in V(C) \setminus \{a, a_{1i}, b, b_{1t}\}$, we have $\{l, a_{1i}\}, \{l, b_{1t}\} \in E(\overline{G_{m,ab}})$. Thus the cycle must be of the following form:



As $\{a,b\} \notin E(\overline{G_{m,ab}})$ and this is a cycle, we must have at least one more vertex in this cycle, say $o$,




where $o \notin \{a, b, a_{1i}, b_{1t}, l\}$.

But then by the definition of $\overline{G_{m,ab}}$, we must have $\{a_{1i}, o\}, \{b_{1t}, o\} \in E(\overline{G_{m,ab}})$, thus no such cycle of length not less than 5 exists in $\overline{G_{m,ab}}$. The case when $v = b_{1i}$ for any $i = 1, \ldots, m-1$, can be proved along the same lines. $\square$

**Proposition 2.4.** *Let $\overline{G}$ be a chordal graph with $|V(\overline{G})| \geqslant 4$, then $\overline{G_{m,ab}}$ is not chordal if and only if there exist $\{c,d\} \in E(G)$ with $c, d \in V(G) \setminus \{a,b\}$.*

Proof. If there exist $\{c,d\} \in E(G)$ with $c,d \in V(G) \setminus \{a,b\}$, Lemma 2.2 guarantees that $\overline{G_{m,ab}}$ contains at least one minimal 4-cycle through $c$ and $d$, thus $\overline{G_{m,ab}}$ is not chordal.

Conversely if $\overline{G_{m,ab}}$ is not chordal and $\overline{G}$ is chordal, Lemma 2.3 ensures that $\overline{G_{m,ab}}$ contains a 4 cycle, say $\{p,q,r,s\}$. As $\overline{G}$ is chordal, some of these vertices do not belong to $V(\overline{G})$. Moreover, $\{q,s\}$ does not belong to $E(\overline{G})$, without loss of generality, we may assume that $p = a_{1i}$ for some $i = 1, \ldots, m-1$, then neither $q$ nor $s$ are $b$. Since $\{a_{1i}, r\} \notin E(\overline{G_{m,ab}})$ we have that $r = b$ or $r = b_{1j}$ for some $j = 1, \ldots, m-1$. By the definition of $\overline{G_{m,ab}}$, we have $q \neq a$, $s \neq a$ so that $\{q,s\}$ is the requested edge, as desired. $\square$

**Theorem 2.5.** *Let $G$ be a simple connected graph and let*

$$I_G = \bigcap_{\{a_i,b_i\} \in E(G)} (x_{a_i}, x_{b_i})$$

*be the Cohen-Macaulay vertex cover ideal of $G$. Then*

$$J = (x_{a_i}^m, x_{b_i}^m) \cap \left( \bigcap_{\{a_j,b_j\} \in E(G),\, i \neq j} (x_{a_j}, x_{b_j}) \right)$$

*is Cohen-Macaulay for no $m > 1$ if and only if there exist an edge $\{c,d\} \in E(G)$ such that $c, d \notin \{a_i, b_i\}$.*

P r o o f. The ideal $I_G$ is Cohen-Macaulay if and only if $\overline{G}$ is chordal. Using primary decomposition and polarization we can observe that the ideal $J$ will be Cohen-Macaulay if and only if the graph $\overline{G_{m,a_ib_i}}$ is chordal. But by Proposition 2.4, $\overline{G_{m,a_ib_i}}$ is not chordal if and only if there exist $\{c,d\} \in E(G)$ with $c,d \in V(G) \setminus \{a_i,b_i\}$. Thus the ideal $J$ will not be Cohen-Macaulay if and only if there exist $\{c,d\} \in E(G)$ with $c,d \in V(G) \setminus \{a_i,b_i\}$, completing the proof. □

