Yukihide Enaka; Masakazu Onitsuka

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INTEGRAL AVERAGING TECHNIQUE FOR OSCILLATION OF DAMPED HALF-LINEAR OSCILLATORS

Yukihide Enaka, Masakazu Onitsuka, Okayama

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Cordially dedicated to Professor Jitsuro Sugie on the occasion of his 60th birthday

Abstract. This paper is concerned with the oscillatory behavior of the damped half-linear oscillator $(a(t)\varphi_p(x'))' + b(t)\varphi_p(x') + c(t)\varphi_p(x) = 0$, where $\varphi_p(x) = |x|^{p-1}\operatorname{sgn} x$ for $x \in \mathbb{R}$ and $p > 1$. A sufficient condition is established for oscillation of all nontrivial solutions of the damped half-linear oscillator under the integral averaging conditions. The main result can be given by using a generalized Young's inequality and the Riccati type technique. Some examples are included to illustrate the result. Especially, an example which asserts that all nontrivial solutions are oscillatory if and only if $p \neq 2$ is presented.

Keywords: damped half-linear oscillator; integral averaging technique; Riccati technique; generalized Young inequality; oscillatory solution

MSC 2010: 34C10, 34C15

1. Introduction

We consider the damped nonlinear oscillator

$$(a(t)\varphi_p(x'))' + b(t)\varphi_p(x') + c(t)\varphi_p(x) = 0 \tag{1.1}$$

for $t \geqslant t_0$, where the prime denotes $\mathrm{d}/\mathrm{d}t$; the coefficients $a(t)$, $b(t)$ and $c(t)$ are continuous for $t \geqslant t_0$, and $a(t) > 0$ for $t \geqslant t_0$; the real-valued function $\varphi_p(x)$ is defined by $\varphi_p(x) = |x|^{p-1}\operatorname{sgn} x$ for $x \in \mathbb{R}$ and $p > 1$. Note that $\varphi_p(xy) = \varphi_p(x)\varphi_p(y)$ holds for any $x, y \in \mathbb{R}$, but $\varphi_p(x+y) = \varphi_p(x) + \varphi_p(y)$ does not hold for any $x, y \in \mathbb{R} \setminus \{0\}$ and $p \neq 2$. Therefore, the solution space of (1.1) is homogeneous, but it is not additive in the case $p \neq 2$. For this reason, this equation is often called "*half-linear*". For example, half-linear differential equation can be found in

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[1], [2], [3], [6], [7], [8], [12], [13], [14], [17]–[26], [30], [31], [35], [37]. Clearly, the equation (1.1) has a trivial solution $x(t) \equiv 0$. It is known that the global existence and uniqueness of solutions of (1.1) are guaranteed for the initial-value problem (see [6], page 170, and [7], pages 8–13). Hence, all nontrivial solutions of (1.1) are divided as follows: a nontrivial solution $x(t)$ of (1.1) is said to be "*oscillatory*" if there exists a sequence $\{t_n\}$ such that $x(t_n) = 0$ and $\lim_{n\to\infty} t_n = \infty$; otherwise, it is said to be "*non-oscillatory*". Multiplying (1.1) by $\exp\bigl(\int_{t_0}^t (b(s)/a(s))\,\mathrm{d}s\bigr)$, we obtain an undamped half-linear differential equation

$$\left(a(t)\exp\left(\int_{t_0}^t \frac{b(s)}{a(s)}\,\mathrm{d}s\right)\varphi_p(x')\right)' + c(t)\exp\left(\int_{t_0}^t \frac{b(s)}{a(s)}\,\mathrm{d}s\right)\varphi_p(x) = 0 \tag{1.2}$$

for $t \geqslant t_0$. It is well known that all nontrivial solutions of second order undamped linear or half-linear differential equations are oscillatory if a nontrivial solution of them is oscillatory by virtue of Sturm's separation theorem (see [1], [5], [6], [7], [11], [29]). Needless to say, the oscillatory behavior of (1.1) and (1.2) are equivalent. Hence, if (1.1) has an oscillatory solution, then all nontrivial solutions of (1.1) are oscillatory. We call the sufficient conditions for all nontrivial solutions of differential equations to be oscillatory "*oscillation theorems*". Many researchers have been interested in the oscillation theorems for linear or nonlinear differential equations. For example, the reader is referred to [1], [2], [4]–[13], [15], [16], [21], [22], [23], [26]–[32], [34], [35], [36], [38]. Our main aim of this paper is to establish conditions on the coefficients $a(t)$, $b(t)$ and $c(t)$ for all nontrivial solutions to be oscillatory.

