Boban Karapetrović
Libera and Hilbert matrix operator on logarithmically weighted Bergman, Bloch and
Hardy-Bloch spaces

LIBERA AND HILBERT MATRIX OPERATOR
ON LOGARITHMICALLY WEIGHTED BERGMAN,
BLOCH AND HARDY-BLOCH SPACES

Boban Karapetrović, Belgrade

Abstract. We show that if $\alpha > 1$, then the logarithmically weighted Bergman space $A^2_{\log^\alpha}$ is mapped by the Libera operator $\mathcal{L}$ into the space $A^2_{\log^{\alpha-1}}$, while if $\alpha > 2$ and $0 < \varepsilon \leqslant \alpha - 2$, then the Hilbert matrix operator H maps $A^2_{\log^\alpha}$ into $A^2_{\log^{\alpha-2-\varepsilon}}$.

We show that the Libera operator $\mathcal{L}$ maps the logarithmically weighted Bloch space $\mathcal{B}_{\log^\alpha}$, $\alpha \in \mathbb{R}$, into itself, while H maps $\mathcal{B}_{\log^\alpha}$ into $\mathcal{B}_{\log^{\alpha+1}}$.

In Pavlović's paper (2016) it is shown that $\mathcal{L}$ maps the logarithmically weighted Hardy-Bloch space $\mathcal{B}^1_{\log^\alpha}$, $\alpha > 0$, into $\mathcal{B}^1_{\log^{\alpha-1}}$. We show that this result is sharp. We also show that H maps $\mathcal{B}^1_{\log^\alpha}$, $\alpha \geqslant 0$, into $\mathcal{B}^1_{\log^{\alpha-1}}$ and that this result is sharp also.

Keywords: Libera operator; Hilbert matrix operator; Hardy space; Bergman space; Bloch space; Hardy-Bloch space

MSC 2010: 47B38, 47G10, 30H25

1. Introduction

We consider the action of the Libera and Hilbert matrix operators on logarithmically weighted Bergman, Bloch and Hardy-Bloch spaces.

We show that if $\alpha > 1$, then the logarithmically weighted Bergman space $A^2_{\log^\alpha}$ is mapped by the Libera operator $\mathcal{L}$ into the space $A^2_{\log^{\alpha-1}}$. In [4] it is shown that if $f \in A^2_{\log^\alpha}$, where $\alpha > 3$, then $Hf \in A^2$. Here H is the Hilbert matrix operator. Also, in [1] it is shown that $H\colon A^2_{\log^\alpha} \to A^2$ for $\alpha > 2$. We improve this result by showing that if $\alpha > 2$ and $0 < \varepsilon \leqslant \alpha - 2$, then H is well defined on $A^2_{\log^\alpha}$ and maps this space into $A^2_{\log^{\alpha-2-\varepsilon}}$.

We show that the Libera operator $\mathcal{L}$ maps the logarithmically weighted Bloch space $\mathcal{B}_{\log^\alpha}$, $\alpha \in \mathbb{R}$, into itself, while H maps $\mathcal{B}_{\log^\alpha}$ into $\mathcal{B}_{\log^{\alpha+1}}$.

In [8] it is shown that $\mathcal{L}$ maps the logarithmically weighted Hardy-Bloch space $\mathcal{B}_{\log^\alpha}^1$, $\alpha > 0$, into $\mathcal{B}_{\log^{\alpha-1}}^1$. We note that this result is sharp. Our main results are given in Theorem 5.3. Among other things, we show that H maps $\mathcal{B}_{\log^\alpha}^1$, $\alpha \geqslant 0$, into $\mathcal{B}_{\log^{\alpha-1}}^1$ and that this result is sharp.

The definitions of logarithmically weighted Bergman, Bloch and Hardy-Bloch spaces will be given in Sections 3, 4 and 5, respectively.

For $0 < p \leqslant \infty$, Hardy space H^p is the space of all functions f holomorphic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ for which

$$\|f\|_{H^p} = \|f\|_p = \sup_{0 \leqslant r < 1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p} \qquad \text{if } 0 < p < \infty,$$

and

$$M_\infty(r, f) = \sup_{0 \leqslant t < 2\pi} |f(re^{it})|.$$

The Lebesgue measure on $\mathbb{D}$ will be denoted by A and will be normalized so as to have $A(\mathbb{D}) = 1$. That is,

$$dA(z) = \frac{1}{\pi} \, dx \, dy = \frac{1}{\pi} r \, dr \, dt, \qquad \text{where } z = x + iy = re^{it}.$$

The Bergman space A^p, $0 < p < \infty$, is the space of holomorphic functions in $L^p(\mathbb{D}, dA)$, that is,

$$A^p = \left\{ f \in \mathcal{H}(\mathbb{D}) \colon \|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty \right\}.$$

A function f holomorphic in the unit disk $\mathbb{D}$ belongs to the Hardy-Bloch space $\mathcal{B}_0^{p,q}$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ (notation from [7]) if

$$\int_0^1 M_p^q(r, f')(1 - r)^{q-1} \, dr < \infty \quad \text{for } 0 < q < \infty,$$

and

$$\sup_{0 \leqslant r < 1} (1 - r) M_p(r, f') < \infty \quad \text{for } q = \infty.$$

Let $\mathcal{H}(\mathbb{D})$ denote the space of all functions holomorphic in the unit disk $\mathbb{D}$ of the complex plane endowed with the topology of uniform convergence on compact

subsets of $\mathbb{D}$. The dual of $\mathcal{H}(\mathbb{D})$ is equal to $\mathcal{H}(\overline{\mathbb{D}})$, where $g \in \mathcal{H}(\overline{\mathbb{D}})$ means that g is holomorphic in a neighborhood of $\overline{\mathbb{D}}$ (depending on g). The duality pairing is given by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)},$$

where $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n)z^n \in \mathcal{H}(\mathbb{D})$ and $g(z) = \sum\limits_{n=0}^{\infty} \widehat{g}(n)z^n \in \mathcal{H}(\overline{\mathbb{D}})$.

It is easy to see that the Libera operator defined by

$$\overline{\mathcal{L}}g(z) = \sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty} \frac{\widehat{g}(k)}{k+1}\right)z^n = \int_0^1 g(t + (1-t)z)\,\mathrm{d}t,$$

$$g(z) = \sum_{n=0}^{\infty} \widehat{g}(n)z^n \in \mathcal{H}(\overline{\mathbb{D}})$$

maps $\mathcal{H}(\overline{\mathbb{D}})$ into $\mathcal{H}(\overline{\mathbb{D}})$.

