Fuyuan Chen
Order of the smallest counterexample to Gallai's conjecture

# ORDER OF THE SMALLEST COUNTEREXAMPLE TO GALLAI'S CONJECTURE

Fuyuan Chen, Anhui

*Abstract.* In 1966, Gallai conjectured that all the longest paths of a connected graph have a common vertex. Zamfirescu conjectured that the smallest counterexample to Gallai's conjecture is a graph on 12 vertices. We prove that Gallai's conjecture is true for every connected graph $G$ with $\alpha'(G) \leqslant 5$, which implies that Zamfirescu's conjecture is true.

*Keywords*: longest path; matching number

*MSC 2010*: 05C38, 05C70, 05C75

## 1. Introduction

Graphs in this paper are simple (without loops or parallel edges), finite and undirected. Let $G$ be a graph with the vertex set $V(G)$ and edge set $E(G)$. Let $v$ be a vertex of $V(G)$. The *neighborhood* of $v$ in $G$, denoted by $N_G(v)$, is the set of vertices in $V(G)$ which are adjacent to $v$. The *degree* of $v$ in $G$, denoted by $d_G(v)$, equals to $|N_G(v)|$. A *matching* in a graph is a set of pairwise nonadjacent edges. A *maximum matching* is a matching with the largest number of edges. The *matching number* of $G$, denoted by $\alpha'(G)$, is the number of edges in the maximum matching of $G$.

The research on the intersection of longest paths in a graph has a long history. In particular, Gallai in [6] proposed the following conjecture in 1966.

**Conjecture 1.1** (Gallai [6]). If $G$ is a connected graph, then all the longest paths of $G$ have a common vertex.

Three years later, Walther in [9] disproved Gallai's conjecture by exhibiting a counterexample on 25 vertices. Up to now, the smallest counterexample to Gallai's

conjecture is a graph on 12 vertices (see Figure 1), which was found by Walther in [10] and Zamfirescu in [12], independently. One may find that this graph is somewhat interesting: for each vertex $v$ of it there is a longest path not containing $v$. Therefore, all the longest paths share no common vertex.
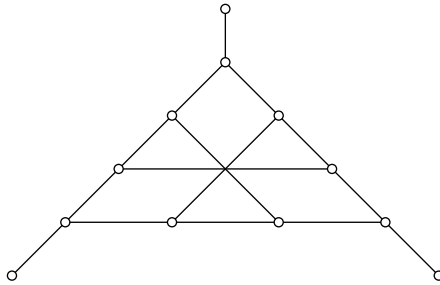


Figure 1. A counterexample to Gallai's conjecture on 12 vertices.

Although Gallai's conjecture has been disproved, finding classes of graphs that support this conjecture is also very meaningful. An obvious such example is the class of trees. In 1990, Klavžar and Petkovšek in [7] proved that Conjecture 1.1 holds on split graphs, and every connected graph such that each block is Hamiltonian-connected, almost Hamiltonian-connected or a cycle. As a corollary, Gallai's conjecture is true for the class of cacti. In 2004, Balister, Győri, Lehel, and Schelp in [1] showed that circular arc graphs support Conjecture 1.1. In 2013, Rezende, Fernandes, Martin and Wakabayashi in [5] proved that Conjecture 1.1 also holds on outer-planar graphs and 2-trees. In 2015, Chen in [3] proved that Gallai's conjecture is true for graphs with small matching number. In 2017, Chen et al. in [4] proved that Gallai's conjecture is true for all series-Parallel graphs ($K_4$-minor-free graphs).

In this paper, we prove the following statement:

**Theorem 1.1.** If $G$ is a connected graph with $\alpha'(G) \leqslant 5$, then all the longest paths of $G$ have a common vertex.

Theorem 1.1 implies that the following conjecture is true, which was verified by Brinkmann and Van Cleemput, see [2], by using computers.

**Conjecture 1.2** (Zamfirescu [11]). A smallest counterexample to Gallai's conjecture is a graph on 12 vertices.

## 2. Proof of Theorem 1.1

We prove by contradiction. Let $G$ be a counterexample. Since $G$ is connected, if $G$ has no cycle, then $G$ is a tree, and therefore all the longest paths of $G$ have a common vertex (a center vertex of $G$). So $G$ has a cycle. Let $C = v_1 v_2 \ldots v_r v_1$, $r \geqslant 3$ be a longest cycle of $G$, and $P = x_0 x_1 \ldots x_s$ be a longest path of $G$. We write $C[v_i, v_j]$ for the longer subpath of $C$ between $v_i$ and $v_j$ (if there are two different longest paths between $v_i$ and $v_j$ in $C$, we choose one for $C[v_i, v_j]$ arbitrarily), and $P[x_m, x_n]$ for the subpath of $P$ between $x_m$ and $x_n$, $1 \leqslant i, j \leqslant r$ and $0 \leqslant m, n \leqslant s$. Since $\alpha'(G) \leqslant 5$, we have that $r \leqslant 11$ and $s \leqslant 10$. If $C$ is a Hamilton cycle, then every longest path of $G$ is a Hamilton path, therefore all the longest paths of $G$ have a common vertex. Thus, $C$ is not a Hamilton cycle. Let $R = G - C$ and $u \in V(R)$.

**Claim 2.1.** $s \geqslant r$.

P r o o f.  Since $G$ is connected and $C$ is not a Hamilton cycle, there is a vertex $y \in V(R)$ such that $y v_i \in E(G)$, where $v_i \in V(C)$. Then $y v_i v_{i+1} \ldots v_{i-1}$ is a path of length $r$. Since $P$ is a longest path of $G$, $s \geqslant r$.  □

**Claim 2.2.** If there is a vertex $v \in V(G)$ such that $N_G(v) = \{v_1, v_2\}$, then every longest path of $G$ containing $v$ must also contain $v_1$ and $v_2$.

P r o o f.  Let $Q$ be a longest path of $G$ such that $v \in V(Q)$. If $v_1 \notin V(Q)$ or $v_2 \notin V(Q)$, then $v$ is an end-vertex of $Q$, since $N_G(v) = \{v_1, v_2\}$. But now $Q \cup v v_1$ or $Q \cup v v_2$ is a path longer than $Q$, a contradiction.  □

**Claim 2.3.** If there is a vertex $v \in V(G) \setminus V(P)$ such that $v x_i \in E(G)$, $1 \leqslant i \leqslant s - 1$, then $v x_{i-1}, v x_{i+1} \notin E(G)$.

P r o o f.  If $v x_{i-1} \in E(G)$ or $v x_{i+1} \in E(G)$, then $(P - x_i x_{i-1}) \cup x_{i-1} v x_i$ or $(P - x_i x_{i+1}) \cup x_{i+1} v x_i$ is a path longer than $P$, a contradiction.  □

**Claim 2.4.** If $Q_1 = y_0 y_1 \ldots y_s$ and $Q_2 = z_0 z_1 \ldots z_s$ are two longest paths of $G$, then $Q_1 \cap Q_2 \neq \emptyset$.

P r o o f.  If $Q_1 \cap Q_2 = \emptyset$, then since $G$ is connected, there is a path $W$ connecting $Q_1$ and $Q_2$. Suppose that $W$ connects $y_j \in Q_1$ and $z_l \in Q_2$, $1 \leqslant j, l \leqslant s - 1$, and $Q_1[y_0, y_j]$ is a longer part of $Q_1$, and $Q_2[z_0, z_l]$ is a longer part of $Q_2$. Now $Q_1[y_0, y_j] \cup W[y_j, z_l] \cup Q_2[z_l, z_0]$ is a path longer than $Q_1$, a contradiction.  □

By Claim 2.1 and $s \leqslant 10$ we have $r \leqslant 10$. Now we distinguish several cases in the following subsections.

**2.1. Proof of the case** $r = 3$**.** If $r = 3$, then by Claim 2.1 we have $s \geqslant 3$.

**Claim 2.5.** $|V(P) \cap V(C)| \geqslant 1$.

P r o o f. If $|V(P) \cap V(C)| = 0$, then since $G$ is connected, there is a path $W$ connecting $P$ and $C$. Suppose that $W$ connects $v_i$ and $x_j$. We assume that $s \geqslant 6$, for otherwise either $v_{i+1}v_{i-1}v_iWx_jPx_s$ or $v_{i+1}v_{i-1}v_iWx_jPx_0$ is a path longer than $P$, a contradiction. If $s \geqslant 9$, then $x_0x_1, x_2x_3, x_4x_5, x_6x_7, x_8x_9, v_1v_2$ are 6 independent edges, a contradiction. Thus $s \leqslant 8$. If $x_j \notin \{x_3, x_{s-3}\}$ or $v_ix_j \notin E(G)$, then $s = 8$ and $x_j = x_4$, for otherwise either $v_{i+1}v_{i-1}v_iWx_jPx_s$ or $v_{i+1}v_{i-1}v_iWx_jPx_0$ is a path longer than $P$, a contradiction. But now $x_0x_1, x_2x_3, x_5x_6, x_7x_8, v_{i-1}v_{i+1}$ and an edge in $x_4Wv_i$ are 6 independent edges, a contradiction. Thus $x_j \in \{x_3, x_{s-3}\}$ and $v_ix_j \in E(G)$.

Now we can check that for each $v \in \{v_{i-1}, v_{i+1}\}$, the followings hold:

(i) $v$ is not adjacent to any vertex in $V(P)$ (since $r = 3$);

(ii) $v$ is not adjacent to any vertex in $V(G) \setminus (V(P) \cup V(C))$ (since if there is a vertex $z \in V(G) \setminus (V(P) \cup V(C))$ such that $zv \in E(G)$, then $zv_{i+1}v_{i-1}v_ix_jPx_s$ or $zv_{i+1}v_{i-1}v_ix_jPx_0$ or $zv_{i-1}v_{i+1}v_ix_jPx_s$ or $zv_{i-1}v_{i+1}v_ix_jPx_0$ is a path longer than $P$, a contradiction);

(iii) $d_G(v) = 2$ (since (i) and (ii)).