Since $\varphi_2(x) = x$, the half-linear differential equation becomes the linear differential equation. If $p = 2$, $a(t) \equiv 1$ and $b(t) \equiv 0$, then (1.1) corresponds to the linear oscillator

$$x'' + c(t)x = 0 \tag{L$_1$}$$

for $t \geqslant t_0$. Wintner in [33] established the following oscillation theorem by using an integral averaging condition.

Theorem A. *If*

$$\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^t \int_{t_0}^s c(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = \infty \tag{1.3}$$

holds, then all nontrivial solutions of (L$_1$) *are oscillatory.*

If $p = 2$ and $b(t) \equiv 0$, then (1.1) becomes the linear oscillator

$$(a(t)x')' + c(t)x = 0 \tag{L$_2$}$$

for $t \geqslant t_0$. The following result was presented by Grace and Lalli in [9]. In the book of Agarwal, Grace and O'Regan, see [1], pages 40–42, it was introduced in the following form.

Theorem B. *If* (1.3) *and*

$$\lim_{t\to\infty}\int^t\Big(\int_{t_0}^s a(\tau)d\tau\Big)^{-1}\,\mathrm{d}s=\infty \tag{1.4}$$

hold, then all nontrivial solutions of (L_2) *are oscillatory.*

Theorems A and B have been extended by many authors to more precise and general results for nonlinear differential equations (see [1], [2], [4], [9], [10], [15], [16], [30], [31], [32], [34], [36], [38]).

If $p=2$, then (1.1) becomes the damped linear oscillator

$$(a(t)x')'+b(t)x'+c(t)x=0 \tag{L_3}$$

for $t\geqslant t_0$. Multiplying (L_3) by $\exp\big(\int_{t_0}^t(b(s)/a(s))\,\mathrm{d}s\big)$, we obtain

$$\bigg(a(t)\exp\bigg(\int_{t_0}^t\frac{b(s)}{a(s)}\,\mathrm{d}s\bigg)x'\bigg)'+c(t)\exp\bigg(\int_{t_0}^t\frac{b(s)}{a(s)}\,\mathrm{d}s\bigg)x=0$$

for $t\geqslant t_0$. We can get the following corollary by using Theorem B.

Corollary 1.1. *If*

$$\lim_{t\to\infty}\int^t\bigg(\int_{t_0}^s a(\tau)\exp\bigg(\int_{t_0}^\tau\frac{b(\sigma)}{a(\sigma)}d\sigma\bigg)d\tau\bigg)^{-1}\,\mathrm{d}s=\infty \tag{1.5}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^t\int_{t_0}^s c(\tau)\exp\bigg(\int_{t_0}^\tau\frac{b(\sigma)}{a(\sigma)}\,\mathrm{d}\sigma\bigg)\,\mathrm{d}\tau\,\mathrm{d}s=\infty \tag{1.6}$$

hold, then all nontrivial solutions of (L_3) *are oscillatory.*

Letting $x=y\exp\big(-\int_{t_0}^t(b(s)/(2a(s)))\,\mathrm{d}s\big)$, we can transform equation (L_3) into the equation

$$(a(t)y')'+\Big(c(t)-\frac{b^2(t)}{4a(t)}-\frac{b'(t)}{2}\Big)y=0 \tag{1.7}$$

for $t \geqslant t_0$. Needless to say, all nontrivial solutions of (L_3) are oscillatory if and only if all nontrivial solutions of (1.7) are oscillatory. By using Theorem B, we can easily establish the following corollary.

Corollary 1.2. *Suppose that $b(t)$ is continuously differentiable for $t \geqslant t_0$. If (1.4) and*

$$\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\Big(c(\tau) - \frac{b^2(\tau)}{4a(\tau)} - \frac{b'(\tau)}{2}\Big)\,\mathrm{d}\tau\,\mathrm{d}s = \infty \tag{1.8}$$

hold, then all nontrivial solutions of (L_3) are oscillatory.

Now, we give two examples to illustrate the differences between Corollaries 1.1 and 1.2. We consider the damped linear oscillator

$$x'' + \frac{1}{t}x' + x = 0$$

for $t \geqslant 1$. In this case, we have

$$\lim_{t\to\infty}\int^{t}\Big(\int_1^s a(\tau)\exp\Big(\int_1^\tau \frac{b(\sigma)}{a(\sigma)}\,\mathrm{d}\sigma\Big)\,\mathrm{d}\tau\Big)^{-1}\,\mathrm{d}s = \lim_{t\to\infty}\Big(\log\frac{t-1}{t+1} + c_1\Big) = c_1,$$

where c_1 is an arbitrary constant. Hence, condition (1.5) in Corollary 1.1 does not hold. However, we can show that conditions (1.4) and (1.8) in Corollary 1.2 hold. Then all nontrivial solutions of this equation are oscillatory. On the other hand, we consider the damped linear oscillator