We denote by $\mathcal{L}$ the operator

$$\mathcal{L}g(z) = \int_0^1 g(t + (1-t)z)\,\mathrm{d}t,$$

$$g(z) = \sum_{n=0}^{\infty} \widehat{g}(n)z^n \in \mathcal{H}(\mathbb{D}),$$

whenever the integral converges uniformly on compact subsets of $\mathbb{D}$. Uniform convergence means that the limit

$$\lim_{r \to 1^-} \int_0^r g(t + (1-t)z)\,\mathrm{d}t$$

is uniform with respect to z in any compact subset of $\mathbb{D}$. This hypothesis guarantees that $\mathcal{L}g$ is a holomorphic function in $\mathbb{D}$. We call $\mathcal{L}$ also the Libera operator since $\mathcal{L} = \overline{\mathcal{L}}$ on $\mathcal{H}(\overline{\mathbb{D}})$.

The Hilbert matrix is an infinite matrix $H = [h_{n,k}]_{n,k=0}^{\infty}$ whose entries are $h_{n,k} = 1/(n+k+1)$ for all nonnegative integers n and k. It can be viewed as an operator on spaces of holomorphic functions by its action on their Taylor coefficients. If

$$f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n$$

is a holomorphic function in the unit disk $\mathbb{D}$, then we define the transformation H by

$$Hf(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} h_{n,k}\widehat{f}(k)z^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n+k+1}z^n.$$

It is possible to write Hf, $f \in H^p$, $1 \leqslant p \leqslant \infty$, in an integral form, which is quite convenient for analyzing the operator. More specifically, by looking at the Taylor series expansion of the function f, we have the following integral representation:

$$Hf(z) = \int_0^1 \frac{f(t)}{1 - tz} \, \mathrm{d}t.$$

It is well known that the Libera operator $\mathcal{L}$ acts as a bounded operator from H^p into H^p if and only if $1 < p \leqslant \infty$ and that $\mathcal{L}$ acts as a bounded operator from A^p into A^p if and only if $2 < p < \infty$ (see [2], [6]). On the other hand, it is well known that the Hilbert matrix operator H acts as a bounded operator from H^p into H^p if and only if $1 < p < \infty$ and that H acts as a bounded operator from A^p into A^p if and only if $2 < p < \infty$ (see [3]).

2. SOME PRELIMINARY RESULTS

In this section we shall collect some results which will be needed in our work. We start with one useful result.

Sublemma 2.1. *Let $\alpha \in \mathbb{R}$ and $a \geqslant 2$. Then*

$$\int_{\log a}^\infty t^\alpha \mathrm{e}^{-t} \, \mathrm{d}t \leqslant C_\alpha \frac{\log^\alpha a}{a},$$

where C_α is a constant independent of a.

Proof. (1) Case $\alpha \leqslant 0$.

$$\int_{\log a}^\infty t^\alpha \mathrm{e}^{-t} \, \mathrm{d}t \leqslant \log^\alpha a \int_{\log a}^\infty \mathrm{e}^{-t} \, \mathrm{d}t = \frac{\log^\alpha a}{a}.$$

(2) Case $\alpha > 0$. In this case, partial integration gives

$$
\begin{aligned}
\int_{\log a}^\infty t^\alpha \mathrm{e}^{-t} \, \mathrm{d}t &= \frac{\log^\alpha a}{a} + \alpha \int_{\log a}^\infty t^{\alpha-1} \mathrm{e}^{-t} \, \mathrm{d}t \\
&= \frac{\log^\alpha a}{a} + \alpha \frac{\log^{\alpha-1} a}{a} + \alpha(\alpha-1) \int_{\log a}^\infty t^{\alpha-2} \mathrm{e}^{-t} \, \mathrm{d}t \\
&\leqslant C_\alpha \frac{\log^\alpha a}{a} + \alpha(\alpha-1) \int_{\log a}^\infty t^{\alpha-2} \mathrm{e}^{-t} \, \mathrm{d}t.
\end{aligned}
$$

562

Continuing on this way, we find that

$$
\int_{\log a}^{\infty} t^{\alpha} e^{-t}\, dt \leqslant C_{\alpha} \frac{\log^{\alpha} a}{a} + \alpha(\alpha - 1) \ldots (\alpha - \lfloor \alpha \rfloor) \int_{\log a}^{\infty} t^{\alpha - \lfloor \alpha \rfloor - 1} e^{-t}\, dt
$$

$$
\leqslant C_{\alpha} \frac{\log^{\alpha} a}{a} + \alpha(\alpha - 1) \ldots (\alpha - \lfloor \alpha \rfloor) \frac{\log^{\alpha - \lfloor \alpha \rfloor - 1} a}{a}
$$

$$
\leqslant C_{\alpha} \frac{\log^{\alpha} a}{a},
$$

where $\lfloor \alpha \rfloor$ is the largest integer less then or equal to α. $\qquad\square$

Consequently, we get the following result.

Lemma 2.2. *Let $\alpha \in \mathbb{R}$ and let n be a nonnegative integer. Then*

$$
\int_0^1 r^n \log^{\alpha} \frac{2}{1 - r}\, dr \asymp \frac{\log^{\alpha}(n + 2)}{n + 1},
$$

where the corresponding constant is independent of n, i.e., there is a constant C_{α} independent of n such that

$$
\frac{1}{C_{\alpha}} \frac{\log^{\alpha}(n + 2)}{n + 1} \leqslant \int_0^1 r^n \log^{\alpha} \frac{2}{1 - r}\, dr \leqslant C_{\alpha} \frac{\log^{\alpha}(n + 2)}{n + 1}.
$$

P r o o f. (1) Case $\alpha \geqslant 0$. First, we find that

$$
\int_0^1 r^n \log^{\alpha} \frac{2}{1 - r}\, dr \geqslant \int_{1 - 1/(n+1)}^1 r^n \log^{\alpha} \frac{2}{1 - r}\, dr
$$

$$
\geqslant \log^{\alpha}(n + 2) \int_{1 - 1/(n+1)}^1 r^n\, dr
$$

$$
= \frac{\log^{\alpha}(n + 2)}{n + 1} \left(1 - \left(1 - \frac{1}{n + 1} \right)^{n+1} \right)
$$

$$
\geqslant C \frac{\log^{\alpha}(n + 2)}{n + 1}.
$$

On the other hand, by using Sublemma 2.1, we have that

$$
\int_{1 - 1/(n+1)}^1 r^n \log^{\alpha} \frac{2}{1 - r}\, dr \leqslant \int_{1 - 1/(n+1)}^1 \log^{\alpha} \frac{2}{1 - r}\, dr
$$

$$
= 2 \int_{\log(2n+2)}^{\infty} t^{\alpha} e^{-t}\, dt
$$

$$
\leqslant 2 \int_{\log(n+2)}^{\infty} t^{\alpha} e^{-t}\, dt
$$

$$
\leqslant C_{\alpha} \frac{\log^{\alpha}(n + 2)}{n + 1},
$$

and

$$\int_0^{1-1/(n+1)} r^n \log^\alpha \frac{2}{1-r} \, dr \leqslant C_\alpha \log^\alpha(n+2) \int_0^{1-1/(n+1)} r^n \, dr$$

$$= C_\alpha \frac{\log^\alpha(n+2)}{n+1} \Big(1 - \frac{1}{n+1} \Big)^{n+1}$$

$$\leqslant C_\alpha \frac{\log^\alpha(n+2)}{n+1}.$$

Therefore,

$$\int_0^1 r^n \log^\alpha \frac{2}{1-r} \, dr \leqslant C_\alpha \frac{\log^\alpha(n+2)}{n+1}.$$

(2) Case $\alpha < 0$. Let $\varphi(r) = r \log^\alpha(2/r)$, $0 < r \leqslant 1$. Then φ is a nonnegative, increasing function on the interval $(0, 1]$ and

$$t^{1-2\alpha}\varphi(r) \leqslant \varphi(tr) \leqslant t\varphi(r)$$

for all $0 < t < 1$. By using Lemma 4.1 in [5], we find that

$$\int_0^1 r^n \frac{\varphi(1-r)}{1-r} \, dr \asymp \varphi\Big(\frac{1}{n+1}\Big).$$

Hence,

$$\int_0^1 r^n \log^\alpha \frac{2}{1-r} \, dr \asymp \frac{\log^\alpha(n+2)}{n+1}.$$

$\square$

The following auxiliary result will be useful.