By Claim 2.2, every longest path containing $v_{i+1}(v_{i-1})$ must also contain $v_i$. Now we prove that if a longest path $Q$ contains $v_i$, then $Q$ contains $x_j$. By Claim 2.4, $P \cap Q \neq \emptyset$. If $Q$ does not contain $x_j$, then there exists a vertex $x_t \neq x_j$ such that $v_iQx_t$ is a segment of $Q$ and $v_iQ \cap P = \emptyset$. By Claim 2.3, $x_t \notin \{x_{j-1}, x_{j+1}\}$. But now $v_iQx_tPx_jv_i$ is a cycle of length at least 4, a contradiction. Thus, every longest path of $G$ containing $v_i$ must also contain $x_j$. Therefore all the longest paths of $G$ contain $x_j$. Since $G$ is a counterexample, $|V(P) \cap V(C)| \geqslant 1$. $\square$

**Claim 2.6.** *For any longest path $Q$ of $G$, $|V(Q) \cap V(C)| = 1$ or $|V(Q) \cap V(C)| = 3$.*

P r o o f. By Claim 2.5, $|V(Q) \cap V(C)| \geqslant 1$. If $|V(Q) \cap V(C)| = 2$, then without loss of generality, suppose that $v_1, v_2 \in V(Q)$. Now $v_1v_2 \in E(Q)$, since if $v_1v_2 \notin E(Q)$, then $v_1v_3v_2Q[v_2, v_1]v_1$ is a cycle longer than $C$, a contradiction. But now $(Q - v_1v_2) \cup v_1v_3v_2$ is a path longer than $Q$, a contradiction. $\square$

**Claim 2.7.** *If there is a longest path $Q = y_0y_1 \ldots y_s$ of $G$ such that $|V(Q) \cap V(C)| = 1$, then all the longest paths of $G$ share a common vertex.*

P r o o f. Suppose that $V(Q) \cap V(C) = \{y_j\}$. Without loss of generality, suppose that $v_1 = y_j$. Let $Q_1$ be a longest path of $G$ containing $v_i$ ($i \in \{2, 3\}$). By Claim 2.4, $Q \cap Q_1 \neq \emptyset$. If $Q_1$ does not contain $y_j$, then there exists a vertex $y_t \neq y_j$ such that

344

$v_i Q_1 y_t$ is a segment of $Q_1$ and $v_i Q_1 \cap Q = \emptyset$. By Claim 2.3, $y_t \notin \{y_{j-1}, y_{j+1}\}$. But now $v_i Q_1 y_t Q y_j v_i$ is a cycle of length at least 4, a contradiction. Thus, every longest path of $G$ containing $v_2$ $(v_3)$ must also contain $v_1 = y_j$. Therefore all the longest paths of $G$ contain $y_j$. $\qquad \square$

Since $G$ is a counterexample, by Claims 2.6 and 2.7, for every longest path $Q$ of $G$, $V(C) \subset V(Q)$. But now all the longest paths of $G$ contain $V(C)$, a contradiction.

**2.2. Proof of the case** $r = 4$. If $r = 4$, then by Claim 2.1, $s \geqslant 4$.

**Claim 2.8.** *For any longest path $Q$ of $G$, $|V(Q) \cap V(C)| \geqslant 2$.*

P r o o f.  Let $Q = y_0 y_1 \ldots y_s$ be a longest path of $G$. If $|V(Q) \cap V(C)| = 0$, then $s \leqslant 6$, for otherwise $y_0 y_1, y_2 y_3, y_4 y_5, y_6 y_7$ and $v_1 v_2, v_3 v_4$ are 6 independent edges, a contradiction. But now we could find a path longer than $Q$, a contradiction.

If $|V(Q) \cap V(C)| = 1$, then $s \geqslant 6$, for otherwise we could find a path longer than $Q$, a contradiction. Suppose that $V(Q) \cap V(C) = \{y_j\}$. Without loss of generality, assume $y_j = v_1$. If $s \geqslant 9$, then $y_0 y_1, y_2 y_3, y_4 y_5, y_6 y_7, y_8 y_9, v_2 v_3$ are 6 independent edges, a contradiction. Thus $6 \leqslant s \leqslant 8$.

We can check that for each vertex $v \in \{v_2, v_3, v_4\}$ the following holds:

(i) $v$ is not adjacent to $y_{j-1}, y_{j+1}$ (since otherwise $(Q - v_1 y_{j-1}) \cup v_1 C[v_1, v] v y_{j-1}$ or $(Q - v_1 y_{j+1}) \cup v_1 C[v_1, v] v y_{j+1}$ is a path longer than $Q$);

(ii) $v$ is not adjacent to any vertex in $Q[y_0, y_{j-2}] \cup Q[y_{j+2}, y_s]$ (since $r = 4$).

Let $Q_1$ be a longest path of $G$ containing $v_i$, $i \in \{2, 3, 4\}$. By Claim 2.4, $Q \cap Q_1 \neq \emptyset$. If $Q_1$ does not contain $v_1$, then there exists a vertex $y_t \neq y_j (v_1)$ such that $v_i Q_1 y_t$ is a segment of $Q_1$ and $v_i Q_1 \cap Q = \emptyset$. By (i) and (ii) we could obtain a cycle of length at least 5, a contradiction.

Thus, every longest path of $G$ containing $v_i$, $i \in \{2, 3, 4\}$ must also contain $v_1$. Therefore all the longest paths of $G$ contain $v_1$. Since $G$ is a counterexample, $|V(Q) \cap V(C)| \geqslant 2$. $\qquad \square$

**Claim 2.9.** *For any longest path $Q$ of $G$, if $v_i \in V(C) \setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$.*

P r o o f.  If not, then there exists a longest path $Q_1$ and $v_i \in V(C) \setminus V(Q_1)$ such that $v_{i-1} \notin V(Q_1)$ or $v_{i+1} \notin V(Q_1)$. Without loss of generality, suppose that $v_{i-1} \notin V(Q_1)$. By Claim 2.8, $v_{i+1}, v_{i-2} \in V(Q_1)$. If $v_{i+1} v_{i-2} \in E(Q_1)$, then $(Q_1 - v_{i+1} v_{i-2}) \cup v_{i+1} v_i v_{i-1} v_{i-2}$ is a path longer than $Q_1$, a contradiction. Thus $v_{i+1} v_{i-2} \notin E(Q_1)$. But now $Q_1[v_{i+1}, v_{i-2}] v_{i-2} v_{i-1} v_i v_{i+1}$ is a cycle longer than $C$, a contradiction. $\qquad \square$

**Claim 2.10.** *If there is a vertex $v_i \in V(C)$ such that $d_G(v_i) = 2$, then all the longest paths of $G$ contain $v_{i-1}$ and $v_{i+1}$.*

P r o o f.  If not, then there is a longest path $Q_1$ such that $v_{i-1} \notin V(Q_1)$ or $v_{i+1} \notin V(Q_1)$. By Claim 2.9, $v_i \in V(Q_1)$. We assume that $v_i$ is not the end-vertex of $Q_1$, since otherwise adding $v_{i-1}$ or $v_{i+1}$ to $Q_1$ results in a longer path, a contradiction. But now $d_G(v_i) \geqslant 3$, a contradiction. $\square$

**Claim 2.11.** *If there is a longest path $Q = y_0 y_1 \ldots y_s$ of $G$ such that $|V(Q) \cap V(C)| \leqslant 3$, then for each vertex $v_i \in V(C) \setminus V(Q)$, $d_Q(v_i) = 2$, and all the longest paths of $G$ share a common vertex.*

P r o o f.  Without loss of generality, suppose that $v_1 \notin V(Q)$. By Claim 2.9, $v_2$, $v_4 \in V(Q)$. Since $r = 4$, $v_2 w_1 v_4$ is a subpath of $Q$ in $G$ ($w_1$ may be a vertex of $V(C)$). Now $v_2$ and $v_4$ are not end-vertices of $Q$, since otherwise adding $v_1$ to $Q$ results in a longer path, a contradiction. Suppose $v_2 = y_k, v_4 = y_j, 1 \leqslant k < j \leqslant s - 1$.

We can check that for each vertex $v \in \{v_1, w_1\}$ the following assertions hold:

(i) $v_1$ is not adjacent to $w_1$ (by Claim 2.3);

(ii) $v$ is not adjacent to any vertex in $Q[y_0, y_{k-1}] \cup Q[y_{j+1}, y_s]$ (since $r = 4$);

(iii) $d_Q(v) = 2$ (since (i) and (ii)).

If $d_G(v) = 2$ ($v \in \{v_1, w_1\}$), then by Claim 2.10, all the longest paths of $G$ contain $v_2$ and $v_4$. Since $G$ is a counterexample, $d_G(v) \geqslant 3$. Thus, there is a vertex $v' \in V(G) \setminus V(Q)$ such that $vv' \in E(G)$. If there is a vertex $v'' \in V(G) \setminus V(Q)$ such that $v'v'' \in E(G)$, then both $Q[y_0, v_2]$ and $Q[v_4, y_s]$ have lengths at least 3, for otherwise either $v''v'v_1v_2Qy_s$ ($v''v'w_1v_2v_1v_4Qy_s$) or $v''v'v_1v_4Qy_0$ ($v''v'w_1v_4v_1v_2Qy_0$) is a path longer than $Q$, a contradiction. But now $y_0y_1$, $v_2y_{k-1}$, $w_1w_1'$, $v_4y_{j+1}$, $y_{s-1}y_s$, $v_1'v_1''$ are 6 independent edges, a contradiction. Thus $N_G(v') = N_Q(v') \cup \{v\}$.