$$\Big(\frac{1}{t}x'\Big)' + \frac{1}{t^2}x' + \frac{1}{t^2}x = 0$$

for $t \geqslant 1$. In this case, we can estimate that

$$\lim_{t\to\infty}\frac{1}{t}\int_1^t\int_1^s\Big(c(\tau) - \frac{b^2(\tau)}{4a(\tau)} - \frac{b'(\tau)}{2}\Big)\,\mathrm{d}\tau\,\mathrm{d}s = \lim_{t\to\infty}\frac{1}{t}\Big(-\log t + \frac{3}{8t} + \frac{11}{8}t - \frac{7}{4}\Big) = \frac{11}{8},$$

that is, condition (1.8) in Corollary 1.2 does not hold. However, since conditions (1.5) and (1.6) in Corollary 1.1 hold, all nontrivial solutions of this equation are oscillatory. From the above mentioned examples, we conclude that Corollaries 1.1 and 1.2 have no relation of inclusion. Moreover, they show that the criteria for the damped equation change according to the transformation to the undamped equation. This means that we cannot use fully a theory for the undamped equation to analyse the damped equation.

If $b(t) \equiv 0$, then (1.1) becomes the undamped half-linear oscillator

$$(a(t)\varphi_p(x'))' + c(t)\varphi_p(x) = 0 \tag{H}$$

for $t \geqslant t_0$. The following result can be found in the book [1], pages 166–167, by Agarwal, Grace and O'Regan.

Theorem C. *Suppose* $p \geqslant 2$ *is satisfied. If* (1.3) *and*

$$\lim_{t\to\infty} \int^t \left(\int_{t_0}^s a(\tau)\,\mathrm{d}\tau\right)^{1/(1-p)} \mathrm{d}s = \infty \tag{1.9}$$

hold, then all nontrivial solutions of (H) *are oscillatory.*

By using Theorem C and (1.2), we can get the following result immediately.

Corollary 1.3. *Suppose* $p \geqslant 2$ *is satisfied. If* (1.6) *and*

$$\lim_{t\to\infty} \int^t \left(\int_{t_0}^s a(\tau)\exp\left(\int_{t_0}^\tau \frac{b(\sigma)}{a(\sigma)}\,\mathrm{d}\sigma\right)\mathrm{d}\tau\right)^{1/(1-p)} \mathrm{d}s = \infty \tag{1.10}$$

hold, then all nontrivial solutions of (1.1) *are oscillatory.*

This corollary is a generalization of Theorems A, B, C and Corollary 1.1. On the other hand, we cannot transform equation (1.1) into the half-linear version of equation (1.7) since $\varphi_p(x+y) = \varphi_p(x) + \varphi_p(y)$ does not hold for any $x, y \in \mathbb{R} \setminus \{0\}$ if $p \neq 2$. The main purpose of this paper is to give an oscillation theorem which includes Corollary 1.2 without requiring the transformation for equation (1.1). Our main result is stated as follows.

Theorem 1.1. *Let* $\lambda_p = \max\{1, p-1\}$. *Suppose that* $a^{2-p}(t)\varphi_p(b(t))$ *is continuously differentiable for* $t \geqslant t_0$. *If* (1.9) *and*

$$\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^t \int_{t_0}^s \left\{c(\tau) - \lambda_p a^{1-p}(\tau)\left|\frac{b(\tau)}{p}\right|^p - \left(a^{2-p}(\tau)\varphi_p\left(\frac{b(\tau)}{p}\right)\right)'\right\}\mathrm{d}\tau\,\mathrm{d}s = \infty \tag{1.11}$$

hold, then all nontrivial solutions of (1.1) *are oscillatory.*

This theorem is a generalization of Theorems A, B, C and Corollary 1.2. Moreover, it becomes the following result when $b(t) \equiv 0$.

Corollary 1.4. *If* (1.3) *and* (1.9) *hold, then all nontrivial solutions of* (1.1) *are oscillatory.*

By using Corollary 1.4 and (1.2), we can obtain a generalization of Corollary 1.3.

Corollary 1.5. *If* (1.6) *and* (1.10) *hold, then all nontrivial solutions of* (1.1) *are oscillatory.*

Before we give the proof of the main theorem, we prepare a generalized Young's inequality in Section 2. The proof of Theorem 1.1 is given in Section 3, which is a core of this paper. A Riccati type substitution plays an important role in the proof. To illustrate the results, we take two concrete examples in Section 4.