Theorem 2.3. (a) *For every real α, the Taylor coefficients $\widehat{F}(n)$ of the function*

$$F(z) = \frac{1}{1-z} \log^\alpha \frac{2}{1-z}$$

have the property

$$\widehat{F}(n) \asymp \log^\alpha(n+2),$$

where the corresponding constant is independent of n.

(b) *For every real α, the Taylor coefficients $\widehat{G}(n)$ of the function*

$$G(z) = \log^\alpha \frac{2}{1-z}$$

have the property

$$\widehat{G}(n) \asymp \frac{\log^{\alpha-1}(n+2)}{n+1},$$

where the corresponding constant is independent of n.

564

This theorem is a consequence of Theorem 2.31 on page 192 of the classical monograph [10], hence we omit its proof. Now, we are ready to prove our next result.

Lemma 2.4. *Let $\alpha \in \mathbb{R}$ and let k be a nonnegative integer. Then*

$$\sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+2)}{(n+1)(n+k+1)} \asymp \frac{\log^{\alpha+1}(k+2)}{k+1},$$

where the corresponding constant is independent of k, i.e., there is a constant C_{α} independent of k such that

$$\frac{1}{C_{\alpha}} \frac{\log^{\alpha+1}(k+2)}{k+1} \leqslant \sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+2)}{(n+1)(n+k+1)} \leqslant C_{\alpha} \frac{\log^{\alpha+1}(k+2)}{k+1}.$$

P r o o f.   By using Lemma 2.2 and Theorem 2.3 (b), we find that

$$\frac{\log^{\alpha+1}(k+2)}{k+1} \asymp \int_{0}^{1} r^{k} \log^{\alpha+1} \frac{2}{1-r} \, \mathrm{d}r$$

$$\asymp \int_{0}^{1} r^{k} \sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+2)}{n+1} r^{n} \, \mathrm{d}r$$

$$= \sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+2)}{n+1} \int_{0}^{1} r^{n+k} \, \mathrm{d}r$$

$$= \sum_{n=0}^{\infty} \frac{\log^{\alpha}(n+2)}{(n+1)(n+k+1)},$$

where the corresponding constant is independent of k. $\square$

3. Libera and Hilbert matrix operator on logarithmically weighted Bergman spaces

For $\alpha \in \mathbb{R}$ we define the logarithmically weighted Bergman spaces $A_{\log^{\alpha}}^{2}$ as follows:

$$A_{\log^{\alpha}}^{2} = \left\{ f \in \mathcal{H}(\mathbb{D}) \colon \|f\|_{A_{\log^{\alpha}}^{2}}^{2} = \int_{\mathbb{D}} |f(z)|^{2} \log^{\alpha} \frac{2}{1-|z|^{2}} \, \mathrm{d}A(z) < \infty \right\}.$$

Note that $A_{\log^{\alpha}}^{2} \subset A^{2}$ for $\alpha > 0$ and $A_{\log^{0}}^{2} = A^{2}$.

Let $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^{n} \in A_{\log^{\alpha}}^{2}$. By using Parseval's formula and Lemma 2.2, we find that

$$\|f\|_{A_{\log^{\alpha}}^{2}}^{2} = \sum_{n=0}^{\infty} |\widehat{f}(n)|^{2} \int_{0}^{1} r^{n} \log^{\alpha} \frac{2}{1-r} \, \mathrm{d}r \asymp \sum_{n=0}^{\infty} |\widehat{f}(n)|^{2} \frac{\log^{\alpha}(n+2)}{n+1},$$

where the corresponding constant is independent of function f, i.e., there is a constant C independent of function f such that

$$\frac{1}{C}\|f\|_{A^2_{\log^\alpha}}^2 \leqslant \sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \frac{\log^\alpha(n+2)}{n+1} \leqslant C\|f\|_{A^2_{\log^\alpha}}^2.$$

3.1. Libera operator on logarithmically weighted Bergman spaces. Our next result describes the action of the Libera operator $\mathcal{L}$ on the logarithmically weighted Bergman space $A^2_{\log^\alpha}$ for $\alpha > 1$.

Theorem 3.1. *If* $\alpha > 1$, *then the operator* $\mathcal{L}$ *is well defined on* $A^2_{\log^\alpha}$ *and maps this space into* $A^2_{\log^{\alpha-1}}$.

P r o o f.   Let $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n)z^n \in A^2_{\log^\alpha}$. Then, by using the Cauchy-Schwarz inequality, we find that

$$\sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|}{n+1} \leqslant \left(\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \frac{\log^\alpha(n+2)}{n+1}\right)^{1/2} \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)\log^\alpha(n+2)}\right)^{1/2} < \infty,$$

because $\alpha > 1$. From this, we get that if $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n)z^n \in A^2_{\log^\alpha}$, then $\sum\limits_{n=0}^{\infty} |\widehat{f}(n)| \times (n+1)^{-1} < \infty$. Hence, the operator $\mathcal{L}$ is well defined on $A^2_{\log^\alpha}$. Using inequality (59) from [6], we find that

$$rM_2^2(r, \mathcal{L}f) \leqslant C(1-r)^{-1} \int_r^1 M_2^2(s, f)\,\mathrm{d}s$$

for all $0 \leqslant r < 1$. Therefore,

$$\begin{aligned}
\int_{\mathbb{D}} |\mathcal{L}f(z)|^2 \log^{\alpha-1} \frac{2}{1-|z|^2}\,\mathrm{d}A(z) &= 2\int_0^1 rM_2^2(r, \mathcal{L}f) \log^{\alpha-1} \frac{2}{1-r^2}\,\mathrm{d}r \\
&\leqslant C\int_0^1 \frac{1}{1-r} \int_r^1 M_2^2(s, f)\,\mathrm{d}s \log^{\alpha-1} \frac{2}{1-r^2}\,\mathrm{d}r \\
&= C\int_0^1 M_2^2(s, f) \int_0^s \frac{1}{1-r} \log^{\alpha-1} \frac{2}{1-r^2}\,\mathrm{d}r\,\mathrm{d}s \\
&\leqslant C\int_0^1 M_2^2(s, f) \int_0^s \frac{1}{1-r} \log^{\alpha-1} \frac{2}{1-r}\,\mathrm{d}r\,\mathrm{d}s \\
&= C\int_0^1 M_2^2(s, f) \left(\log^\alpha \frac{2}{1-s} - \log^\alpha 2\right)\mathrm{d}s \\
&\leqslant C\int_0^1 M_2^2(s, f) \log^\alpha \frac{2}{1-s}\,\mathrm{d}s
\end{aligned}$$