Since $N_Q(v) \subseteq \{v_2, v_4\}$, $N_Q(v') \subseteq \{v_2, v_4\} \cup \{v \in Q\}$. By Claim 2.3, $v'v_2, v'v_4 \notin E(G)$. Thus $d_G(v') = 1$.

Now if there is a longest path $Q_1$ not containing $v_2$, then by Claim 2.9, $v_1, w_1 \in V(Q_1)$. Since $d_Q(v_1) = d_Q(w_1) = 2$, there are two vertices $u_1, u_2 \in V(G) \setminus V(Q)$ such that $u_1v_1v_4$ and $u_2w_1v_4$ are two subpaths of $Q_1$. Since $d_G(u_1) = d_G(u_2) = 1$, $Q_1 = u_1v_1v_4w_1u_2$. But now $u_1v_1v_2w_1v_4Qy_s$ is a path longer than $Q_1$, a contradiction. Thus, all the longest paths of $G$ contain $v_2$. $\square$

Since $G$ is a counterexample, by Claim 2.11, for every longest path $Q$ of $G$, $V(C) \subset V(Q)$. But now all the longest paths of $G$ contain $V(C)$, a contradiction.

**2.3. Proof of the case $r = 5$.** If $r = 5$, then by Claim 2.1 we have that $s \geqslant 5$.

**Claim 2.12.** *For any longest path $Q$ of $G$, $|V(Q) \cap V(C)| \geqslant 3$.*

P r o o f.  Let $Q = y_0 y_1 \ldots y_s$ be a longest path of $G$. If $|V(Q) \cap V(C)| \leqslant 1$, then $s \leqslant 6$, for otherwise $y_0 y_1$, $y_2 y_3$, $y_4 y_5$, $y_6 y_7$ and two independent edges in $C \setminus V(Q)$ are 6 independent edges, a contradiction. But now, we could find a path longer than $Q$, a contradiction. If $|V(Q) \cap V(C)| = 2$, then at least two vertices of $V(C) \setminus V(Q)$ are consecutive in $C$. Suppose $v_i W v_j$ is a maximum consecutive segment in $C \setminus V(Q)$, and $N_{C \cap Q}(v_i) = \{\overline{v}_i\}$, $N_{C \cap Q}(v_j) = \{\overline{v}_j\}$. Since $Q$ is a longest path of $G$, the length of $Q[\overline{v}_i, \overline{v}_j]$ is at least 3. But now $v_i W v_j \overline{v}_j Q[\overline{v}_j, \overline{v}_i] \overline{v}_i v_i$ is a cycle of length at least 6, a contradiction. Thus $|V(Q) \cap V(C)| \geqslant 3$. $\qquad \square$

**Claim 2.13.** *For any longest path $Q$ of $G$, if $v_i \in V(C) \setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$.*

P r o o f.  We could obtain this result by the proof of Claim 2.12. $\qquad \square$

**Claim 2.14.** *If there is a vertex $v_i \in V(C)$ such that $d_G(v_i) = 2$, then all the longest paths of $G$ contain $v_{i-1}$ and $v_{i+1}$.*

P r o o f.  Similar to the proof of Claim 2.10, we could obtain this result. $\qquad \square$

**Claim 2.15.** *If there is a longest path $Q = y_0 y_1 \ldots y_s$ of $G$ such that $|V(Q) \cap V(C)| \leqslant 4$, then for each vertex $v_i \in V(C) \setminus V(Q)$, $d_Q(v_i) = 2$, and all the longest paths of $G$ share a common vertex.*

P r o o f.  Without loss of generality, suppose that $v_1 \notin V(Q)$. By Claim 2.13, $v_2, v_5 \in V(Q)$. Since $r = 5$, $v_2 w_1 v_5$ or $v_2 w_1 w_2 v_5$ is a subpath of $Q$ ($w_1, w_2$ may be vertices of $V(C)$). Now $v_2$ and $v_5$ are not end-vertices of $Q$, since otherwise adding $v_1$ to $Q$ results in a longer path, a contradiction. Suppose that $v_2 = y_k$, $v_5 = y_j$, $1 \leqslant k < j \leqslant s - 1$.

We can check that for the vertex $v_1 \in V(C) \setminus V(Q)$, the following assertions hold:

(i) $v_1$ is not adjacent to $w_1$, $y_{k-1}$, $y_{j+1}$ (if $v_2 w_1 v_5$ is a segment of $Q$) and $w_1$, $w_2$, $y_{k-1}$, $y_{j+1}$ (if $v_2 w_1 w_2 v_5$ is a segment of $Q$) (by Claim 2.3);

(ii) $v_1$ is not adjacent to any vertex in $Q[y_0, y_{k-2}] \cup Q[y_{j+2}, y_s]$ (since $r = 5$);

(iii) $d_Q(v_1) = 2$ (since (i) and (ii)).

If $d_G(v_1) = 2$, then by Claim 2.14, all the longest paths of $G$ contain $v_2$ and $v_5$. If $d_G(v_1) \geqslant 3$, then there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1 v_1' \in E(G)$. If there exists a vertex $v_1'' \in V(G) \setminus V(Q)$ such that $v_1' v_1'' \in E(G)$, then $s = 8$ and $Q = y_0 y_1 y_2 v_2 w_1 v_5 y_6 y_7 y_8$, for otherwise the 5 independent edges in $Q$ together with $v_1 v_1'$ are 6 independent edges, a contradiction.

We can check that for the vertex $w_1$, the following assertions hold:

(i) $w_1$ is not adjacent to $\{v_1', v_1''\}$ (by Claim 2.3);

(ii) $w_1$ is not adjacent to any vertex in $Q[y_0, y_{k-1}] \cup Q[y_{j+1}, y_8]$ (since if there exists a vertex $z \in Q[y_0, y_{k-1}] \cup Q[y_{j+1}, y_8]$ such that $w_1 z \in E(G)$, then $y_8 Q w_1 z Q v_2 v_1 v_1' v_1''$ or $y_0 Q w_1 z Q v_5 v_1 v_1' v_1''$ is a path of length at least 9, a contradiction);

(iii) $w_1$ is not adjacent to any vertex in $V(G) \setminus V(Q)$ (since if there exists a vertex $z \in V(G) \setminus V(Q)$ such that $z w_1 \in E(G)$, then $y_0 y_1$, $y_2 v_2$, $w_1 z$, $v_5 y_6$, $y_7 y_8$, $v_1' v_1''$ are 6 independent edges);

(iv) $d_G(w_1) = 2$ (since (i), (ii) and (iii)).

If there is a longest path $Q_1$ not containing $v_2$, then by Claim 2.13, $v_1 \in V(Q_1)$. By Claim 2.2, $w_1 \notin V(Q_1)$. If $v_5 \in V(Q_1)$, then $C_1 = v_1 Q_1[v_1, v_5] v_5 w_1 v_2 v_1$ is a cycle of length at least 6, a contradiction. Thus $v_5 \notin V(Q_1)$. By Claim 2.13, $v_4 \in V(Q_1)$. But now $C_2 = v_1 Q_1[v_1, v_4] v_4 v_5 w_1 v_2 v_1$ is a cycle of length at least 8, a contradiction. Thus, all the longest paths of $G$ contain $v_2$.

Since $G$ is a counterexample, for each vertex $w \in V(G) \setminus V(Q)$ such that $v_1 w \in E(G)$ we must have $N_G(w) \cap (V(G) \setminus V(Q)) = \{v_1\}$.

Now we prove that all the longest paths of $G$ contain $v_2$. If there exists a longest path $Q_2$ not containing $v_2$, then by Claim 2.13, $v_1 \in V(Q_2)$. If $w_1 \notin V(Q_2)$, then there exists a vertex $u \in Q[w_1, v_5]$ such that $V(Q[w_1, u]) \cap V(Q_2) = \{u\}$, for otherwise by Claim 2.13, $v_4 \in V(Q_2)$ and $C_3 = v_1 Q_2[v_1, v_4] v_4 v_5 Q w_1 v_2 v_1$ is a cycle of length at least 8, a contradiction. But now $C_4 = v_1 Q_2[v_1, u] u Q w_1 v_2 v_1$ is a cycle of length at least 6, a contradiction. Thus $w_1 \in V(Q_2)$. By the above, $y_{k-1} \notin V(Q_2)$ and $N_G(y_{k-1}) \cap (V(G) \setminus V(Q_2)) = \{v_2\}$. Thus $y_{k-2} \in V(Q_2)$. But now $C_5 = v_1 Q_2[v_1, y_{k-2}] y_{k-2} y_{k-1} v_2 v_1$ is a cycle of length at least 6, a contradiction. $\square$

Since $G$ is a counterexample, by Claim 2.15, for every longest path $Q$ of $G$, $V(C) \subset V(Q)$. But now all the longest paths of $G$ contain $V(C)$, a contradiction.

**2.4. Proof of the case $r = 6$.** If $r = 6$, then by Claim 2.1 we have that $s \geqslant 6$.

**Claim 2.16.** *For any longest path $Q$ of $G$, $|V(Q) \cap V(C)| \geqslant 3$.*

P r o o f.    Let $Q = y_0 y_1 \ldots y_s$ be a longest path of $G$. Similar to the proof of Claim 2.12, we could obtain that $|V(Q) \cap V(C)| \geqslant 2$.