2. Generalized Young inequality

We now define a real-valued function F on $\mathbb{R}^2$ by

$$F(u,v) = \frac{|u|^p}{p} - uv + \frac{|v|^q}{q}, \tag{2.1}$$

where q is the positive number satisfying

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Since $p > 1$, the number q is also greater than 1. Note that if $1 < p < 2$ or $1 < q < 2$, then the number q or p, respectively, is greater than 2, and the function φ_q is the inverse function of φ_p. It is well known that the inequality

$$F(u,v) \geqslant 0$$

holds on $\mathbb{R}^2$. This inequality is often called the "*Young inequality*". In this section, we give a generalization of the Young inequality.

Theorem 2.1. *Let* $F(u,v)$ *be the function given by* (2.1). *Then the following inequalities hold:*

(i) *there exists a constant* $0 < \varepsilon_0 \leqslant 1/q$ *such that*

$$F(u,v) \geqslant \varepsilon_0 |\varphi_p(u) - v|^q \tag{2.2}$$

for $(u,v) \in \mathbb{R}^2$ *and* $1 < p \leqslant 2$;

(ii) *there exists a constant* $0 < \varepsilon_0 \leqslant 1/q$ *such that*

$$F(u,v) \geqslant \varepsilon_0 |\varphi_p(u) - v|^q - \frac{p-2}{p}|u|^p \tag{2.3}$$

for $(u,v) \in \mathbb{R}^2$ *and* $p \geqslant 2$.

Proof. If $p=2$, then we have

$$F(u,v)=\frac{1}{2}(u-v)^2$$

for any $(u,v)\in\mathbb{R}^2$ when $\varepsilon_0=1/2$. Clearly, inequalities (2.2) and (2.3) hold. Moreover, if $u=0$ holds, then we have

$$F(0,v)=\frac{|v|^q}{q}=\frac{1}{q}|\varphi_p(0)-v|^q=\frac{1}{q}|\varphi_p(0)-v|^q-\frac{p-2}{p}|0|^p$$

for $u\in\mathbb{R}$. This means that inequalities (2.2) and (2.3) hold when $u=0$. Thus, we have only to consider the case that $p\neq 2$ and $u\neq 0$.

Next, we prove the assertion (i) with $1<p<2$ and $u\neq 0$. We consider two real-valued functions

$$f(v)=F(1,v)=\frac{|v|^q}{q}-v+\frac{1}{p}\quad\text{and}\quad g(v)=|1-v|^q$$

for $v\in\mathbb{R}$. Since $f(1)=0$ and

$$\frac{\mathrm{d}}{\mathrm{d}v}f(v)=\varphi_q(v)-1\quad\text{for } v\in\mathbb{R},$$

we see that $f(v)$ is decreasing for $v<1$ and increasing for $v>1$, and $f(v)>0$ for $v\in\mathbb{R}\setminus\{1\}$. Therefore, the inequality

$$\frac{g(v)}{f(v)}>0\quad\text{for } v\in\mathbb{R}\setminus\{1\}$$

holds. Since $q>2$, we get

$$\lim_{v\to 1}\frac{g(v)}{f(v)}=\lim_{v\to 1}\frac{q\varphi_q(v-1)}{\varphi_q(v)-1}=\lim_{v\to 1}\frac{q|v-1|^{q-2}}{|v|^{q-2}}=0.$$

In addition, we have

$$\lim_{v\to\pm\infty}\frac{g(v)}{f(v)}=\lim_{v\to\pm\infty}\frac{|v^{-1}-1|^q}{q^{-1}-\varphi_q^{-1}(v)+p^{-1}|v|^{-q}}=q.$$

Hence, we see that the function $g(v)/f(v)$ is bounded from above. Let

$$\varepsilon_0=\frac{1}{\sup\{g(v)/f(v)\colon\ v\in\mathbb{R}\}}\leqslant\frac{1}{q}.$$

Then we get

$$F\Big(1, \frac{v}{\varphi_p(u)}\Big) \geqslant \varepsilon_0 \Big|1 - \frac{v}{\varphi_p(u)}\Big|^q.$$

Multiplying this inequality by $|u|^p = |\varphi_p(u)|^q$, we obtain inequality (2.2).

Next, we prove the assertion (ii) with $p > 2$ and $u \neq 0$. Since $p > 2$ and $f(v) \geqslant 0$ for $v \in \mathbb{R}$, we see that

$$\frac{g(v)}{f(v) + (p-2)p^{-1}} > 0 \quad \text{for } v \in \mathbb{R},$$

where f and g are the functions given in the previous part. In addition, we obtain

$$\lim_{v\to 1} \frac{g(v)}{f(v) + (p-2)p^{-1}} = 0$$

and

$$\lim_{v\to\pm\infty} \frac{g(v)}{f(v) + (p-2)p^{-1}} = \lim_{v\to\pm\infty} \frac{|v^{-1} - 1|^q}{q^{-1} - \varphi_q^{-1}(v) + q^{-1}|v|^{-q}} = q.$$

