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# REMARKS ON BALANCED NORM ERROR ESTIMATES FOR SYSTEMS OF REACTION-DIFFUSION EQUATIONS

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*Abstract.* Error estimates of finite element methods for reaction-diffusion problems are often realized in the related energy norm. In the singularly perturbed case, however, this norm is not adequate. A different scaling of the $H^1$ seminorm leads to a balanced norm which reflects the layer behavior correctly. We discuss the difficulties which arise for systems of reaction-diffusion problems.

*Keywords*: singular perturbation; finite element method; layer-adapted mesh; balanced norm

*MSC 2010*: 65N30

## 1. Introduction

We will examine the finite element method for the numerical solution of systems of reaction-diffusion equations

$$-Eu'' + Au = f \quad \text{in } \Omega = (0,1), \tag{1.1a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $E = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_l)$ with small real parameters $\varepsilon_1, \ldots, \varepsilon_l$. $A$ is a symmetric, strictly diagonally dominant matrix with sufficiently smooth components $a_{ij}$ and $a_{ii} > 0$; we assume also $f$ to be sufficiently smooth.

The finite element discretization uses the bilinear form

$$B(u,v) := \sum_m \varepsilon_m (u'_m, v'_m) + \sum_m \sum_{i=1}^{m} (a_{mi} u_i, v_m).$$

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The related energy norm is

$$\|v\|_e^2 := \sum_m \varepsilon_m (v_m', v_m') + \|v\|_0^2.$$

Linß [7] proved error estimates for linear elements on Bakhvalov and Shishkin meshes, for instance,

$$\|u - u^N\|_e \leqslant CN^{-1} \ln N$$

for a Shishkin mesh.

However, the typical boundary layer function $\exp(-x/\varepsilon_l^{1/2})$ measured in the norm $\|\cdot\|_\varepsilon$ is of order $\mathcal{O}(\varepsilon_l^{1/4})$. Consequently, error estimates in the energy norm are less valuable. Therefore, we ask the fundamental question:

*Is it possible to prove error estimates in the balanced norm*

$$\|v\|_b^2 := \sum_m \varepsilon_m^{1/2} (v_m', v_m') + \|v\|_0^2\,? \tag{1.2}$$

The first balanced error estimate was presented by Lin and Stynes [5] using a first order system least squares (FOSLS) mixed method. But it is also possible to use a direct mixed method [10]. Several further results concerning balanced norm estimates for finite element methods and second order reaction-diffusion problems are presented in [11] (for instance, weakly nonlinear problems, different classes of layer-adapted meshes, the 3D case, supercloseness). For the $hp$-FEM on spectral boundary layer meshes we refer to [8] and [2].

Convection-diffusion problems with different layers in the $x$- and $y$-direction are examined in [3], fourth order problems discretized by mixed finite element methods in [4].

As discussed in [11], it is open how to handle problems with different layers in one coordinate direction or systems of reaction-diffusion equations.

In Section 2 we will repeat a basic idea to prove error estimates in a balanced norm from [12].

In Section 3.1 we start to discuss systems in the case $\varepsilon_1 = \varepsilon_2$ (for simplicity, we only discuss two equations, i.e., $l = 2$), and in Section 3.2 sketch the difficulties for different parameters.

## 2. The basic error estimate in a balanced norm in the scalar case

We consider Shishkin meshes. In the scalar case (replace the matrix $A$ by a scalar $c$) the mesh distributes $N/4$ points (assuming $N$ is divisible by 4) equidistantly within each of the subintervals $[0, \lambda_x]$, $[1 - \lambda_x, 1]$ and the remaining points within the third subinterval. For simplicity, assume

$$\lambda = \lambda_x = \lambda_y = \min\left\{1/4, \lambda_0\sqrt{\varepsilon/c^*}\ln N\right\} \quad \text{with } \lambda_0 = 2 \text{ and } c^* < c.$$

We remark that the choice of $\lambda_0$ mainly depends on the polynomial degree of the finite element space. For systems, see [7] for the description of a related Shishkin mesh.

We use the step sizes

$$h := \frac{4\lambda}{N} \quad \text{and} \quad H := \frac{2(1-2\lambda)}{N}.$$

Let $V^N \subset H_0^1(\Omega)$ be the space of linear finite elements on $\Omega^N$. A standard weak formulation of the scalar version of problem (1.1) reads: Find $u \in V$ such that

$$\varepsilon(u', v') + (cu, v) = (f, v) \quad \forall\, v \in V. \tag{2.1}$$

Replacing $V$ in (2.1) by $V^N$ one obtains a standard discretization that yields the FEM-solution $u^N$.