**Corollary 2.6.** *If $\{a_i,b_i\}$ is a minimal vertex cover of $G$, then the ideal*

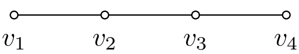
$$J = (x_{a_i}^m, x_{b_i}^m) \cap \Bigg( \bigcap_{\{a_j,b_j\} \in E(G),\, i \neq j} (x_{a_j}, x_{b_j}) \Bigg)$$

*is Cohen-Macaulay for all $m \in \mathbb{Z}^+$.*

P r o o f. As $\{a_i,b_i\}$ is a minimal vertex cover of $G$, there exists no edge $\{c,d\} \in E(G)$ such that $c,d \notin \{a_i,b_i\}$, so that $J$ is Cohen-Macaulay. □

**Example 2.7.**

(1) Consider the graph $G$ with the vertex set $V(G) = \{v_1,v_2,v_3,v_4\}$ and edge set $E(G) = \{\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\}\}$:




Here the vertex cover ideal will be

$$I_G = (x_1,x_2) \cap (x_2,x_3) \cap (x_3,x_4).$$

Now, there exists an edge $\{v_3,v_4\}$ such that $v_3,v_4 \notin \{v_1,v_2\}$, thus Theorem 2.5 guarantees that the ideal

$$J = (x_1^m, x_2^m) \cap (x_2,x_3) \cap (x_3,x_4)$$

will not be Cohen-Macaulay for any $m > 1$. The same is true with $\{v_1,v_2\}$ and $\{v_3,v_4\}$ interchanged. On the other hand the edge $\{v_2,v_3\}$ is a minimal vertex cover for this graph so by Theorem 2.5, the ideal

$$K = (x_1,x_2) \cap (x_2^m, x_3^m) \cap (x_3,x_4)$$

is Cohen-Macaulay for all choices of $m \in \mathbb{Z}^+$.

(2) Let us consider the graph $G$ with $V(G) = \{v_1,v_2,v_3,v_4,v_5\}$ and $E(G) = \{\{v_1,v_2\},\{v_2,v_3\},\{v_2,v_5\},\{v_3,v_4\}\}$. Then

$$I_G = (x_1,x_2) \cap (x_2,x_3) \cap (x_3,x_4) \cap (x_2,x_5).$$

By Theorem 2.5, all the following ideals are not Cohen-Macaulay for any $m > 1$,

$$J_1 = (x_1^m, x_2^m) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_2, x_5);$$
$$J_2 = (x_1, x_2) \cap (x_2, x_3) \cap (x_3^m, x_4^m) \cap (x_2, x_5);$$
$$J_3 = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_2^m, x_5^m).$$

On the other hand, the ideal

$$J_4 = (x_1, x_2) \cap (x_2^m, x_3^m) \cap (x_3, x_4) \cap (x_2, x_5)$$

is Cohen-Macaulay for all choices of $m > 1$, $m \in \mathbb{Z}$ because $\{v_2, v_3\}$ is a minimal vertex cover of $G$.

(3) Consider the graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{\{v_1, v_i\}\colon 2 \leqslant i \leqslant n\}$, the so called *bouquet graph.* Then

$$J = \bigcap_{\{v_1, v_i\} \in E(G)} (x_1^{m_i}, x_i^{m_i})$$

is Cohen-Macaulay for all $m_i > 1$, $m_i \in \mathbb{Z}$.

(4) Finally, for the graph $K_3$, all its edges are minimal vertex covers of $K_3$. Thus the ideal

$$J = (x_1^l, x_2^l) \cap (x_2^m, x_3^m) \cap (x_3^n, x_1^n)$$

is Cohen-Macaulay for all choices of $l, m, n \in \mathbb{Z}^+$.