$$= C \int_0^1 u M_2^2(u^2, f) \log^\alpha \frac{2}{1 - u^2} \, \mathrm{d}u$$

$$\leqslant C \int_0^1 u M_2^2(u, f) \log^\alpha \frac{2}{1 - u^2} \, \mathrm{d}u$$

$$= C \int_{\mathbb{D}} |f(z)|^2 \log^\alpha \frac{2}{1 - |z|^2} \, \mathrm{d}A(z) < \infty.$$

Hence, $\mathcal{L}f \in A^2_{\log^{\alpha-1}}$. $\qquad\square$

3.2. Hilbert matrix operator on logarithmically weighted Bergman spaces. In [4] it is shown that if $f \in A^2_{\log^\alpha}$, where $\alpha > 3$, then $Hf \in A^2$. Also, in [1] it is shown that $H \colon A^2_{\log^\alpha} \to A^2$ for $\alpha > 2$. Our next theorem improves this result.

Theorem 3.2. *If $\alpha > 2$ and $0 < \varepsilon \leqslant \alpha - 2$, then H is well defined on $A^2_{\log^\alpha}$ and maps this space into $A^2_{\log^{\alpha-2-\varepsilon}}$.*

Proof. For $\alpha > 2$, we have that if $f(z) = \sum\limits_{n=0}^\infty \widehat{f}(n) z^n \in A^2_{\log^\alpha}$, then we get $\sum\limits_{n=0}^\infty |\widehat{f}(n)|/(n+1) < \infty$. Therefore, the operator H is well defined on $A^2_{\log^\alpha}$. On the other hand, if $f(z) = \sum\limits_{n=0}^\infty \widehat{f}(n) z^n \in A^2_{\log^\alpha}$, then by using the Cauchy-Schwarz inequality and Lemma 2.4, we find that

$$\|Hf\|^2_{A^2_{\log^{\alpha-2-\varepsilon}}} \asymp \sum_{n=0}^\infty |\widehat{Hf}(n)|^2 \frac{\log^{\alpha-2-\varepsilon}(n+2)}{n+1}$$

$$= \sum_{n=0}^\infty \left| \sum_{k=0}^\infty \frac{\widehat{f}(k)}{n+k+1} \right|^2 \frac{\log^{\alpha-2-\varepsilon}(n+2)}{n+1}$$

$$\leqslant \sum_{n=0}^\infty \sum_{k=0}^\infty \frac{|\widehat{f}(k)|^2 \log^\alpha(k+2)}{n+k+1}$$

$$\times \sum_{k=0}^\infty \frac{1}{(n+k+1)\log^\alpha(k+2)} \frac{\log^{\alpha-2-\varepsilon}(n+2)}{n+1}$$

$$\leqslant \sum_{k=0}^\infty |\widehat{f}(k)|^2 \frac{\log^\alpha(k+2)}{k+1} \sum_{k=0}^\infty \frac{1}{\log^\alpha(k+2)} \sum_{n=0}^\infty \frac{\log^{\alpha-2-\varepsilon}(n+2)}{(n+1)(n+k+1)}$$

$$\leqslant C\|f\|^2_{A^2_{\log^\alpha}} \sum_{k=0}^\infty \frac{1}{\log^\alpha(k+2)} \frac{\log^{\alpha-1-\varepsilon}(k+2)}{k+1}$$

$$= C\|f\|^2_{A^2_{\log^\alpha}} \sum_{k=0}^\infty \frac{1}{(k+1)\log^{1+\varepsilon}(k+2)} < \infty.$$

Therefore, $Hf \in A^2_{\log^{\alpha-2-\varepsilon}}$. $\qquad\square$

We note that for $\alpha \in (1, 2]$ the operator H is well defined on $A^2_{\log^\alpha}$. We do not know whether Theorem 3.2 holds in this case. A natural question is: Does Theorem 3.2 hold for $\varepsilon = 0$?

4. Libera and Hilbert matrix operator on logarithmically weighted Bloch spaces

For $\alpha \in \mathbb{R}$ we define the logarithmically weighted Bloch spaces $\mathcal{B}_{\log^\alpha}$ as follows,

$$\mathcal{B}_{\log^\alpha} = \left\{ f \in \mathcal{H}(\mathbb{D})\colon |f'(z)|(1 - |z|) = \mathcal{O}\left(\log^\alpha \frac{2}{1 - |z|} \right) \right\}.$$

The norm in the space $\mathcal{B}_{\log^\alpha}$ is defined by

$$\|f\|_{\mathcal{B}_{\log^\alpha}} = |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)|(1 - |z|) \log^{-\alpha} \frac{2}{1 - |z|}.$$

Note that $\mathcal{B}_{\log^0}$ is the Bloch space $\mathcal{B}_0^{\infty,\infty} = \mathcal{B}$.

4.1. Libera operator on logarithmically weighted Bloch spaces. Now, we have the following theorem.

Theorem 4.1. *Let $\alpha \in \mathbb{R}$. Then $\mathcal{L}$ is well defined on $\mathcal{B}_{\log^\alpha}$ and maps this space into $\mathcal{B}_{\log^\alpha}$.*

P r o o f.   By Theorem 2.1 (a) in [8], we have that if $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathcal{B}_{\log^\alpha}$, then $\sum\limits_{n=0}^{\infty} |\widehat{f}(n)|/(n + 1) < \infty$. Therefore, the Libera operator $\mathcal{L}$ is well defined on $\mathcal{B}_{\log^\alpha}$.

By using Lemma 22 from [6], for $\nu = 1$ and $p = \infty$, we find that

$$M_\infty(r, (\mathcal{L}f)') \leqslant (1 - r)^{-2} \int_r^1 (1 - s) M_\infty(s, f') \, \mathrm{d}s$$

for all $0 \leqslant r < 1$. Then, by using Sublemma 2.1, we have that

$$
\begin{aligned}
(1 - r) M_\infty(r, (\mathcal{L}f)') &\leqslant \frac{1}{1 - r} \int_r^1 (1 - s) M_\infty(s, f') \, \mathrm{d}s \\
&\leqslant C \frac{1}{1 - r} \int_r^1 \log^\alpha \frac{2}{1 - s} \, \mathrm{d}s \\
&= C \frac{1}{1 - r} \int_{\log(2/(1-r))}^{\infty} t^\alpha \mathrm{e}^{-t} \, \mathrm{d}t \\
&\leqslant C \log^\alpha \frac{2}{1 - r}.
\end{aligned}
$$

Hence, we obtain $\mathcal{L}f \in \mathcal{B}_{\log^\alpha}$. $\qquad\square$

568

4.2. Hilbert matrix operator on logarithmically weighted Bloch spaces.
Our next result describes the action of the Hilbert matrix operator H on the log-aritmically weighted Bloch space $\mathcal{B}_{\log^\alpha}$ for $\alpha \in \mathbb{R}$. We improve results given in Proposition 5.1 and Proposition 5.2 in [4].