Certain assumptions on $f$ allow a decomposition of $u$ into smooth components $S$ and layer terms $E$ such that the following estimates for the interpolation error of the Lagrange interpolant hold true (see [13]):

$$\|u - u^I\|_0 \preceq N^{-2}, \quad \varepsilon^{1/4}|u - u^I|_1 \preceq N^{-1}\ln N \tag{2.2}$$

and

$$\|u - u^I\|_{\infty,\Omega_0} \preceq N^{-2}, \quad \|u - u^I\|_{\infty,\Omega\setminus\Omega_0} \preceq (N^{-1}\ln N)^2, \tag{2.3}$$

where $\Omega_0 = (\lambda_x, 1 - \lambda_x)$. Let us also introduce $\Omega_f := \Omega \setminus \Omega_0$. We have used the notation that if $a \preceq b$ there exists a constant $C$ independent of $\varepsilon$ such that $a \leqslant Cb$.

Instead of the Lagrange interpolant we introduce into the error analysis the $L_2$ projection $\pi u \in V^N$ from $u$. Based on

$$u - u^N = u - \pi u + \pi u - u^N$$

we estimate $\xi := \pi u - u^N$:

$$\|\xi\|_e^2 \preceq \varepsilon|\nabla\xi|_1^2 + c\,\|\xi\|_0^2 = \varepsilon(\nabla(\pi u - u), \nabla\xi) + (c\,(\pi u - u), \xi).$$

Because our projection is defined by $(c\,(\pi u - u), \xi) = 0$, it follows that

$$|\pi u - u^N|_1 \preceq |u - \pi u|_1. \tag{2.4}$$

If we now could prove an estimate similar to (2.2) for the error of the $L_2$ projection, we would obtain an estimate in the balanced norm if also a fitting estimate of $\|u - u_N\|_0$ is available. The standard error estimation in the energy norm yields for the $L_2$ part $\|u - u_N\|_0 \preceq \varepsilon^{1/4}(N^{-1}\ln N + N^{-2})$, which is sufficient for our aims. Alternatively, one can also prove $\|u - u_N\|_0 \preceq N^{-2}$, very easily in 1D, while in 2D one uses the supercloseness techniques assuming additionally $\lambda_0 \geqslant 2.5$.

If $\pi u$ has some representation $\pi u = \sum_i V_i\varphi_i$, the $V_j$ satisfy the tridiagonal system (with $\bar{h}_i := (h_i + h_{i+1})/2$)

$$\frac{1}{6}\frac{h_i}{\bar{h}_i}\tilde{c}_{i-1}V_{i-1} + \frac{2}{3}\tilde{c}_iV_i + \frac{1}{6}\frac{h_{i+1}}{\bar{h}_i}\tilde{c}_{i+1}V_{i+1} = \frac{1}{\bar{h}_i}\int_{x_{i-1}}^{x_{i+1}} cu\varphi_i. \tag{2.5}$$

The coefficient matrix $M$ of this system is strictly diagonal dominant. It follows that $|V_i| \preceq \|u\|_\infty$, hence we have the stability property

$$\|\pi u\|_\infty \preceq \|u\|_\infty. \tag{2.6}$$

As a consequence we obtain

**Lemma 1.** *Assuming the validity of* (2.2) *and* (2.3), *the error of the* $L_2$ *projection on the Shishkin mesh satisfies*

$$\|u - \pi u\|_\infty \preceq \|u - u^I\|_\infty, \quad \varepsilon^{1/4}|u - \pi u|_1 \preceq N^{-1}(\ln N)^{3/2}. \tag{2.7}$$

From (2.4), Lemma 1 and the estimates for $\|u - u_N\|_0$ we get

**Theorem 1.** *Assuming* (2.2) *and* (2.3), *the error of the Galerkin finite element method with linear elements on a Shishkin mesh satisfies*

$$\|u - u^N\|_b \preceq N^{-1}(\ln N)^{3/2} + N^{-2}. \tag{2.8}$$

In 2D, the $L_\infty$ stability of $L_2$ projection is an interesting topic [1], we used in [12] a result of Oswald [9] for meshes with a special structure. Inverse inequalities are used to move from estimates in $W^1_\infty$ to $L_\infty$, for details see [12]. Finally, in 2D one obtains the estimate (2.8) for linear as well as bilinear finite elements.

**3.1. The case $\varepsilon_1 = \varepsilon_2$.** First let us remark that for systems

$$-\varepsilon u'' + Au = f \quad \text{in } \Omega = (0,1), \tag{3.1a}$$

$$u(0) = u(1) = 0 \quad \text{on } \partial\Omega, \tag{3.1b}$$

so far there exists only a result of Lin and Stynes [6] in a balanced norm. Following the basic idea from [5], but using $C^1$ elements instead of mixed finite elements, they introduce the bilinear form

$$\varepsilon(w', v') + (Aw, v) + \varepsilon^{3/2}(w'', v'') + \varepsilon^{1/2}((Aw)', v')$$

and analyze the finite element method for quadratic $C^1$ elements. The analysis for the Galerkin method with $C^0$ elements is open.