## 3. Modifications of second type

In this section we consider the modifications of the form

$$J = (x_{a_i}^m, x_{b_i}) \cap \Bigg( \bigcap_{\{a_j, b_j\} \in E(G),\, i \neq j} (x_{a_j}, x_{b_j}) \Bigg).$$

We recall that a subset $T \subset V(G)$ is called an *independent set* of $G$, if for all $v_i, v_j \in T$ it holds that $\{v_i, v_j\} \notin E(G)$ . An independent set $T$ is called *maximal*, if it is not a proper subset of any independent set, see [8]. The set of vertices adjacent to $v_i$ will be denoted by $N_G(v_i)$. In [1], the first author proved the following result:

**Theorem 3.1.** *Suppose that $I_G$ is Cohen-Macaulay. Let $W = \{v_{i_1}, \ldots, v_{i_r}\}$ be a set of pairwise distinct vertices of $G$ with the property that each $v_{i_k}$ belongs to exactly one maximal independent set $T_k$ of $G$, where $T_k \neq T_l$ for $k \neq l$. For each*

*$v_{i_k} \in W$ choose a nonempty subset $A_k \subset N_G(v_{i_k})$ with the property that, if some $v_{i_k} \in A_j$ then $v_{i_j} \notin A_k$ and $A_k \cap A_l = \emptyset$ for $k \neq l$, and let*

$$J = \bigcap_{\{v_i,v_j\}} (x_i, x_j) \cap \bigcap_{k=1}^{r} \bigcap_{v_j \in A_k} (x_{i_k}, x_j^{aj}),$$

*where the first intersection is taken over all edges $\{v_i, v_j\}$ different from the edges $\{v_{i_k}, v_j\}$ with $v_j \in A_k$, and where each $a_j$ is a positive integer. Then $J$ is Cohen-Macaulay.*

We will now prove the converse.

**Definition 3.2.** Let $G$ be a graph, $v \in V(G)$ and $w$ a new vertex not belonging to $V(G)$. We let $G_v$ be the graph with $V(G_v) = V(G) \cup \{w\}$ and $E(G_v) = E(G) \cup \{\{u, w\}\colon\ u \in V(G),\ u \neq v\}$.

Let $c(G)$ be the maximum length of a chord-less cycle in $G$.

**Lemma 3.3** ([1], Lemma 3.2)**.** *Suppose $G$ is chordal. Then $c(G_v) \leqslant 4$.*

**Definition 3.4.** Let $v \in V(G)$ and $N_G(v) = \{v_1, \dots, v_r\}$. Then we define $c_G(v)$ to be the cardinality of the set

$$\{\{v_i, v_j\}\colon\ \{v_i, v_j\} \notin E(G);\ 1 \leqslant i < j \leqslant r\},$$

and call $c_G(v)$ the cycle number of $v$ in $G$.

**Remark 3.5.** Note that $c_G(v) = 0$ if and only if the restriction of $G$ to the vertex set $\{v\} \cup N_G(v) \subset V(G)$ is a clique in $G$ (a complete subgraph of $G$). Observe that if $\{v\} \cup N_G(v)$ is a clique, it is indeed a maximal clique in $V(G)$, since it contains all neighbors of $v$.

Let us see an immediate consequence of Lemma 3.3.

**Lemma 3.6.** *$G_v$ is chordal if and only if $c_G(v) = 0$.*

Now we are ready to state and prove our main theorem of this section.

**Theorem 3.7.** *Let $G$ be a graph such that the vertex cover ideal*

$$I_G = \bigcap_{\{v_i,v_j\} \in E(G)} (x_i, x_j)$$

*is Cohen-Macaulay. If for some $v_i \in V(G)$ there exists $v_k, v_l \in V(G)$ with $\{v_k, v_l\} \in E(G)$ and $\{v_i, v_k\}, \{v_i, v_l\} \notin E(G)$, then for any $v_m \in N_G(v_i)$*

$$J = \bigcap_{\{v_i,v_j\}\in E(G);\, j\neq m} (x_i, x_j) \cap (x_i, x_m^n)$$

*is not Cohen-Macaulay, for any $n \geqslant 2$.*

P r o o f. It is enough to prove the theorem for $n = 2$. Since, by assumption, $I_G$ is Cohen-Macaulay, it follows that $\overline{G}$ is chordal. Suppose for some $v_i \in V(G)$ there exists $v_k, v_l \in V(G)$ such that $\{v_k, v_l\} \in E(G)$ with $\{v_i, v_k\}, \{v_i, v_l\} \notin E(G)$. Then $\{v_k, v_l\} \notin E(\overline{G})$ and $\{v_i, v_k\}, \{v_i, v_l\} \in E(\overline{G})$.