If $|V(Q) \cap V(C)| = 2$, then at least two vertices of $V(C) \setminus V(Q)$ are consecutive in $C$. Suppose $v_i W v_j$ is a maximum consecutive segment in $C \setminus V(Q)$, and $N_{C \cap Q}(v_i) = \{\overline{v}_i\}$, $N_{C \cap Q}(v_j) = \{\overline{v}_j\}$. If the length of $W[v_i, v_j]$ is at least 2, then as $Q$ is a longest path of $G$, the length of $Q[\overline{v}_i, \overline{v}_j]$ is at least 4. But now $v_i W v_j \overline{v}_j Q[\overline{v}_j, \overline{v}_i] \overline{v}_i v_i$ is a cycle of length at least 8, a contradiction. If the length of $W[v_i, v_j]$ is at most 1, then as $r = 6$, there are two independent edges in $C \setminus V(Q)$. We assume that $s \leqslant 6$,

since otherwise $y_0y_1$, $y_2y_3$, $y_4y_5$, $y_6y_7$ and two independent edges in $C \setminus V(Q)$ are 6 independent edges, a contradiction. But now, we could find a path longer than $Q$, a contradiction. □

**Claim 2.17.** *For any longest path $Q$ of $G$, if $v_i \in V(C) \setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$.*

P r o o f. If not, then there exists a longest path $Q_1$ and $v_i \in V(C) \setminus V(Q_1)$ such that $v_{i-1} \notin V(Q_1)$ or $v_{i+1} \notin V(Q_1)$. Without loss of generality, suppose that $v_{i-1} \notin V(Q_1)$. By the proof of Claim 2.16, $v_{i+1}, v_{i-2} \in V(Q_1)$. Now $v_{i+1}$, $v_{i-2}$ are not end-vertices of $Q_1$, since otherwise adding $v_i$, $v_{i-1}$ to $Q_1$ results in a longer path, a contradiction. Suppose that $v_{i+1} = y_k$, $v_{i-2} = y_j$, $k < j$. Since $Q_1$ is a longest path of $G$ and $r = 6$, $v_{i+1}w_1w_2v_{i-2}$ is a segment of $Q_1$ ($w_1$, $w_2$ may be vertices of $V(C)$). If $s \geqslant 9$, then $y_0y_1$, $y_2y_3$, $y_4y_5$, $y_6y_7$, $y_8y_9$, $v_iv_{i-1}$ are 6 independent edges, a contradiction. Thus $s \leqslant 8$.

We can check that for each vertex $v \in \{v_i, v_{i-1}\}$, the following assertions hold:

(i) $v$ is not adjacent to $w_1$, $w_2$, $y_{k-1}$, $y_{j+1}$ (by Claim 2.3);

(ii) $v$ is not adjacent to any vertex in $Q_1[y_0, y_{k-2}] \cup Q_1[y_{j+2}, y_s]$ (since $r = 6$);

(iii) $N_{Q_1}(v) \subseteq \{v_{i+1}, v_{i-2}\}$ (since (i) and (ii)).

If there is a vertex $v_i' \in V(G) \setminus V(Q_1)$ such that $v_iv_i' \in E(G)$, then as $s \leqslant 8$, $Q_1 = y_0y_1v_{i+1}w_1w_2v_{i-2}y_6y_7y_8$. We assume that $N_G(v_i') = N_{Q_1}(v_i') \cup \{v_i\}$. Since if there is a vertex $v_i'' \in V(G) \setminus V(Q_1)$ such that $v_i'v_i'' \in E(G)$, then $y_0y_1v_{i+1}w_1w_2v_{i-2}v_{i-1}v_iv_i'v_i''$ is a path of length 9, and if $v_i'v_{i-1} \in E(G)$, then $y_0y_1v_{i+1}v_iv_i'v_{i-1}v_{i-2}y_6y_7y_8$ is a path of length 9, a contradiction. Furthermore, we assume that $N_G(v_i') \subseteq \{v_i, v_{i-2}\}$. Since by (iii), we could obtain that $N_{Q_1}(v_i') \subseteq \{v_{i+1}, v_{i-2}\}$, and if $v_i'v_{i+1} \in E(G)$, then $y_0y_1v_{i+1}v_i'v_iv_{i-1}v_{i-2}y_6y_7y_8$ is a path of length 9, a contradiction. Now $v_{i-1}v_{i+1} \notin E(G)$ and there is no vertex $v_{i-1}' \in V(G) \setminus V(Q_1)$ such that $v_{i-1}v_{i-1}' \in E(G)$. Since otherwise $v_i'v_iv_{i-1}v_{i+1}Qy_8$ or $v_{i-1}'v_{i-1}v_iv_{i+1}Qy_8$ is a path of length 9, a contradiction. Thus $d_G(v_{i-1}) = 2$.

Now if there is a longest path $Q_2$ not containing $v_{i-2}$, then by Claim 2.2, $v_{i-1} \notin V(Q_2)$. By the proof of Claim 2.16, $v_i, v_{i-3} \in V(Q_2)$. We can check that $N_{Q_2}(v_{i-2}) \subseteq \{v_i, v_{i-3}\}$. Now $w_2 \notin V(Q_2)$ or $y_6 \notin V(Q_2)$. Without loss of generality, suppose that $w_2 \notin V(Q_2)$. As above, we could prove that $N_G(w_2) \subseteq \{v_{i-2}, v_i\}$. By (i), $v_iw_2 \notin E(G)$. Now $d_G(w_2) = 1$, a contradiction to that $Q_1 = y_0y_1v_{i+1}w_1w_2v_{i-2}y_6y_7y_8$. Therefore all the longest paths of $G$ contain $v_{i-2}$.

Since $G$ is a counterexample, $N_G(v_i) \subseteq \{v_{i+1}, v_{i-2}, v_{i-1}\}$ and $N_G(v_{i-1}) \subseteq \{v_{i+1}, v_{i-2}, v_i\}$. If $v_iv_{i-2} \in E(G)$, then all the longest paths of $G$ contain $v_{i-2}$. Since if there is a longest path $Q_3$ not containing $v_{i-2}$, then as $d_G(v_{i-2}) \geqslant 4$, $v_{i-1} \in V(Q_3)$. Now $v_{i-1}$, $v_i$ (if they exist) are not end-vertices of $Q_3$, otherwise

adding $v_{i-2}$ to $Q_3$ results in a longer path, a contradiction. But now $v_{i+1}v_{i-1}v_iv_{i+1}$ is a segment of $Q_3$, a contradiction. Since $G$ is a counterexample, $d_G(v_i) = 2$. Similarly, $d_G(v_{i-1}) = 2$. Now all the longest paths of $G$ contain $v_{i-2}$. Since if there is a longest path $Q_4$ not containing $v_{i-2}$, then by Claim 2.2, $v_i, v_{i-1} \notin V(Q_4)$. But now by the proof of Claim 2.16, we could obtain that $Q_4$ is a path of length at least 10, a contradiction. Since $G$ is a counterexample, for any longest path $Q$ of $G$, if $v_i \in V(C) \setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$. □

**Claim 2.18.** *If there is a vertex $v_i \in V(C)$ such that $d_G(v_i) = 2$, then all the longest paths of $G$ contain $v_{i-1}$ and $v_{i+1}$.*

P r o o f. Similar to the proof of Claim 2.10, we could obtain this result. □

**Claim 2.19.** *If there is a longest path $Q = y_0y_1 \ldots y_s$ of $G$ such that $|V(Q) \cap V(C)| \leqslant 5$, then for each vertex $v \in V(C) \setminus V(Q)$, $d_Q(v) = 2$.*

P r o o f. Without loss of generality, suppose that $v_1 \in V(C) \setminus V(Q)$. By Claim 2.17, $v_2, v_6 \in V(Q)$. Since $r = 6$, $v_2w_1v_6$ or $v_2w_1w_2v_6$ or $v_2w_1w_2w_3v_6$ is a subpath of $Q$ in $G$ ($w_1, w_2, w_3$ may be vertices of $V(C)$). Now $v_2$ and $v_6$ are not end-vertices of $Q$, since otherwise adding $v_1$ to $Q$ results in a longer path, a contradiction. Suppose $v_2 = y_k$, $v_6 = y_j$, $1 \leqslant k < j \leqslant s - 1$.

**Case 2.4.1.** $v_2w_1v_6$ **is a subpath of** $Q$. In this case, we can check that for the vertex $v_1 \in V(C) \setminus V(Q)$, the following assertions hold:
 (i) $v_1$ is not adjacent to $w_1, y_{k-1}, y_{j+1}$ (by Claim 2.3);
 (ii) $v_1$ is not adjacent to any vertex in $Q[y_0, y_{k-3}] \cup Q[y_{j+3}, y_s]$ (since $r = 6$);
 (iii) $N_Q(v_1) \subseteq \{v_2, v_6, y_{k-2}, y_{j+2}\}$ (since (i) and (ii)).

**Claim 2.20.** $v_1y_{j+2} \notin E(G)$.

P r o o f. If $v_1y_{j+2} \in E(G)$, then since $r = 6$, $v_1y_{k-2} \notin E(G)$. Now we can check that for each vertex $v \in \{w_1, y_{j+1}\}$, the following assertions hold:
 (i) $v$ is not adjacent to any vertex in $Q[y_0, y_{k-1}]$ (since if there exists a vertex $z \in Q[y_0, y_{k-1}]$ such that $vz \in E(G)$, then $zQv_2v_1y_{j+2}Qw_1z$ or $zQy_jv_1y_{j+2}y_{j+1}z$ is a cycle of length at least 7);
 (ii) $v$ is not adjacent to any vertex in $Q[y_{j+3}, y_s]$ (since if there exists a vertex $z \in Q[y_{j+3}, y_s]$ such that $vz \in E(G)$, then $zQy_jv_1v_2w_1z$ or $zQy_{j+2}v_1v_2Qy_{j+1}z$ is a cycle of length at least 7);
 (iii) $w_1$ is not adjacent to $y_{j+1}$ (since otherwise $y_0Qw_1y_{j+1}y_jv_1y_{j+2}Qy_s$ is a path longer than $Q$);
 (iv) $N_Q(v) \subseteq \{v_2, v_6, y_{j+2}\}$ (since (i), (ii) and (iii)).