Hence, we see that the function $g(v)/(f(v) + (p-2)p^{-1})$ is bounded from above. Let

$$\varepsilon_0 = \frac{1}{\sup\{g(v)/(f(v) + (p-2)p^{-1})\colon\ v \in \mathbb{R}\}} \leqslant \frac{1}{q}.$$

Then we have

$$F\Big(1, \frac{v}{\varphi_p(u)}\Big) + \frac{p-2}{p} \geqslant \varepsilon_0 \Big|1 - \frac{v}{\varphi_p(u)}\Big|^q.$$

Multiplying this inequality by $|\varphi_p(u)|^q$, we get inequality (2.3). This completes the proof of Theorem 2.1. □

3. Proof of the main theorem

In this section we present the proof of the main theorem. The method of the proof is based on the undamped linear case in [1], pages 40–42.

P r o o f of Theorem 1.1. The proof is by contradiction. Suppose that (1.1) has a nontrivial non-oscillatory solution $x(t)$. We may assume without loss of generality that there exists a $t_1 > \max\{0, t_0\}$ such that $x(t) > 0$ for $t \geqslant t_1$, because the function $-x(t)$ is also a solution of (1.1). Define a function $r(t) = a(t)\varphi_p(x'(t)/x(t))$. Note that the function φ_q is the inverse function of φ_p. Since

$x'(t)/x(t) = \varphi_q(r(t)/a(t))$ and $1/p + 1/q = 1$, we have

$$\begin{aligned} r'(t) &= -(p-1)a(t)\Big|\frac{x'(t)}{x(t)}\Big|^p - b(t)\varphi_p\Big(\frac{x'(t)}{x(t)}\Big) - c(t) \\ &= -(p-1)a(t)\Big|\frac{r(t)}{a(t)}\Big|^q - b(t)\frac{r(t)}{a(t)} - c(t) \\ &= -pa^{1-q}(t)\Big(\frac{|r(t)|^q}{q} + \frac{a^{q-2}(t)b(t)}{p}r(t)\Big) - c(t) \\ &= -pa^{1-q}(t)F\Big(\frac{a^{q-2}(t)b(t)}{p}, -r(t)\Big) + \lambda_p a^{1-p}(t)\Big|\frac{b(t)}{p}\Big|^p - c(t) \end{aligned}$$

for $t \geqslant t_1$, where $F(u,v)$ is the function given by (2.1). Let

$$w(t) = a^{2-p}(t)\varphi_p\Big(\frac{b(t)}{p}\Big) + r(t);$$

it follows from Theorem 2.1 that

$$\begin{aligned} r'(t) &\leqslant -p\varepsilon_0 a^{1-q}(t)\Big|\varphi_p\Big(\frac{a^{q-2}(t)b(t)}{p}\Big) + r(t)\Big|^q + \lambda_p a^{1-p}(t)\Big|\frac{b(t)}{p}\Big|^p - c(t) \\ &= -p\varepsilon_0 a^{1-q}(t)|w(t)|^q + \lambda_p a^{1-p}(t)\Big|\frac{b(t)}{p}\Big|^p - c(t) \end{aligned}$$

for $t \geqslant t_1$. Therefore, we get a generalized Riccati inequality

$$w'(t) = r'(t) + \Big(a^{2-p}(t)\varphi_p\Big(\frac{b(t)}{p}\Big)\Big)' \leqslant -p\varepsilon_0 a^{1-q}(t)|w(t)|^q - \psi(t)$$

for $t \geqslant t_1$, where

$$\psi(t) = c(t) - \lambda_p a^{1-p}(t)\Big|\frac{b(t)}{p}\Big|^p - \Big(a^{2-p}(t)\varphi_p\Big(\frac{b(t)}{p}\Big)\Big)'.$$

Integrating twice the above inequality from t_1 to $t \geqslant t_1$, we obtain

$$\int_{t_1}^t w(s)\,\mathrm{d}s - w(t_1)(t-t_1) \leqslant -p\varepsilon_0\int_{t_1}^t\int_{t_1}^s a^{1-q}(\tau)|w(\tau)|^q\,\mathrm{d}\tau\,\mathrm{d}s - \int_{t_1}^t\int_{t_1}^s \psi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

for $t \geqslant t_1$. Hence, we have

$$\begin{aligned} \int_{t_1}^t w(s)\,\mathrm{d}s &+ p\varepsilon_0\int_{t_1}^t\int_{t_1}^s a^{1-q}(\tau)|w(\tau)|^q\,\mathrm{d}\tau\,\mathrm{d}s \\ &\leqslant -t_1 w(t_1) + t\Big(w(t_1) - \frac{1}{t}\int_{t_1}^t\int_{t_1}^s \psi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Big) \end{aligned} \tag{3.1}$$