Theorem 4.2. *Let $\alpha \in \mathbb{R}$. Then H is well defined on $\mathcal{B}_{\log^\alpha}$ and maps this space into $\mathcal{B}_{\log^{\alpha+1}}$.*

P r o o f. By Theorem 2.1 (a) in [8], we have that if $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathcal{B}_{\log^\alpha}$, then $\sum\limits_{n=0}^{\infty} |\widehat{f}(n)|/(n+1) < \infty$. Hence, the Hilbert matrix operator H is well defined on $\mathcal{B}_{\log^\alpha}$.

Now, let $f \in \mathcal{B}_{\log^\alpha}$, where without loss of generality, we can additionally assume that $f(0) = 0$. Then, by Lemma 4.2.8 in [9], we can write

$$f(z) = \int_{\mathbb{D}} \frac{f'(w)(1 - |w|^2)}{\overline{w}(1 - \overline{w}z)^2} \, \mathrm{d}A(w),$$

for all $z \in \mathbb{D}$. Also, we have that

$$\int_{\mathbb{D}} \frac{z^k |z|^{2n}}{1 - \overline{z}} \, \mathrm{d}A(z) = \frac{1}{n + k + 1},$$

for all nonnegative integers n and k. Therefore,

$$\int_{\mathbb{D}} f(z) \frac{|z|^{2n}}{1 - \overline{z}} \, \mathrm{d}A(z) = \sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n + k + 1} = \widehat{Hf}(n).$$

Consequently,

$$
\begin{aligned}
|\widehat{Hf}(n)| &= \left| \int_{\mathbb{D}} f(z) \frac{|z|^{2n}}{1 - \overline{z}} \, \mathrm{d}A(z) \right| \\
&= \left| \int_{\mathbb{D}} \frac{f'(w)(1 - |w|^2)}{\overline{w}} \int_{\mathbb{D}} \frac{|z|^{2n}}{(1 - \overline{w}z)^2(1 - \overline{z})} \, \mathrm{d}A(z) \, \mathrm{d}A(w) \right| \\
&= \frac{1}{\pi} \left| \int_{\mathbb{D}} \frac{f'(w)(1 - |w|^2)}{\overline{w}} \int_0^1 r^{2n+1} \int_0^{2\pi} \frac{1}{(1 - \overline{w}re^{i\theta})^2(1 - re^{-i\theta})} \, \mathrm{d}\theta \, \mathrm{d}r \, \mathrm{d}A(w) \right| \\
&= \frac{1}{\pi} \left| \int_{\mathbb{D}} \frac{f'(w)(1 - |w|^2)}{\overline{w}} \int_0^1 r^{2n+1} \frac{2\pi}{(1 - r^2\overline{w})^2} \, \mathrm{d}r \, \mathrm{d}A(w) \right| \\
&\leqslant C \int_{\mathbb{D}} \frac{|f'(w)|(1 - |w|)}{|w|} \int_0^1 \frac{r^{2n+1}}{|1 - r^2 w|^2} \, \mathrm{d}r \, \mathrm{d}A(w) \\
&\leqslant C \int_{\mathbb{D}} \frac{\log^\alpha \frac{2}{1 - |w|}}{|w|} \int_0^1 \frac{r^{2n+1}}{|1 - r^2 w|^2} \, \mathrm{d}r \, \mathrm{d}A(w)
\end{aligned}
$$

569

$$= C \int_0^1 r^{2n+1} \int_{\mathbb{D}} \frac{\log^\alpha \frac{2}{1-|w|}}{|w||1 - r^2 w|^2} \, \mathrm{d}A(w) \, \mathrm{d}r$$

$$= C \int_0^1 r^{2n+1} \int_0^1 \log^\alpha \frac{2}{1-\varrho} \int_0^{2\pi} \frac{1}{|1 - r^2 \varrho \mathrm{e}^{i\theta}|^2} \, \mathrm{d}\theta \, \mathrm{d}\varrho \, \mathrm{d}r$$

$$\leqslant C \int_0^1 r^{2n+1} \int_0^1 \frac{1}{1 - r^2\varrho} \log^\alpha \frac{2}{1-\varrho} \, \mathrm{d}\varrho \, \mathrm{d}r$$

$$= C \int_0^1 \log^\alpha \frac{2}{1-\varrho} \int_0^1 \frac{r^n}{1 - r\varrho} \, \mathrm{d}r \, \mathrm{d}\varrho.$$

On the other hand, we find that

$$\int_0^1 \frac{r^n}{1 - r\varrho} \, \mathrm{d}r = \sum_{k=0}^{\infty} \frac{\varrho^k}{n + k + 1}.$$

Hence, by using Lemma 2.2 and Lemma 2.4, we obtain

$$|\widehat{Hf}(n)| \leqslant C \sum_{k=0}^{\infty} \frac{1}{n+k+1} \int_0^1 \varrho^k \log^\alpha \frac{2}{1-\varrho} \, \mathrm{d}\varrho$$

$$\leqslant C \sum_{k=0}^{\infty} \frac{\log^\alpha(k+2)}{(k+1)(n+k+1)}$$

$$\leqslant C \frac{\log^{\alpha+1}(n+2)}{n+1}.$$

Therefore,

$$|(Hf)'(z)| = \left| \sum_{n=1}^{\infty} n\widehat{Hf}(n) z^{n-1} \right|$$

$$\leqslant \sum_{n=1}^{\infty} n|\widehat{Hf}(n)||z|^{n-1}$$

$$\leqslant C \sum_{n=1}^{\infty} \frac{n}{n+1} \log^{\alpha+1}(n+2)|z|^{n-1}$$

$$\leqslant C \sum_{n=0}^{\infty} \log^{\alpha+1}(n+3)|z|^n$$

$$\leqslant C \sum_{n=0}^{\infty} \log^{\alpha+1}(n+2)|z|^n.$$

By using Theorem 2.3 (a), we find that

$$\frac{1}{1-|z|} \log^{\alpha+1} \frac{2}{1-|z|} \asymp \sum_{n=0}^{\infty} \log^{\alpha+1}(n+2)|z|^n.$$

570

Finally,

$$|(Hf)'(z)| \leqslant C \sum_{n=0}^{\infty} \log^{\alpha+1}(n+2)|z|^n$$

$$\leqslant C \frac{1}{1-|z|} \log^{\alpha+1} \frac{2}{1-|z|}.$$

Hence, $Hf \in \mathcal{B}_{\log^{\alpha+1}}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