If there exists a vertex $w_1' \in V(G) \setminus V(Q)$ such that $w_1 w_1' \in E(G)$, then $w_1' w_1 v_2 v_1 v_6 Q y_s$ is a path of length at least 7 and there is no vertex $w_1'' \in V(G) \setminus V(Q)$ such that $w_1' w_1'' \in E(G)$. Since if there exists such vertex, then $Q[y_0, v_6]$ has length at least 5, and $y_0 y_1 Q v_2 v_1 y_{j+2} y_{j+1} v_6 w_1 w_1' w_1''$ is a path of length at least 9, and we could find 6 independent edges, a contradiction. Thus $N_G(w_1') = N_Q(w_1')$. Since $N_Q(w_1) \subseteq \{v_2, v_6, y_{j+2}\}$, $N_Q(w_1') \subseteq \{w_1, v_2, v_6, y_{j+2}\}$. By Claim 2.3, $w_1' v_2, w_1' v_6 \notin E(G)$. If $w_1' y_{j+2} \in E(G)$, then $w_1' w_1 v_2 v_1 v_6 y_{j+1} y_{j+2} w_1'$ is a cycle of length 7, a contradiction. Thus $d_G(w_1') = 1$. Similarly, we could obtain that for any vertex $y_{j+1}' \in V(G) \setminus V(Q)$ such that $y_{j+1} y_{j+1}' \in E(G)$, $d_G(y_{j+1}') = 1$.

If there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1' v_1 \in E(G)$, then $s \geqslant 8$, for otherwise either $v_1' v_1 v_2 Q y_s$ or $v_1' v_1 y_{j+2} Q y_0$ is a path longer than $Q$, a contradiction. Furthermore $s = 8$, for otherwise $y_0 y_1$, $y_2 y_3$, $y_4 y_5$, $y_6 y_7$, $y_8 y_9$, $v_1 v_1'$ are 6 independent edges, a contradiction. We assume that $N_G(v_1') = N_Q(v_1') \cup \{v_1\}$. Since if there exists a vertex $v_1'' \in V(G) \setminus V(Q)$ such that $v_1' v_1'' \in E(G)$, then $v_1'' v_1' v_1 v_2 Q y_s$ is a path of length at least 10, a contradiction. As $N_Q(v_1) = \{v_2, v_6, y_{j+2}\}$, $N_Q(v_1') \subseteq \{v_2, v_6, y_{j+2}\}$. By Claim 2.3, $v_1' v_2, v_1' v_6, v_1' y_{j+2} \notin E(G)$. Thus $d_G(v_1') = 1$.

If there is a longest path $Q_1$ not containing $v_6$, then by Claim 2.17, $v_1, w_1, y_{j+1} \in V(Q_1)$. Now $v_1, w_1, y_{j+1}$ are not end-vertices of $Q_1$, otherwise adding $v_6$ to $Q_1$ results in a longer path, a contradiction. If $v_2 v_1 y_{j+2}$ is a subpath of $Q_1$, then there are two vertices $u_1, u_2 \in V(G) \setminus V(Q)$ such that $u_1 w_1 v_2$, $u_2 y_{j+1} y_{j+2}$ or $u_1 y_{j+1} v_2$, $u_2 w_1 y_{j+2}$ are two subpaths of $Q_1$, since otherwise $v_2 v_1 y_{j+2} w_1 v_2$ or $v_2 v_1 y_{j+2} y_{j+1} v_2$ is a subpath of $Q_1$, a contradiction. But now $Q_1 = u_1 w_1 v_2 v_1 y_{j+2} y_{j+1} u_2$ or $Q_1 = u_1 y_{j+1} v_2 v_1 y_{j+2} w_1 u_2$ is a path of length 6, a contradiction. Since $N_Q(v_1) = \{v_2, v_6, y_{j+2}\}$, there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1' v_1 v_2$ or $v_1' v_1 y_{j+2}$ is a segment of $Q_1$. Without loss of generality, suppose that $v_1' v_1 v_2$ is a segment of $Q_1$. Now there is no vertex $u_1 \in V(G) \setminus V(Q)$ such that $u_1 w_1 v_2$ or $u_1 y_{j+1} v_2$ is a subpath of $Q_1$, for otherwise $Q_1 = v_1' v_1 v_2 w_1 u_1$ or $Q_1 = v_1' v_1 v_2 y_{j+1} u_1$, a contradiction. If there is a vertex $u_1 \in V(G) \setminus V(Q)$ such that $u_1 w_1 y_{j+2}$ is a subpath of $Q_1$, then $v_2 y_{j+1} y_{j+2}$ is a subpath of $Q_1$, for otherwise $Q_1 = u_1 w_1 y_{j+2} y_{j+1} u_2$ ($u_2 \in V(G) \setminus V(Q)$), a contradiction. But now $Q_1 = v_1' v_1 v_2 y_{j+1} y_{j+2} w_1 u_1$, a contradiction. Since $N_Q(w_1) \subseteq \{v_2, v_6, y_{j+2}\}$, $v_2 w_1 y_{j+2}$ is a subpath of $Q_1$. Similarly, we could prove that $v_2 y_{j+1} y_{j+2}$ is a subpath of $Q_1$. But now $v_2 w_1 y_{j+2} y_{j+1} v_2$ is a subpath of $Q_1$, a contradiction. Thus all the longest paths of $G$ contain $v_6$. Since $G$ is a counterexample, $v_1 y_{j+2} \notin E(G)$. $\qquad \square$

Similarly, we could obtain that $v_1 y_{k-2} \notin E(G)$. Therefore $d_Q(v_1) = 2$.

**Case 2.4.2.** $v_2 w_1 w_2 v_6$ **is a subpath of** $Q$**.** In this case, we can check that for the vertex $v_1 \in V(C) \setminus V(Q)$, the following assertions hold:

(i) $v$ is not adjacent to any vertex in $\{w_1, w_2, y_{k-1}, y_{j+1}\}$ (by Claim 2.3);

(ii) $v$ is not adjacent to any vertex in $Q[y_0, y_{k-2}] \cup Q[y_{j+2}, y_s]$ (since $r = 6$);

(iii) $d_Q(v) = 2$ (since (i) and (ii)).

**Case 2.4.3.** $v_2 w_1 w_2 w_3 v_6$ **is a subpath of** $Q$. In this case, similar to the proof of Case 2.4.1 we could obtain that $d_Q(v_1) = 2$. □

**Claim 2.21.** *If there is a longest path* $Q = y_0 y_1 \ldots y_s$ *of* $G$ *such that* $|V(Q) \cap V(C)| \leqslant 5$, *then all the longest paths of* $G$ *have a common vertex.*

P r o o f.     Without loss of generality, suppose that $v_1 \in V(C) \setminus V(Q)$. By Claim 2.19, $d_Q(v_1) = 2$. If $d_G(v_1) = 2$, then by Claim 2.18, all the longest paths of $G$ share a common vertex.

If $d_G(v_1) \geqslant 3$, then similar to the proof of Claim 2.15 in the third, forth, fifth paragraphs, we could obtain that for each vertex $w \in V(G) \setminus V(Q)$ such that $v_1 w \in E(G)$ we must have $N_G(w) \cap (V(G) \setminus V(Q)) = \{v_1\}$.

Now we prove that all the longest paths of $G$ contain $v_2$. If there exists a longest path $Q_2$ not containing $v_2$, then by Claim 2.17, $v_1 \in V(Q_2)$. If $w_1 \notin V(Q_2)$, then there exists a vertex $u \in Q[w_1, v_6]$ such that $u \in V(Q_2)$, for otherwise by Claim 2.17, $v_5 \in V(Q_2)$ and $C_3 = v_1 Q_2[v_1, v_5] v_5 v_6 Q w_1 v_2 v_1$ is a cycle of length at least 8, a contradiction. But now as $r = 6$, $C_4 = v_1 Q_2[v_1, u] u w_1 v_2 v_1$ is a cycle of length 6. Now for $C_4$, $v_2, w_1 \in V(C_4) \setminus V(Q_2)$, a contradiction to Claim 2.17. Thus $w_1 \in V(Q_2)$. By Claim 2.19, $y_{k-1} \notin V(Q_2)$. By the above, $N_G(y_{k-1}) \cap (V(G) \setminus V(Q_2)) = \{v_2\}$. Thus $y_{k-2} \in V(Q_2)$. But now $C_5 = v_1 Q_2[v_1, y_{k-2}] y_{k-2} y_{k-1} v_2 v_1$ is a cycle of length 6, a contradiction to Claim 2.17. □

Since $G$ is a counterexample, by Claim 2.21, for every longest path $Q$ of $G$, $V(C) \subset V(Q)$. But now all the longest paths of $G$ contain $V(C)$, a contradiction.

**2.5. Proof of the case** $r = 7$. If $r = 7$, then by Claim 2.1 we have that $s \geqslant 7$.

**Claim 2.22.** *For any longest path* $Q$ *of* $G$, $|V(Q) \cap V(C)| \geqslant 3$.

P r o o f.     Let $Q = y_0 y_1 \ldots y_s$ be a longest path of $G$. If $|V(Q) \cap V(C)| \leqslant 2$, then $y_0 y_1$, $y_2 y_3$, $y_4 y_5$, $y_6 y_7$ and two independent edges in $C \setminus V(Q)$ are 6 independent edges, a contradiction. □

**Claim 2.23.** *For any longest path* $Q$ *of* $G$, *if* $v_i \in V(C) \setminus V(Q)$, *then* $v_{i-1}, v_{i+1} \in V(Q)$.