for $t \geqslant t_1$. Taking into account that

$$\frac{1}{t}\int_{t_1}^{t}\int_{t_1}^{s}\psi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = \frac{1}{t}\Big(\int_{t_0}^{t}\int_{t_1}^{s}\psi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s - \int_{t_0}^{t_1}\int_{t_1}^{s}\psi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Big)$$
$$= \frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\psi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s - \frac{t-t_0}{t}\int_{t_0}^{t_1}\psi(\tau)\,\mathrm{d}\tau - \frac{1}{t}\int_{t_0}^{t_1}\int_{t_1}^{s}\psi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

for $t \geqslant t_1$, from (1.11) and (3.1) we can choose a $t_2 > t_1$ such that

$$\int_{t_1}^{t} w(s)\,\mathrm{d}s + p\varepsilon_0\int_{t_1}^{t}\int_{t_1}^{s} a^{1-q}(\tau)|w(\tau)|^q\,\mathrm{d}\tau\,\mathrm{d}s < 0 \quad \text{for } t \geqslant t_2.$$

Let

$$R(t) = p\varepsilon_0\int_{t_1}^{t}\int_{t_1}^{s} a^{1-q}(\tau)|w(\tau)|^q\,\mathrm{d}\tau\,\mathrm{d}s \geqslant 0 \quad \text{for } t \geqslant t_2.$$

Then we have

$$0 \leqslant R(t) < -\int_{t_1}^{t} w(s)\,\mathrm{d}s \leqslant \left|\int_{t_1}^{t} w(s)\,\mathrm{d}s\right| \leqslant \int_{t_1}^{t}|w(s)|\,\mathrm{d}s \quad \text{for } t \geqslant t_2.$$

Using Hölder's inequality, we obtain

$$0 \leqslant R^q(t) < \Big(\int_{t_1}^{t}|w(s)|\,\mathrm{d}s\Big)^q \leqslant \Big\{\Big(\int_{t_1}^{t}\frac{|w(s)|^q}{a^{q-1}(s)}\,\mathrm{d}s\Big)^{1/q}\Big(\int_{t_1}^{t}a(s)\,\mathrm{d}s\Big)^{1/p}\Big\}^q$$
$$= \frac{R'(t)}{p\varepsilon_0}\Big(\int_{t_1}^{t}a(s)\,\mathrm{d}s\Big)^{q-1}$$

for $t \geqslant t_2$. Since $\big(\int_{t_1}^{t}a(s)\,\mathrm{d}s\big)^{q-1}$ is positive for $t \geqslant t_2$, we see that $R'(t)$ is also positive for $t \geqslant t_2$. For this reason, we can find a $t_3 \geqslant t_2$ such that $R(t)$ is positive for $t \geqslant t_3$. Thus, we get

$$\int_{t_3}^{t}\Big(\int_{t_1}^{s}a(\tau)\,\mathrm{d}\tau\Big)^{1-q}\mathrm{d}s \leqslant \frac{1}{p\varepsilon_0}\int_{t_3}^{t}R^{-q}(s)R'(s)\,\mathrm{d}s < \frac{1}{q\varepsilon_0}R^{1-q}(t_3) \tag{3.2}$$

for $t \geqslant t_3$. From (1.9) it follows that the left-hand side of (3.2) diverges to infinity as $t \to \infty$, which contradicts the fact that the right-hand side of (3.2) is finite. This completes the proof of Theorem 1.1. □

To illustrate our main theorem, we give two examples. For the sake of simplicity, let

$$\psi(t) = c(t) - \lambda_p a^{1-p}(t)\left|\frac{b(t)}{p}\right|^p - \left(a^{2-p}(t)\varphi_p\left(\frac{b(t)}{p}\right)\right)'.$$

Example 4.1. Consider the damped half-linear oscillator

$$((p-1)\,t^{p-2}\varphi_p(x'))' + \frac{1}{t}\varphi_p(x') + \varphi_p(x) = 0 \tag{4.1}$$

for $t \geqslant 1$. Then all nontrivial solutions of (4.1) are oscillatory. Moreover, in the case that $p = 2$, we cannot use Corollaries 1.1, 1.3 and 1.5.