Let us consider the case of two equations and let us write the system (2.5) as

$$M(\pi u) = g.$$

Now we also define in the matrix case the generalized vector-valued $L_2$ projection by

$$(A(\Pi u), \xi) = (A\, u, \xi).$$

If $A$ is a constant matrix, we get the desired $L_\infty$ stability immediately. But if not, we get for the vector of the values of $\Pi u$ in the mesh points a linear system, where the coefficient matrix $\widehat{M}$ has the structure

$$\widehat{M} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{3.2}$$

Here every matrix $A_{ij}$ has the structure of $M$ corresponding to (2.5), one has just to replace $c$ by the components of $A$, i.e., $a_{11}, a_{12}, \dots$

The question is: *Which assumptions on $A$ guarantee that $\|\widehat{M}^{-1}\|_\infty$ is bounded?*

Remark that in the case of constant coefficients we have

$$\widehat{M} = \begin{bmatrix} a_{11}M & a_{12}M \\ a_{21}M & a_{22}M \end{bmatrix}.$$

Now $\widehat{M}$ is the product of the matrices

$$\begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11}E & a_{12}E \\ a_{21}E & a_{22}E \end{bmatrix}.$$

Because $M$ is strictly diagonally dominant, we have $\|\widehat{M}^{-1}\|_\infty \leqslant C$, and the same properties of $A$ imply $\|\widehat{M}^{-1}\|_\infty \leqslant C$. We conjecture that perturbation arguments should yield results in the case of nonconstant coefficients.

**3.2. Different small parameters.** If the two small parameters are different there appear new difficulties. Consider the simplest case of two equations with *constant* coefficients:

$$-\varepsilon_1 u_1'' + u_1 + a_{12}u_2 = f_1, \tag{3.3a}$$

$$-\varepsilon_2 u_2'' + a_{21}u_1 + a_{22}u_2 = f_2 \tag{3.3b}$$

and discretize by the Galerkin method on the corresponding Shishkin mesh. We assume $\varepsilon_1 < \varepsilon_2$.

With some projections $\hat{u}_1, \hat{u}_2$ into the finite element space we introduce $\xi_1 = u_1^N - \hat{u}_1$ and $\xi_2 = u_2^N - \hat{u}_2$. Then

$$\begin{aligned} \varepsilon_1|\xi_1|_1^2 + \varepsilon_2|\xi_2|_1^2 + \|\xi\|_0^2 \preceq{}& \varepsilon_1((\hat{u}_1 - u_1)', \xi_1') + \varepsilon_2((\hat{u}_2 - u_2)', \xi_2') \\ &+ (\hat{u}_1 - u_1, \xi_1) + a_{12}(\hat{u}_2 - u_2, \xi_1) + a_{21}(\hat{u}_1 - u_1, \xi_2) + a_{22}(\hat{u}_2 - u_2, \xi_2). \end{aligned}$$

First we choose for the projections the $L_2$ projections of $u_1, u_2$.

Then the four terms on the second line disappear and we get

$$|\xi_2|_1 \preceq \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{1/2} |u_1 - \hat{u}_1|_1 + |u_2 - \hat{u}_2|_1,$$

consequently we obtained the desired estimate for $\varepsilon_2^{1/4}|\xi_2|_1$ because $\varepsilon_1 < \varepsilon_2$.

*But, unfortunately, this approach does not yield the desired estimate for* $\varepsilon_1^{1/4}|\xi_1|_1$.

In the second step we define $\hat{u}_1$ and $\hat{u}_2$ by

$$(\hat{u}_1 + a_{12}\hat{u}_2, \xi_1) = (u_1 + a_{12}u_2, \xi_1) \tag{3.4}$$

and

$$\varepsilon_2(\hat{u}_2', \xi_2') + (a_{21}\hat{u}_1 + a_{22}\hat{u}_2, \xi_2) = \varepsilon_2(u_2', \xi_2') + (a_{21}u_1 + a_{22}u_2, \xi_2). \tag{3.5}$$

Using (3.4), we can eliminate the first component and get with $\beta = a_{22} - a_{12}a_{21}$

$$\varepsilon_2(\hat{u}_2', \xi_2') + \beta(\hat{u}_2, \xi_2) = \varepsilon_2(u_2', \xi_2') + \beta(u_2, \xi_2). \tag{3.6}$$

This means: $\hat{u}_2$ is just the Ritz projection of a standard scalar reaction-diffusion operator, and we have the desired estimate for $\varepsilon_2^{1/4}|u_2 - \hat{u}_2|_1$. From (3.4) we get (introducing $L_2$ projections)

$$u_1 - \hat{u}_1 = (u_1 - \pi u_1) + a_{12}(u_2 - \pi u_2) + a_{12}(\hat{u}_2 - u_2).$$

This yields the desired estimate for $\varepsilon_1^{1/4}|u_1 - \hat{u}_1|_1$.

Hence, the following natural question arises:

*How that basic idea can be generalized to problems with nonconstant coefficients?*

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