P r o o f.     If not, then there exists a longest path $Q_1$ and $v_i \in V(C) \setminus V(Q_1)$ such that $v_{i-1} \notin V(Q_1)$ or $v_{i+1} \notin V(Q_1)$. Without loss of generality, suppose that $v_{i-1} \notin V(Q_1)$. Now similar to the proof of Claim 2.17 in the first paragraph, we

352

could obtain that $v_{i+1}w_1w_2v_{i-2}$ or $v_{i+1}w_1w_2w_3v_{i-2}$ is a segment of $Q_1$ ($w_1$, $w_2$, $w_3$ may be vertices of $V(C)$) and $7 \leqslant s \leqslant 8$.

We can check that for each vertex $v \in \{v_i, v_{i-1}\}$ the following assertions hold:

(i) $v$ is not adjacent to $w_1, w_2, y_{k-1}, y_{j+1}$ (if $v_{i+1}w_1w_2v_{i-2}$ is a segment of $Q_1$) or $w_1, w_2, w_3, y_{k-1}, y_{j+1}$ (if $v_{i+1}w_1w_2w_3v_{i-2}$ is a segment of $Q_1$) (by Claim 2.3);

(ii) $v$ is not adjacent to any vertex in $Q_1[y_0, y_{k-3}] \cup Q_1[y_{j+3}, y_s]$ (since $r = 7$);

(iii) $v$ is not adjacent to any vertex in $\{y_{k-2}, y_{j+2}\}$ (since otherwise $v_i v_{i-1} y_{k-2} Q_1 y_s$ or $v_{i-1} v_i y_{k-2} Q_1 y_s$ or $v_i v_{i-1} y_{j+2} Q_1 y_0$ or $v_{i-1} v_i y_{j+2} Q_1 y_0$ is a path of length at least 9, a contradiction);

(iv) $N_{Q_1}(v) \subseteq \{v_{i+1}, v_{i-2}\}$ (since (i), (ii) and (iii)).

Now similar to the proof of Claim 2.17 in the third, forth and fifth paragraphs, we could obtain that for any longest path $Q$ of $G$, if $v_i \in V(C) \setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$. $\qquad\square$

**Claim 2.24.** *If there is a vertex $v_i \in V(C)$ such that $d_G(v_i) = 2$, then all the longest paths of $G$ contain $v_{i-1}$ and $v_{i+1}$.*

P r o o f.  We could obtain this result similarly as in the proof of Claim 2.10. $\quad\square$

**Claim 2.25.** *If there is a longest path $Q = y_0 y_1 \ldots y_s$ of $G$ such that $|V(Q) \cap V(C)| \leqslant 6$, then for each vertex $v \in V(C) \setminus V(Q)$, $d_Q(v) = 2$.*

P r o o f.  Without loss of generality, suppose that $v_1 \in V(C) \setminus V(Q)$. By Claim 2.23, $v_2, v_7 \in V(Q)$. Since $r = 7$, $v_2 w_1 v_7$ or $v_2 w_1 w_2 v_7$ or $v_2 w_1 w_2 w_3 v_7$ or $v_2 w_1 w_2 w_3 w_4 v_7$ is a subpath of $Q$ ($w_1$, $w_2$, $w_3$, $w_4$ may be vertices of $V(C)$). Now $v_2$ and $v_7$ are not end-vertices of $Q$, since otherwise adding $v_1$ to $Q$ results in a longer path, a contradiction. Suppose $v_2 = y_k, v_7 = y_j, 1 \leqslant k < j \leqslant s - 1$.

**Case 2.5.1.** $v_2 w_1 v_7$ **is a subpath of** $Q$. In this case, we can check that for the vertex $v_1 \in V(C) \setminus V(Q)$, the following assertions hold:

(i) $v_1$ is not adjacent to $w_1, y_{k-1}, y_{j+1}$ (by Claim 2.3);

(ii) $v_1$ is not adjacent to any vertex in $Q[y_0, y_{k-4}] \cup Q[y_{j+4}, y_s]$ (since $r = 7$);

(iii) $N_Q(v_1) \subseteq \{v_2, v_7, y_{k-2}, y_{k-3}, y_{j+2}, y_{j+3}\}$ (since (i) and (ii)).

**Claim 2.26.** $v_1 y_{j+3} \notin E(G)$.

P r o o f.  If $v_1 y_{j+3} \in E(G)$, then by Claim 2.3, $v_1 y_{j+2} \notin E(G)$. Furthermore, as $r = 7$, $v_1 y_{k-2}, v_1 y_{k-3} \notin E(G)$. We assume that $d_G(v_1) = 3$. Since if there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1 v_1' \in E(G)$, then both $Q[y_0, v_2]$ and $Q[y_{j+3}, y_s]$ have lengths at least 2, for otherwise $v_1' v_1 v_2 Q y_s$ or $v_1' v_1 y_{j+3} Q y_0$ is a path longer

than $Q$, a contradiction. But now $s \geqslant 9$ and $y_0y_1$, $y_2y_3$, $y_4y_5$, $y_6y_7$, $y_8y_9$, $v_1v_1'$ are 6 independent edges, a contradiction.

Now we can check that for the vertex $w_1$, the following assertions hold:

(i) $w_1$ is not adjacent to any vertex in $Q[y_0, y_{k-1}] \cup Q[y_{j+4}, y_s]$ (since if there exists a vertex $z \in Q[y_0, y_{k-1}] \cup Q[y_{j+4}, y_s]$ such that $w_1z \in E(G)$, then $zQv_2v_1y_{j+3}Qw_1z$ or $zQv_7v_1v_2w_1z$ is a cycle of length at least 8, a contradiction);

(ii) $w_1$ is not adjacent to any vertex in $\{y_{j+1}, y_{j+2}\}$. Since otherwise

$$y_0Qv_2v_1v_7w_1y_{j+1}Qy_s \quad \text{or} \quad y_0Qv_2w_1y_{j+2}y_{j+1}y_jv_1y_{j+3}Qy_s$$

is a path longer than $Q$;

(iii) $w_1$ is not adjacent to any vertex in $V(G) \setminus V(Q)$ (If there exists a vertex $z \in V(G) \setminus V(Q)$ such that $w_1z \in E(G)$, then both $Q[y_0, v_2]$ and $Q[y_{j+3}, y_s]$ have lengths at least 2, for otherwise $zw_1v_2v_1v_7Qy_s$ or $zw_1Qy_{j+3}v_1v_2Qy_0$ is a path longer than $Q$, a contradiction. But now $y_0y_1$, $v_2v_1$, $w_1z$, $v_7y_{j+1}$, $y_{j+2}y_{j+3}$, $y_{s-1}y_s$ are 6 independent edges.);

(iv) $N_G(w_1) \subseteq \{v_2, v_7, y_{j+3}\}$ (since (i), (ii) and (iii)).

Now we prove that all the longest paths of $G$ contain $v_7$. If there exists a longest path $Q_1$ not containing $v_7$, then by Claim 2.23, $w_1, v_1 \in V(Q_1)$. Now $w_1, v_1$ are not end-vertices of $Q_1$, since otherwise adding $v_7$ to $Q_1$ results in a longer path, a contradiction. But now $v_2v_1y_{j+3}w_1v_2$ is a subpath of $Q_1$, a contradiction. Since $G$ is a counterexample, $v_1y_{j+3} \notin E(G)$. $\qquad\square$

Similarly, we could obtain that $v_1y_{k-3} \notin E(G)$.

**Claim 2.27.** $v_1y_{j+2} \notin E(G)$.

P r o o f.  If $v_1y_{j+2} \in E(G)$, then as $r = 7$, $v_1y_{k-2} \notin E(G)$. We can check that for the vertex $w_1$, the following assertions hold:

(i) $w_1$ is not adjacent to any vertex in $Q[y_0, y_{k-2}] \cup Q[y_{j+4}, y_s]$ (since if there exists a vertex $z \in Q[y_0, y_{k-2}] \cup Q[y_{j+4}, y_s]$ such that $w_1z \in E(G)$, then $zQv_2v_1y_{j+2}Qw_1z$ or $zQv_7v_1v_2w_1z$ is a cycle of length at least 8, a contradiction);

(ii) $w_1$ is not adjacent to any vertex in $\{y_{k-1}, y_{j+1}, y_{j+3}\}$ (since otherwise $y_0Qy_{k-1}$ $w_1v_2v_1v_7Qy_s$ or $y_0Qv_2v_1v_7w_1y_{j+1}Qy_s$ or $y_0Qv_2v_1y_{j+2}Qw_1y_{j+3}Qy_s$ is a path longer than $Q$, a contradiction);

(iii) $N_Q(w_1) \subseteq \{v_2, v_7, y_{j+2}\}$ (since (i) and (ii)).

Similarly, we could prove that $N_Q(y_{j+1}) \subseteq \{v_2, v_7, y_{j+2}\}$.

If there exists a vertex $w_1' \in V(G) \setminus V(Q)$ such that $w_1w_1' \in E(G)$, then $w_1'w_1v_2v_1v_7Qy_s$ is a path of length at least 7. If there is a vertex $w_1'' \in V(G) \setminus V(Q)$ such that $w_1'w_1'' \in E(G)$, then $Q[y_0, v_7]$ has length at least 5, for otherwise $w_1''w_1'w_1v_2v_1v_7Qy_s$ is a path longer than $Q$, a contradiction. But now

$y_0 Q v_2 v_1 y_{j+2} y_{j+1} v_7 w_1 w_1' w_1''$ is a path of length at least 10, and we could find 6 independent edges, a contradiction. Thus $N_G(w_1') = N_Q(w_1')$. By (iii), $N_Q(w_1') \subseteq \{w_1, v_2, v_7, y_{j+2}\}$. By Claim 2.3, $w_1' v_2, w_1' v_7 \notin E(G)$. If $w_1' y_{j+2} \in E(G)$, then $y_0 Q v_2 v_1 v_7 w_1 w_1' y_{j+2} Q y_s$ is a path longer than $Q$, a contradiction. Thus $d_G(w_1') = 1$. Similarly, we could prove that $d_G(y_{j+1}') = 1$ holds for any vertex $y_{j+1}' \in V(G) \setminus V(Q)$ such that $y_{j+1} y_{j+1}' \in E(G)$.