Since $p > 1$ and $a(t) = (p-1)\,t^{p-2}$, we have

$$\left(\int_1^t a(s)\,\mathrm{d}s\right)^{1/(1-p)} = (t^{p-1}-1)^{1/(1-p)} > \frac{1}{t}$$

for $t > 1$. From this inequality, we can estimate that

$$\lim_{t\to\infty}\int_2^t \left(\int_1^s a(\tau)\,\mathrm{d}\tau\right)^{1/(1-p)} \mathrm{d}s = \infty.$$

Hence, condition (1.9) holds. Since $p > 1$, $a(t) = (p-1)t^{p-2}$, $b(t) = 1/t$ and $c(t) = 1$, we have

$$\psi(t) = 1 - \frac{\lambda_p t^{(1-p)(p-2)-p}}{p^p(p-1)^{p-1}} + \frac{\{(p-2)^2+p-1\}(p-1)^{2-p}t^{-(p-2)^2-p}}{p^{p-1}} > 1 - \frac{c_1}{t}$$

for $t \geqslant 1$, where c_1 is a positive number satisfying

$$c_1 = \frac{\lambda_p}{p^p(p-1)^{p-1}}.$$

Consequently, we get

$$\begin{aligned}\frac{1}{t}\int_1^t\int_1^s \psi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s &\geqslant \frac{1}{t}\left\{\frac{(t-1)^2}{2} - c_1(t\log t - t + 1)\right\}\\ &= t\left\{\frac{(1-t^{-1})^2}{2} - c_1\left(\frac{\log t}{t} - \frac{1}{t} + \frac{1}{t^2}\right)\right\}\end{aligned}$$

for $t \geqslant 1$. From this inequality we can easily see that condition (1.11) holds. Thus, by using Theorem 1.1, we conclude that all nontrivial solutions of (4.1) are oscillatory.

In the case that $p = 2$, (4.1) becomes the damped linear oscillator

$$x'' + \frac{1}{t}x' + x = 0.$$

We have already shown that condition (1.5) does not hold in Section 1.

Finally, we take the other example which is very delicate.

Example 4.2. Consider the damped half-linear oscillator

$$((p-1)t^{p-2}\varphi_p(x'))' + pt^{-p^2+5p-7+1/p}\varphi_p(x') + \Big(\frac{1}{t} - \frac{1}{2t\sqrt{t}}\Big)\varphi_p(x) = 0 \tag{4.2}$$

for $t \geqslant 1$. Then all nontrivial solutions of (4.2) are oscillatory if and only if $p \neq 2$. Moreover, in the case that $1 < p < \frac{1}{2}(3+\sqrt{5})$, we cannot use Corollaries 1.3 and 1.5.

If $p = 2$, then equation (4.2) becomes the damped linear oscillator

$$x'' + \frac{2}{\sqrt{t}}x' + \Big(\frac{1}{t} - \frac{1}{2t\sqrt{t}}\Big)x = 0$$

for $t \geqslant 1$. It is easy to see that the function $x(t) = \mathrm{e}^{-2\sqrt{t}}$ is a nontrivial solution of this equation. That is, (4.2) has a non-oscillatory solution when $p = 2$.

Since $a(t)$ is the same as in Example 4.1, condition (1.9) holds. From $p > 1$, $a(t) = (p-1)t^{p-2}$, $b(t) = pt^{-p^2+5p-7+1/p}$ and $c(t) = t^{-1} - \frac{1}{2}t^{-3/2}$, we have

$$\psi(t) = t^{-1} - \frac{t^{-3/2}}{2} - \frac{\lambda_p t^{-p(p-2)^2-1}}{(p-1)^{p-1}} - (p-1)^{2-p}(t^{-(p-1)(p-2)^2-1/p})'.$$

Integrating both sides of this equality from 1 to $t \geqslant 1$ and using $p \neq 2$, we obtain

$$\begin{aligned}\int_1^t \psi(s)\,\mathrm{d}s = {} & \log t + \frac{1}{\sqrt{t}} - 1 + \frac{\lambda_p(t^{-p(p-2)^2} - 1)}{p(p-2)^2(p-1)^{p-1}} \\ & - (p-1)^{2-p}(t^{-(p-1)(p-2)^2-1/p} - 1) > \log t - c_2\end{aligned}$$

for $t \geqslant 1$, where c_2 is a positive number satisfying

$$c_2 = 1 + \frac{\lambda_p}{p(p-2)^2(p-1)^{p-1}}.$$

Therefore, we get

$$\frac{1}{t}\int_1^t\int_1^s \psi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \geqslant \frac{1}{t}(t\log t - t - c_2 t + 1 + c_2) = \log t - 1 - c_2 + \frac{1+c_2}{t}$$

for $t \geqslant 1$. From this inequality, we can estimate that

$$\lim_{t\to\infty} \frac{1}{t}\int_1^t\int_1^s \psi(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = \infty.$$

Hence, condition (1.11) holds. Thus, by virtue of Theorem 1.1, we conclude that all nontrivial solutions of (4.2) are oscillatory.