5. Libera and Hilbert matrix operator on logarithmically weighted Hardy-Bloch spaces

For $\alpha \in \mathbb{R}$ we define the logarithmically weighted Hardy-Bloch spaces $\mathcal{B}^1_{\log^\alpha}$ in the following way:

$$\mathcal{B}^1_{\log^\alpha} = \left\{ f \in \mathcal{H}(\mathbb{D}) \colon \|f\|_{\mathcal{B}^1_{\log^\alpha}} = |f(0)| + \int_{\mathbb{D}} |f'(z)| \log^\alpha \frac{2}{1-|z|}\, \mathrm{d}A(z) < \infty \right\}.$$

For $\alpha = 0$, $\mathcal{B}^1_{\log^0}$ is the Hardy-Bloch space $\mathcal{B}^{1,1}_0$ (notation from [7]). We note that if $\alpha \geqslant 0$, then $\mathcal{B}^1_{\log^\alpha} \subseteq \mathcal{B}^{1,1}_0 \subseteq H^1$ and if $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n \in H^1$, then $\sum_{n=0}^{\infty} |\widehat{f}(n)|/(n+1) < \infty$.

5.1. Libera operator on logarithmically weighted Hardy-Bloch spaces. Action of the Libera operator on the logarithmically weighted Hardy-Bloch spaces has been considered in [8] and the following two theorems are proved.

Theorem 5.1 ([8]). *Let $\alpha \geqslant -1$ and let $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n \in \mathcal{H}(\mathbb{D})$, where $\widehat{f}(n) \downarrow 0$ as $n \to \infty$. Then*

$$f \in \mathcal{B}^1_{\log^\alpha} \qquad \text{if and only if} \qquad \sum_{n=0}^{\infty} \widehat{f}(n) \frac{\log^\alpha(n+2)}{n+1} < \infty.$$

Moreover, there is a constant C independent of the function f, such that

$$\frac{1}{C}\|f\|_{\mathcal{B}^1_{\log^\alpha}} \leqslant \sum_{n=0}^{\infty} \widehat{f}(n) \frac{\log^\alpha(n+2)}{n+1} \leqslant C\|f\|_{\mathcal{B}^1_{\log^\alpha}}.$$

Theorem 5.2 ([8]). *Let $\alpha > 0$.*

(a) *Then $\mathcal{L}$ is well defined on $\mathcal{B}^1_{\log^\alpha}$ and maps this space into $\mathcal{B}^1_{\log^{\alpha-1}}$.*

(b) *If $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n)z^n$, where $\widehat{f}(n) \downarrow 0$ as $n \to \infty$ and $\sum\limits_{n=0}^{\infty} \widehat{f}(n)/(n+1) < \infty$, then $\mathcal{L}f \in \mathcal{B}^1_{\log^{\alpha-1}}$ implies $f \in \mathcal{B}^1_{\log^\alpha}$.*

(c) *If $\alpha < 0$, then $\overline{\mathcal{L}}$ cannot be extended to a continuous operator from $\mathcal{B}^1_{\log^\alpha}$ to $\mathcal{H}(\mathbb{D})$.*

For (a) see Theorem 2.3 in [8]. Item (b) follows from Theorem 1.1 and Theorem 1.2 in [8]. For (c) see Theorem 2.1 (c) in [8].

5.2. Hilbert matrix operator on logarithmically weighted Hardy-Bloch spaces. Now we are ready to state the main theorem of this section.

Theorem 5.3. *Let $\alpha \geqslant 0$.*

(a) *Then H is well defined on $\mathcal{B}^1_{\log^\alpha}$ and maps this space into $\mathcal{B}^1_{\log^{\alpha-1}}$.*

(b) *If $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n)z^n$, where $\widehat{f}(n) \downarrow 0$ as $n \to \infty$ and $\sum\limits_{n=0}^{\infty} \widehat{f}(n)/(n+1) < \infty$, then $Hf \in \mathcal{B}^1_{\log^{\alpha-1}}$ implies $f \in \mathcal{B}^1_{\log^\alpha}$.*

(c) *The result in (a) is sharp in the sense that for any $\varepsilon > 0$ there exists $f \in \mathcal{B}^1_{\log^\alpha}$ such that $Hf \notin \mathcal{B}^1_{\log^{\alpha-1+\varepsilon}}$.*

(d) *If $\alpha < 0$, then H cannot be extended to a continuous operator from $\mathcal{B}^1_{\log^\alpha}$ to $\mathcal{H}(\mathbb{D})$.*

P r o o f.   (a) By Theorem 2.1 (b) in [8], we have that if $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n)z^n \in \mathcal{B}^1_{\log^\alpha}$, then $\sum\limits_{n=0}^{\infty} |\widehat{f}(n)|/(n+1) < \infty$. Therefore, the operator H is well defined on $\mathcal{B}^1_{\log^\alpha}$.

Now, let $f \in \mathcal{B}^1_{\log^\alpha}$, where without loss of generality, we can additionally assume that $f(0) = 0$. Then, by Lemma 4.2.8 in [9], we have

$$f(z) = \int_{\mathbb{D}} \frac{f'(w)(1 - |w|^2)}{\overline{w}(1 - \overline{w}z)^2}\, \mathrm{d}A(w),$$

for all $z \in \mathbb{D}$. Let $S = \int_{\mathbb{D}} |(Hf)'(z)| \log^{\alpha-1}(2/(1 - |z|))\, \mathrm{d}A(z)$. Then, by using the integral representation of the Hilbert matrix operator $Hf(z) = \int_0^1 (f(t)/(1 - tz))\, \mathrm{d}t$,

we find that

$$
\begin{aligned}
S &\leqslant \int_{\mathbb{D}} \int_0^1 \frac{|f(t)|}{|1-tz|^2} \log^{\alpha-1} \frac{2}{1-|z|} \, \mathrm{d}t \, \mathrm{d}A(z) \\
&= \int_{\mathbb{D}} \int_0^1 \frac{1}{|1-tz|^2} \left| \int_{\mathbb{D}} \frac{f'(w)(1-|w|^2)}{\overline{w}(1-\overline{w}t)^2} \, \mathrm{d}A(w) \right| \log^{\alpha-1} \frac{2}{1-|z|} \, \mathrm{d}t \, \mathrm{d}A(z) \\
&\leqslant C \int_{\mathbb{D}} \frac{|f'(w)|(1-|w|)}{|w|} \int_{\mathbb{D}} \log^{\alpha-1} \frac{2}{1-|z|} \int_0^1 \frac{\mathrm{d}t}{|1-tz|^2|1-tw|^2} \, \mathrm{d}A(z) \, \mathrm{d}A(w) \\
&\leqslant C \int_{\mathbb{D}} \frac{|f'(w)|(1-|w|)}{|w|} \int_{\mathbb{D}} \frac{1}{|1-zw|^3} \log^{\alpha-1} \frac{2}{1-|z|} \, \mathrm{d}A(z) \, \mathrm{d}A(w) \\
&\leqslant C \int_{\mathbb{D}} \frac{|f'(w)|}{|w|} \int_{\mathbb{D}} \frac{1}{|1-zw|^2} \log^{\alpha-1} \frac{2}{1-|z|} \, \mathrm{d}A(z) \, \mathrm{d}A(w) \\
&\leqslant C \int_{\mathbb{D}} \frac{|f'(w)|}{|w|} \int_0^1 \int_0^{2\pi} \frac{\mathrm{d}\theta}{|1-r\mathrm{e}^{\mathrm{i}\theta}w|^2} \log^{\alpha-1} \frac{2}{1-r} \, \mathrm{d}r \, \mathrm{d}A(w) \\
&\leqslant C \int_{\mathbb{D}} \frac{|f'(w)|}{|w|} \int_0^1 \frac{1}{1-r|w|} \log^{\alpha-1} \frac{2}{1-r} \, \mathrm{d}r \, \mathrm{d}A(w).
\end{aligned}
$$