Now if there is a longest path $Q_2$ not containing $v_7$, then by Claim 2.23, $v_1, v_6 \in V(Q_2)$. If $w_1 \notin V(Q_2)$, then by the proof of Claim 2.16, $y_{j+1}, v_2 \in V(Q_2)$. By the proof of Claim 2.23 we could check that $N_{Q_2}(v_7) \subseteq \{v_2, y_{j+1}\}$. But now $v_1 \notin V(Q_2)$, a contradiction. Thus $w_1 \in V(Q_2)$. Similarly, we could obtain that $y_{j+1} \in V(Q_2)$. Now $v_1, w_1, y_{j+1}$ are not end-vertices of $Q_2$, since otherwise adding $v_7$ to $Q_2$ results in a longer path, a contradiction. If $v_2 v_1 y_{j+2}$ is a subpath of $Q_2$, then there are two vertices $u_1, u_2 \in V(G) \setminus V(Q)$ such that $u_1 w_1 v_2, u_2 y_{j+1} y_{j+2}$ or $u_1 w_1 y_{j+2}$, $u_2 y_{j+1} v_2$ are two subpaths of $Q_2$, for otherwise $y_{j+2} v_1 v_2 w_1 y_{j+2}$ or $y_{j+2} v_1 v_2 y_{j+1} y_{j+2}$ is a subpath of $Q_2$, a contradiction. But now $Q_2 = u_1 w_1 v_2 v_1 y_{j+2} y_{j+1} u_2$ or $Q_2 = u_2 y_{j+1} v_2 v_1 y_{j+2} w_1 u_1$, a contradiction. Thus, there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1' v_1 v_2$ or $v_1' v_1 y_{j+2}$ is a segment of $Q_2$. Without loss of generality, suppose that $v_1' v_1 v_2$ is a segment of $Q_2$. Now both $Q[y_0, v_2]$ and $Q[y_{j+2}, y_s]$ have lengths at least 2, for otherwise either $v_1' v_1 v_2 Q y_s$ or $v_1' v_1 y_{j+2} Q y_0$ is a path longer than $Q$, a contradiction. Furthermore, both $Q[y_0, v_2]$ and $Q[y_{j+2}, y_s]$ have lengths exactly 2, for otherwise $s \geqslant 9$ and $y_0 y_1, y_2 y_3, y_4 y_5, y_6 y_7, y_8 y_9, v_1 v_1'$ are 6 independent edges, a contradiction. We assume that $N_G(v_1') = N_Q(v_1') \cup \{v_1\}$. Since if there exists a vertex $v_1'' \in V(G) \setminus V(Q)$ such that $v_1' v_1'' \in E(G)$, then $v_1'' v_1' v_1 v_2 Q y_s$ is a path of length at least 9, a contradiction. Now as $N_Q(v_1) = \{v_2, v_7, y_{j+2}\}$, $N_Q(v_1') \subseteq \{v_2, v_7, y_{j+2}\}$. By Claim 2.3, $v_1' v_2, v_1' v_7, v_1' y_{j+2} \notin E(G)$. Thus $d_G(v_1') = 1$.

Since $v_1' v_1 v_2$ is a subpath of $Q_2$, there is no vertex $u_1 \in V(G) \setminus V(Q)$ such that $u_1 w_1 v_2$ or $u_1 y_{j+1} v_2$ is a subpath of $Q_2$, for otherwise $Q_2 = v_1' v_1 v_2 w_1 u_1$ or $Q_2 = v_1' v_1 v_2 y_{j+1} u_1$, a contradiction. If there is a vertex $u_2 \in V(G) \setminus V(Q)$ such that $u_2 w_1 y_{j+2}$ is a subpath of $Q_2$, then $v_2 y_{j+1} y_{j+2}$ is a subpath of $Q_2$, for otherwise $Q_2 = u_2 w_1 y_{j+2} y_{j+1} u_3$ ($u_3 \in V(G) \setminus V(Q)$), a contradiction. But now $Q_2 = v_1' v_1 v_2 y_{j+1} y_{j+2} w_1 u_2$, a contradiction. Since $N_Q(w_1) \subseteq \{v_2, v_7, y_{j+2}\}$, $v_2 w_1 y_{j+2}$ is a subpath of $Q_2$. Similarly, we could prove that $v_2 y_{j+1} y_{j+2}$ is a subpath of $Q_2$. But now $v_2 w_1 y_{j+2} y_{j+1} v_2$ is a subpath of $Q_2$, a contradiction. Thus, all the longest paths of $G$ contain $v_7$. Since $G$ is a counterexample, $v_1 y_{j+2} \notin E(G)$. $\square$

Similarly, we could obtain that $v_1 y_{k-2} \notin E(G)$. Therefore $d_Q(v_1) = 2$.

**Case 2.5.2.** $v_2 w_1 w_2 v_7$ **is a subpath of** $Q$. In this case, we can check that for the vertex $v_1 \in V(G) \setminus V(Q)$, the following assertions hold:

(i) $v_1$ is not adjacent to any vertex in $\{w_1, w_2, y_{k-1}, y_{j+1}\}$ (by Claim 2.3);

355

(ii) $v_1$ is not adjacent to any vertex in $Q[y_0, y_{k-3}] \cup Q[y_{j+3}, y_s]$ (since $r = 7$);

(iii) $N_Q(v_1) \subseteq \{v_2, v_7, y_{k-2}, y_{j+2}\}$ (since (i) and (ii)).

If $v_1 y_{k-2} \in E(G)$ or $v_1 y_{j+2} \in E(G)$, then similar to the proof of Case 2.5.1 we could obtain that $d_Q(v_1) = 2$.

**Case 2.5.3.** $v_2 w_1 w_2 w_3 v_7$ **is a subpath of** $Q$**.** In this case, similar to the proof of Case 2.5.1 we could obtain that $d_Q(v_1) = 2$.

**Case 2.5.4.** $v_2 w_1 w_2 w_3 w_4 v_7$ **is a subpath of** $Q$**.** In this case, similar to the proof of Case 2.5.1 we could obtain that $d_Q(v_1) = 2$. □

**Claim 2.28.** *If there is a longest path $Q = y_0 y_1 \ldots y_s$ of $G$ such that $|V(Q) \cap V(C)| \leqslant 6$, then all the longest paths of $G$ have a common vertex.*

P r o o f.   Without loss of generality, suppose that $v_1 \in V(C) \setminus V(Q)$. By Claim 2.25, $d_Q(v_1) = 2$. If $d_G(v_1) = 2$, then by Claim 2.24, all the longest paths of $G$ share a common vertex.

If $d_G(v_1) \geqslant 3$, then similar to the proof of Claim 2.15 in the third, forth, fifth paragraphs, we could obtain that for any vertex $w \in V(G) \setminus V(Q)$ such that $v_1 w \in E(G)$, $N_G(w) \cap (V(G) \setminus V(Q)) = \{v_1\}$.

Now we prove that all the longest paths of $G$ contain $v_2$. If there exists a longest path $Q_2$ not containing $v_2$, then by Claim 2.23, $v_1, v_3 \in V(Q_2)$. If $w_1 \notin V(Q_2)$, then there exists a vertex $u \in Q[w_1, v_7]$ such that $u \in V(Q_2)$, for otherwise by Claim 2.23, $v_6 \in V(Q_2)$. Now $C_2 = v_1 Q_2[v_1, v_6] v_6 v_7 Q w_1 v_2 v_1$ is a cycle of length at least 8, a contradiction. Thus $w_1 u \in E(Q)$ and $u \in V(Q_2)$. Now we could check that $N_{Q_2}(v_2) \subseteq \{v_1, u\}$. Thus $u = v_3$. By Claim 2.25, $y_{k-1} \notin V(Q_2)$. By the above, $y_{k-2} \in V(Q_2)$. But now $C_3 = y_{k-2} Q_2[y_{k-2}, u] u w_1 v_2 y_{k-1} y_{k-2}$ is a cycle of length at least 8, a contradiction. Thus $w_1 \in V(Q_2)$. By Claim 2.25, $w_1 = v_3$ and $y_{k-1} \notin V(Q_2)$. By the above, $y_{k-2} \in V(Q_2)$. But now we could check that $N_{Q_2}(v_2) \subseteq \{v_3, y_{k-2}\}$, a contradiction.

Thus, all the longest paths of $G$ contain $v_2$. □

Since $G$ is a counterexample, by Claim 2.28, for any longest path $Q$ of $G$, $V(C) \subset V(Q)$. But now all the longest paths of $G$ contain $V(C)$, a contradiction.