Next, we will show that condition (1.10) in Corollaries 1.3 and 1.5 does not hold when $1 < p < \frac{1}{2}(3+\sqrt{5})$. Suppose that $1 < p < \frac{1}{2}(3+\sqrt{5})$. From $p > 1$, $a(t) = (p-1)t^{p-2}$ and $b(t) = pt^{-p^2+5p-7+1/p}$, we have

$$\frac{b(t)}{a(t)} = \frac{p}{p-1}t^{-p^2+4p-5+1/p} = \frac{p}{p-1}t^{h(p)/p-1}, \tag{4.3}$$

where $h(p) = -p(p-2)^2+1$. Note here that the equation $h(p) = 0$ has three real roots

$$p = 1, \quad \frac{3 \pm \sqrt{5}}{2}.$$

Moreover, we can easily see that $h(p)$ is increasing for $1 < p < 2$ and decreasing for $2 < p < \frac{1}{2}(3+\sqrt{5})$. Thus, we obtain

$$0 < h(p) \leqslant 1 \quad \text{for } 1 < p < \frac{3+\sqrt{5}}{2}. \tag{4.4}$$

From this inequality and $p > 1$ we get

$$p^2 - (p-1)h(p) \geqslant \Big(p - \frac{1}{2}\Big)^2 + \frac{3}{4} > 0 \quad \text{for } 1 < p < \frac{3+\sqrt{5}}{2},$$

and

$$\int_1^\tau \frac{b(\sigma)}{a(\sigma)}\,\mathrm{d}\sigma = \frac{p^2}{(p-1)h(p)}(\tau^{h(p)/p} - 1) \geqslant 0 \quad \text{for } \tau \geqslant 1.$$

Using these inequalities, (4.4) and the Taylor series expansion of the exponential function, we can estimate that

$$\begin{aligned}
\exp\Big(\int_1^\tau \frac{b(\sigma)}{a(\sigma)}\,\mathrm{d}\sigma\Big) &\geqslant 1 + \frac{p^2}{(p-1)h(p)}(\tau^{h(p)/p} - 1) \\
&= \frac{p^2}{(p-1)h(p)}\Big\{\tau^{h(p)/p} - \frac{p^2-(p-1)h(p)}{p^2}\Big\} \\
&= \frac{p^2}{(p-1)h(p)}\Big\{1 - \frac{p^2-(p-1)h(p)}{p^2}\tau^{-h(p)/p}\Big\}\tau^{h(p)/p} \geqslant \tau^{h(p)/p}
\end{aligned}$$

for $\tau \geqslant 1$. Multiplying this inequality by $a(\tau)$ and integrating both sides from 1 to $s \geqslant 1$, we have

$$\begin{aligned}
\int_1^s a(\tau)\exp\Big(\int_1^\tau \frac{b(\sigma)}{a(\sigma)}\,\mathrm{d}\sigma\Big)\,\mathrm{d}\tau &\geqslant (p-1)\int_1^s \tau^{p-2+h(p)/p}\,\mathrm{d}\tau \\
&= \frac{(p-1)(s^{p-1+h(p)/p} - 1)}{p-1+h(p)p^{-1}}
\end{aligned}$$

for $s \geqslant 1$. Since the right-hand side of this inequality is positive for $s \geqslant 2$ and $1/(1-p)$ is negative, it follows that

$$\begin{aligned}\left(\int_1^s a(\tau)\exp\left(\int_1^\tau \frac{b(\sigma)}{a(\sigma)}\,\mathrm{d}\sigma\right)\mathrm{d}\tau\right)^{1/(1-p)} &\leqslant \left\{\frac{(p-1)(s^{p-1+h(p)/p}-1)}{p-1+h(p)p^{-1}}\right\}^{-1/(p-1)}\\ &\leqslant H(p)s^{-1-h(p)/\{p(p-1)\}}\end{aligned}$$

for $s \geqslant 2$, where

$$H(p)=\left\{\frac{(p-1)(1-2^{-(p-1+h(p)/p)})}{p-1+h(p)p^{-1}}\right\}^{-1/(p-1)}>0.$$

Therefore, we obtain

$$\int_2^t\left(\int_1^s a(\tau)\exp\left(\int_1^\tau \frac{b(\sigma)}{a(\sigma)}\,\mathrm{d}\sigma\right)\mathrm{d}\tau\right)^{1/(1-p)}\mathrm{d}s < \frac{p(p-1)H(p)}{h(p)}2^{-h(p)/\{p(p-1)\}}$$

for $t \geqslant 2$. This means that (1.10) in Corollaries 1.3 and 1.5 does not hold when $1 < p < \frac{1}{2}(3+\sqrt{5})$.

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Authors' address: Yukihide Enaka, Masakazu Onitsuka (corresponding author), Department of Applied Mathematics, Okayama University of Science, 1-1 Ridaicho, Kita-ku, Okayama-shi 700-0005, Japan, e-mail: `onitsuka@xmath.ous.ac.jp`.