On the other hand, by using Lemma 2.2 and Theorem 2.3 (b), we have that

$$
\begin{aligned}
\int_0^1 \frac{1}{1-r|w|} \log^{\alpha-1} \frac{2}{1-r} \, \mathrm{d}r &= \sum_{n=0}^{\infty} |w|^n \int_0^1 r^n \log^{\alpha-1} \frac{2}{1-r} \, \mathrm{d}r \\
&\asymp \sum_{n=0}^{\infty} \frac{\log^{\alpha-1}(n+2)}{n+1} |w|^n \\
&\asymp \log^{\alpha} \frac{2}{1-|w|}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
S &\leqslant C \int_{\mathbb{D}} \frac{|f'(w)|}{|w|} \log^{\alpha} \frac{2}{1-|w|} \, \mathrm{d}A(w) \\
&\leqslant C \int_{\mathbb{D}} |f'(w)| \log^{\alpha} \frac{2}{1-|w|} \, \mathrm{d}A(w) \\
&< \infty
\end{aligned}
$$

and we get that $Hf \in \mathcal{B}^1_{\log^{\alpha-1}}$.

(b) We have $Hf(z) = \sum_{n=0}^{\infty} \widehat{Hf}(n)z^n$, where $\widehat{Hf}(n) = \sum_{k=0}^{\infty} \widehat{f}(k)/(n+k+1) \downarrow 0$ as $n \to \infty$. Then, by using Theorem 5.1, we find that

$$
\|Hf\|_{\mathcal{B}^1_{\log^{\alpha-1}}} \asymp \sum_{n=0}^{\infty} \widehat{Hf}(n) \frac{\log^{\alpha-1}(n+2)}{n+1},
$$

where the corresponding constant is independent of f. On the other hand, by using Lemma 2.4, we have that

$$\sum_{n=0}^{\infty} \widehat{Hf}(n)\frac{\log^{\alpha-1}(n+2)}{n+1} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\widehat{f}(k)}{n+k+1}\frac{\log^{\alpha-1}(n+2)}{n+1}$$

$$= \sum_{k=0}^{\infty} \widehat{f}(k) \sum_{n=0}^{\infty} \frac{\log^{\alpha-1}(n+2)}{(n+1)(n+k+1)}$$

$$\asymp \sum_{k=0}^{\infty} \widehat{f}(k)\frac{\log^{\alpha}(k+2)}{k+1}.$$

Therefore,

$$\|Hf\|_{\mathcal{B}^1_{\log^{\alpha-1}}} \asymp \sum_{n=0}^{\infty} \widehat{f}(n)\frac{\log^{\alpha}(n+2)}{n+1},$$

where the corresponding constant is independent of f. Then $\sum_{n=0}^{\infty} \widehat{f}(n)\log^{\alpha}(n+2) \times (n+1)^{-1} < \infty$ and by using Theorem 5.1 we find that $f \in \mathcal{B}^1_{\log^{\alpha}}$.

(c) Let $\varepsilon > 0$ and let $\widehat{f}(n) = (\log^{\alpha+1+\varepsilon/2}(n+2))^{-1}$ for all $n \geqslant 0$. Then $\sum_{n=0}^{\infty} \widehat{f}(n) \times (n+1)^{-1} < \infty$ and $\widehat{f}(n) \downarrow 0$ as $n \to \infty$. Also, we find that

$$\sum_{n=0}^{\infty} \widehat{f}(n)\frac{\log^{\alpha}(n+2)}{n+1} < \infty \qquad \text{and} \qquad \sum_{n=0}^{\infty} \widehat{f}(n)\frac{\log^{\alpha+\varepsilon}(n+2)}{n+1} = \infty.$$

Let $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n$. Then $f \in \mathcal{B}^1_{\log^{\alpha}}$ by Theorem 5.1 and $Hf \notin \mathcal{B}^1_{\log^{\alpha-1+\varepsilon}}$, because otherwise we would have $\sum_{n=0}^{\infty} \widehat{f}(n) \log^{\alpha+\varepsilon}(n+2)(n+1)^{-1} < \infty$ by part (b) of this theorem. A contradiction.

(d) Since $\mathcal{B}^1_{\log^{\alpha}} \subset \mathcal{B}^1_{\log^{\beta}}$ for $\beta < \alpha$, we may assume that $-1 < \alpha < 0$. Let

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\log(n+2)}.$$

For every $r \in (0,1)$ the function $f_r(z) = f(rz)$ belongs to $\mathcal{H}(\overline{\mathbb{D}})$ and by Theorem 5.1, the set $\{f_r \colon r \in (0,1)\}$ is bounded in $\mathcal{B}^1_{\log^{\alpha}}$. On the other hand,

$$Hf_r(0) = \sum_{k=0}^{\infty} \frac{r^k}{(k+1)\log(k+2)} \to \infty \quad \text{as } r \uparrow 1.$$

This contradicts the fact that if a set $X \subset \mathcal{B}^1_{\log^{\alpha}}$ is bounded and H is bounded on $\mathcal{B}^1_{\log^{\alpha}}$, then the set $\{Hf(0) \colon f \in X\}$ is bounded, because the functional $h \to h(0)$ is continuous on $\mathcal{H}(\mathbb{D})$. This completes the proof. $\qquad\square$

Remark 5.4. We note that the result stated in Theorem 5.2 (a) is sharp in the sense that for any $\varepsilon > 0$ and $\alpha > 0$ there exists $f \in \mathcal{B}^1_{\log^\alpha}$ such that $\mathcal{L}f \notin \mathcal{B}^1_{\log^{\alpha-1+\varepsilon}}$. As above, we have that

$$f(z) = \sum_{n=0}^\infty \frac{z^n}{\log^{\alpha+1+\varepsilon/2}(n+2)} \in \mathcal{B}^1_{\log^\alpha},$$

while $\mathcal{L}f \notin \mathcal{B}^1_{\log^{\alpha-1+\varepsilon}}$.