As $\{v_i\} \cup N_{\overline{G}}(v_i)$ is not a clique in $\overline{G}$, $c_{\overline{G}}(v_i) \neq 0$ and $\overline{G}_{v_i}$ is not chordal.

Since

$$J = \bigcap_{\{v_i,v_j\}\in E(G);\, j\neq m} (x_i, x_j) \cap (x_i, x_m^2)$$

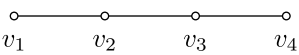
we have

$$J^p = \bigcap_{\{v_i,v_j\}\in E(G);\, j\neq m} (x_i, x_j) \cap (x_i, x_m) \cap (x_i, w) = \bigcap_{\{v_i,v_j\}\in E(G)} (x_i, x_j) \cap (x_i, w).$$

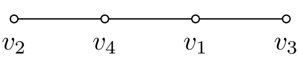
Let $H$ be the graph obtained from $G$ by adding a whisker with vertex $i$. Then $J^p = I_H$, where $H$ is the complementary graph of $\overline{G}_{v_i}$; this implies that $J^p$ is not Cohen-Macaulay and hence $J$ is not Cohen-Macaulay. □

Now we will formulate a complete example to demonstrate the result.

**Example 3.8.** Consider the graph shown in the figure:




The vertex cover ideal associated to this graph is $I_G = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4)$. The complementary graph $\overline{G}$ of $G$ is




As this graph is chordal, the ideal $I_G$ is Cohen-Macaulay. Now, $\{v_3, v_4\} \in E(G)$ and $\{v_1, v_3\}, \{v_1, v_4\} \notin E(G)$. Moreover, $N_G(v_1) = \{v_2\}$, whence by Theorem 3.7, the ideal

$$I_G = (x_1, x_2^n) \cap (x_2, x_3) \cap (x_3, x_4)$$

is not Cohen-Macaulay for any $n$ greater than 1.

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*References*

[1] *S. Ahmad*: Cohen-Macaualy intersections. Arch. Math. *92* (2009), 228–236. zbl MR doi
[2] *S. Ahmad, M. Naeem*: Cohen-Macaulay monomial ideals with given radical. J. Pure Appl. Algebra. *214* (2010), 1812–1817. zbl MR doi
[3] *S. Ahmad, M. Naeem*: Classes of simplicial complexes which admit non-trivial Cohen-Macaulay modifications. Stud. Sci. Math. Hung. *52* (2015), 423–433. zbl MR doi
[4] *J. A. Eagon, V. Reiner*: Resolutions of Stanley-Reisner rings and Alexander duality. J. Pure Appl. Algebra *130* (1998), 265–275. zbl MR doi
[5] *R. Fröberg*: Rings with monomial relations having linear resolutions. J. Pure Appl. Algebra *38* (1985), 235–241. zbl MR doi
[6] *J. Herzog, T. Hibi*: Monomial Ideals. Graduate Texts in Mathematics 260. Springer, London, 2011. zbl MR doi
[7] *J. Herzog, Y. Takayama, N. Terai*: On the radical of a monomial ideal. Arch. Math. *85* (2005), 397–408. zbl MR doi
[8] *R. H. Villarreal*: Monomial Algebras. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, 2015. zbl MR doi

*Authors' addresses*: Safyan Ahmad, Shamsa Kanwal, Abdus Salam School of Mathematical Sciences, GC University Lahore, Lahore, Pakistan, e-mail: safyank@gmail.com, lotus_zone16@yahoo.com; Talat Firdous, University of Gujrat, Gujrat, Pakistan, e-mail: talatfirdous25@gmail.com.