Corollary 5.5. Let $\alpha \geqslant 0$ and let $f(z) = \sum_{n=0}^\infty \widehat{f}(n)z^n$, where $\widehat{f}(n) \geqslant 0$ for all nonnegative integers n and $\sum_{n=0}^\infty \widehat{f}(n)/(n+1) < \infty$. Then

$$Hf \in \mathcal{B}^1_{\log^{\alpha-1}} \qquad \text{if and only if} \qquad \sum_{n=0}^\infty \widehat{f}(n)\frac{\log^\alpha(n+2)}{n+1} < \infty.$$

Moreover, there is a constant C independent of the function f, such that

$$\frac{1}{C}\|Hf\|_{\mathcal{B}^1_{\log^{\alpha-1}}} \leqslant \sum_{n=0}^\infty \widehat{f}(n)\frac{\log^\alpha(n+2)}{n+1} \leqslant C\|Hf\|_{\mathcal{B}^1_{\log^{\alpha-1}}}.$$

P r o o f. We have that $\widehat{Hf}(n) = \sum_{k=0}^\infty \widehat{f}(k)/(n+k+1) \downarrow 0$ as $n \to \infty$, because $\widehat{f}(k) \geqslant 0$ for all $k \geqslant 0$. Now the proof follows from the proof of part (b) of Theorem 5.3. $\qquad\square$

Corollary 5.6. Let $\alpha \geqslant 0$ and let $f(z) = \sum_{n=0}^\infty \widehat{f}(n)z^n \in \mathcal{H}(\mathbb{D})$, such that $\sum_{n=0}^\infty |\widehat{f}(n)|\log^\alpha(n+2)(n+1)^{-1} < \infty$. Then $Hf \in \mathcal{B}^1_{\log^{\alpha-1}}$.

P r o o f. Let $x_n = \operatorname{Re}\widehat{f}(n)$ and $y_n = \operatorname{Im}\widehat{f}(n)$ for all nonnegative integers n. Then, functions $g(z) = \sum_{n=0}^\infty x_n z^n$ and $h(z) = \sum_{n=0}^\infty y_n z^n$ are holomorphic in the unit disk $\mathbb{D}$. Now, let $x_n^+ = (|x_n| + x_n)/2$ and $x_n^- = (|x_n| - x_n)/2$ for all $n = 0, 1, \ldots$. Then, $x_n^\pm \geqslant 0$, $x_n^\pm \leqslant |x_n| \leqslant |\widehat{f}(n)|$ and $x_n^+ - x_n^- = x_n$. Therefore, functions $g^+(z) = \sum_{n=0}^\infty x_n^+ z^n$ and $g^-(z) = \sum_{n=0}^\infty x_n^- z^n$ are holomorphic in the unit disk $\mathbb{D}$, and

$$\sum_{n=0}^\infty x_n^\pm \frac{\log^\alpha(n+2)}{n+1} \leqslant \sum_{n=0}^\infty |\widehat{f}(n)|\frac{\log^\alpha(n+2)}{n+1} < \infty.$$

Hence, by using Corollary 5.5, we find that

$$Hg^+ \in \mathcal{B}^1_{\log^{\alpha-1}} \quad \text{and} \quad Hg^- \in \mathcal{B}^1_{\log^{\alpha-1}}.$$

Then, we have

$$\int_{\mathbb{D}} |(Hg)'(z)| \log^{\alpha-1} \frac{2}{1-|z|} \, \mathrm{d}A(z) = \int_{\mathbb{D}} |(Hg^+)'(z) - (Hg^-)(z)| \log^{\alpha-1} \frac{2}{1-|z|} \, \mathrm{d}A(z)$$
$$\leqslant \int_{\mathbb{D}} \left(|(Hg^+)'(z)| + |(Hg^-)(z)| \right) \log^{\alpha-1} \frac{2}{1-|z|} \, \mathrm{d}A(z) < \infty,$$

and we get $Hg \in \mathcal{B}^1_{\log^{\alpha-1}}$. In the same way, we prove that $Hh \in \mathcal{B}^1_{\log^{\alpha-1}}$. Then, we have that $Hf = Hg + \mathrm{i}Hh$, because of $f = g + \mathrm{i}h$. Consequently,

$$\int_{\mathbb{D}} |(Hf)'(z)| \log^{\alpha-1} \frac{2}{1-|z|} \, \mathrm{d}A(z) = \int_{\mathbb{D}} |(Hg)'(z) + \mathrm{i}(Hh)'(z)| \log^{\alpha-1} \frac{2}{1-|z|} \, \mathrm{d}A(z)$$
$$\leqslant \int_{\mathbb{D}} (|(Hg)'(z)| + |(Hh)'(z)|) \log^{\alpha-1} \frac{2}{1-|z|} \, \mathrm{d}A(z)$$
$$< \infty.$$

Therefore, $Hf \in \mathcal{B}^1_{\log^{\alpha-1}}$. $\qquad\qquad\square$

References

[1] *M. Jevtić, B. Karapetrović*: Hilbert matrix operator on Besov spaces. Publ. Math. *90* (2017), 359–371. `MR` `doi`

[2] *M. Jevtić, B. Karapetrović*: Libera operator on mixed norm spaces $H_\nu^{p,q,\alpha}$ when $0 < p < 1$. Filomat *31* (2017), 4641–4650. `MR`

[3] *M. Jevtić, D. Vukotić, M. Arsenović*: Taylor Coefficients and Coefficient Multipliers of Hardy and Bergman-Type Spaces. RSME Springer Series 2, Springer, Cham, 2016. `zbl` `MR` `doi`

[4] *B. Łanucha, M. Nowak, M. Pavlović*: Hilbert matrix operator on spaces of analytic functions. Ann. Acad. Sci. Fenn., Math. *37* (2012), 161–174. `zbl` `MR` `doi`

[5] *M. Mateljević, M. Pavlović*: L^p-behaviour of the integral means of analytic functions. Stud. Math. *77* (1984), 219–237. `zbl` `MR` `doi`

[6] *M. Pavlović*: Definition and properties of the libera operator on mixed norm spaces. The Scientific World Journal *2014* (2014), Article ID 590656, 15 pages. `doi`

[7] *M. Pavlović*: Function Classes on the Unit Disc. An introduction. De Gruyter Studies in Mathematics 52, De Gruyter, Berlin, 2014. `zbl` `MR` `doi`

[8] *M. Pavlović*: Logarithmic Bloch space and its predual. Publ. Inst. Math. (Beograd) (N.S.) *100(114)* (2016), 1–16. `zbl` `MR` `doi`

[9] *K. Zhu*: Operator Theory in Function Spaces. Monographs and Textbooks in Pure and Applied Mathematics 139, Marcel Dekker, New York, 1990. `zbl` `MR`

[10] *A. Zygmund*: Trigonometric Series. Vol. I, II. Cambridge University Press, Cambridge, 1959. `zbl` `MR`

Author's address: B o b a n K a r a p e t r o v i ć, Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Belgrade, Serbia, e-mail: `bkarapetrovic@matf.bg.ac.rs`.