**2.6. Proof of the case $r = 8$.** If $r = 8$, then by Claim 2.1 we have that $s \geqslant 8$.

**Claim 2.29.** *For any longest path $Q$ of $G$, $|V(Q) \cap V(C)| \geqslant 4$.*

356

P r o o f. Let $Q = y_0 y_1 \ldots y_s$ be a longest path of $G$. If $|V(Q) \cap V(C)| \leqslant 2$, then $y_0 y_1, y_2 y_3, y_4 y_5, y_6 y_7$ and two independent edges in $C \setminus V(Q)$ are 6 independent edges, a contradiction. If $|V(Q) \cap V(C)| = 3$, then at least two vertices of $V(C) \setminus V(Q)$ are consecutive in $C$. Suppose $W[v_i, v_j]$ is a maximum consecutive segment in $C \setminus V(Q)$, and $N_{C \cap Q}(v_i) = \{\overline{v}_i\}$, $N_{C \cap Q}(v_j) = \{\overline{v}_j\}$. If $W[v_i, v_j]$ has length 1, then there are two independent edges in $C \setminus V(Q)$. But now $y_0 y_1, y_2 y_3, y_4 y_5, y_6 y_7$ and the two independent edges are 6 independent edges, a contradiction. If $W[v_i, v_j]$ has length at least 2, then as $Q$ is a longest path of $G$, the length of $Q[\overline{v}_i, \overline{v}_j]$ is at least 4. Suppose that $\overline{v}_i = y_k$, $\overline{v}_j = y_l$, $k < l$. We assume that both $Q[y_0, y_k]$ and $Q[y_l, y_s]$ have lengths at least 3, for otherwise $v_j W v_i \overline{v}_i Q[\overline{v}_i, y_s]$ or $v_i W v_j \overline{v}_j Q[\overline{v}_j, y_0]$ is a path longer than $Q$, a contradiction. But now the length of $Q$ is at least 10, and $y_0 y_1, y_2 y_3$, $y_4 y_5, y_6 y_7, y_8 y_9$ with an edge in $W[v_i, v_j]$ are 6 independent edges, a contradiction. $\square$

**Claim 2.30.** *For any longest path $Q$ of $G$, if $v_i \in V(C) \setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$.*

P r o o f. If not, then there exists a longest path $Q_1$ and $v_i \in V(C) \setminus V(Q_1)$ such that $v_{i-1} \notin V(Q_1)$ or $v_{i+1} \notin V(Q_1)$. Without loss of generality, suppose that $v_{i-1} \notin V(Q_1)$. Now similar to the first paragraph of the proof of Claim 2.17 we could obtain that $v_{i+1} w_1 w_2 v_{i-2}$ or $v_{i+1} w_1 w_2 w_3 v_{i-2}$ is a segment of $Q_1$ and $s = 8$.

We can check that for each vertex $v \in \{v_i, v_{i-1}\}$, the following assertions hold:
 (i) $v$ is not adjacent to $w_1, w_2, y_{k-1}, y_{j+1}$ (if $v_{i+1} w_1 w_2 v_{i-2}$ is a segment of $Q_1$) or $w_1, w_2, w_3, y_{k-1}, y_{j+1}$ (if $v_{i+1} w_1 w_2 w_3 v_{i-2}$ is a segment of $Q_1$) (by Claim 2.3);
 (ii) $v$ is not adjacent to any vertex in $Q_1[y_0, y_{k-2}] \cup Q_1[y_{j+2}, y_s]$ (since if there exists a vertex $z \in Q_1[y_0, y_{k-2}] \cup Q_1[y_{j+2}, y_s]$ such that $vz \in E(G)$, then $v_i v_{i-1} z Q_1 y_s$ or $v_{i-1} v_i z Q_1 y_s$ or $v_i v_{i-1} z Q_1 y_0$ or $v_{i-1} v_i z Q_1 y_0$ is a path of length at least 9, a contradiction);
 (iii) $N_{Q_1}(v) \subseteq \{v_{i+1}, v_{i-2}\}$ (since (i) and (ii)).

Now similar to the proof of Claim 2.17 in the third, forth and fifth paragraphs we could obtain that for any longest path $Q$ of $G$, if $v_i \in V(C) \setminus V(Q)$, then $v_{i-1}, v_{i+1} \in V(Q)$. $\square$

**Claim 2.31.** *If there is a vertex $v_i \in V(C)$ such that $d_G(v_i) = 2$, then all the longest paths of $G$ contain $v_{i-1}$ and $v_{i+1}$.*

P r o o f. Similar to the proof of Claim 2.10 we could obtain this result. $\square$

**Claim 2.32.** *If there is a longest path $Q = y_0 y_1 \ldots y_s$ of $G$ such that $|V(Q) \cap V(C)| \leqslant 7$, then for each vertex $v \in V(C) \setminus V(Q)$, $d_Q(v) = 2$.*

P r o o f.    Without loss of generality, suppose that $v_1 \in V(C) \setminus V(Q)$. By Claim 2.31, $v_2, v_8 \in V(Q)$. Since $r = 8$, $v_2 w_1 v_8$ or $v_2 w_1 w_2 v_8$ or $v_2 w_1 w_2 w_3 v_8$ or $v_2 w_1 w_2 w_3 w_4 v_8$ or $v_2 w_1 w_2 w_3 w_4 w_5 v_8$ is a subpath of $Q$ in $G$ ($w_1, w_2, w_3, w_4, w_5$ may be vertices of $V(C)$). Now $v_2$ and $v_8$ are not end-vertices of $Q$, since otherwise adding $v_1$ to $Q$ results in a longer path, a contradiction. Suppose $v_2 = y_k$, $v_8 = y_j$, $1 \leqslant k < j \leqslant s - 1$.

**Case 2.6.1.** $v_2 w_1 v_8$ **is a subpath of** $Q$. In this case, we can check that for the vertex $v_1 \in V(C) \setminus V(Q)$, the following assertions hold:

(i) $v_1$ is not adjacent to $w_1$, $y_{k-1}$, $y_{j+1}$ (by Claim 2.3);

(ii) $v_1$ is not adjacent to any vertex in $Q[y_0, y_{k-5}] \cup Q[y_{j+5}, y_s]$ (since $r = 8$);

(iii) $N_Q(v_1) \subseteq \{v_2, v_8, y_{k-2}, y_{k-3}, y_{k-4}, y_{j+2}, y_{j+3}, y_{j+4}\}$ (since (i) and (ii)).

**Claim 2.33.** $v_1 y_{j+4} \notin E(G)$.

P r o o f.    If $v_1 y_{j+4} \in E(G)$, then $v_1 y_{k-2}, v_1 y_{k-3}, v_1 y_{k-4} \notin E(G)$, otherwise we could find a cycle longer than $C$, a contradiction. By Claim 2.3, $v_1 y_{j+3} \notin E(G)$. If there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1 v_1' \in E(G)$, then $v_1' v_1 v_2 w_1 v_8 y_{j+1} y_{j+2} y_{j+3} y_{j+4} y_{j+5}$ is a path of length 9. Since $Q$ is a longest path, $s \geqslant 9$. But now $y_0 y_1$, $y_2 y_3$, $y_4 y_5$, $y_6 y_7$, $y_8 y_9$, $v_1 v_1'$ are 6 independent edges, a contradiction. Thus $N_G(v_1) \subseteq \{v_2, v_8, y_{j+2}, y_{j+4}\}$.

Now we can check that for the vertex $w_1$, the following assertions hold:

(i) $w_1$ is not adjacent to any vertex in $Q[y_0, y_{k-1}] \cup Q[y_{j+5}, y_s]$ (since if there exists a vertex $z \in Q[y_0, y_{k-1}] \cup Q[y_{j+5}, y_s]$ such that $w_1 z \in E(G)$, then $z Q v_2 v_1 y_{j+4} Q w_1 z$ or $z Q y_{j+4} Q v_8 v_1 v_2 w_1 z$ is a cycle of length at least 9);

(ii) $w_1$ is not adjacent to any vertex in $\{y_{j+1}, y_{j+3}\}$. Since otherwise

$$y_0 Q v_2 v_1 v_8 w_1 y_{j+1} Q y_s \quad \text{or} \quad y_0 Q w_1 y_{j+3} Q v_8 v_1 y_{j+4} Q y_s$$

is a path longer than $Q$;

(iii) $w_1$ is not adjacent to any vertex in $V(G) \setminus V(Q)$ (If there exists a vertex $z \in V(G) \setminus V(Q)$ such that $w_1 z \in E(G)$, then $Q[y_0, v_8]$ has length at least 4, otherwise $z w_1 v_2 v_1 v_8 Q y_s$ is a path longer than $Q$. But now $y_0 y_1$, $v_2 v_1$, $w_1 z$, $v_8 y_{j+1}$, $y_{j+2} y_{j+3}$, $y_{j+4} y_{j+5}$ are 6 independent edges.);

(iv) $N_G(w_1) \subseteq \{v_2, v_8, y_{j+2}, y_{j+4}\}$ (since (i), (ii) and (iii)).

If $v_1 y_{j+2} \in E(G)$ or $w_1 y_{j+2} \in E(G)$, then we can check that for the vertex $y_{j+1}$, the following assertions hold:

(i) $y_{j+1}$ is not adjacent to any vertex in $Q[y_0, y_{k-1}] \cup Q[y_{j+5}, y_s]$ (since $r = 8$);

(ii) $y_{j+1}$ is not adjacent to $y_{j+3}$ (since otherwise $y_0 Q v_2 v_1 v_8 w_1 y_{j+2} y_{j+1} y_{j+3} Q y_s$ or $y_0 Q y_{j+1} y_{j+3}\, y_{j+2} v_1 y_{j+4} Q y_s$ is a path longer than $Q$, a contradiction);

358

(iii) $y_{j+1}$ is not adjacent to any vertex in $V(G) \setminus V(Q)$ (If there exists a vertex $z \in V(G) \setminus V(Q)$ such that $y_{j+1}z \in E(G)$, then $Q[y_0, y_{j+2}]$ has length at least 6, otherwise $zy_{j+1}v_8v_1v_2w_1y_{j+2}Qy_s$ or $zy_{j+1}v_8w_1v_2v_1y_{j+2}Qy_s$ is a path longer than $Q$. But now $y_0y_1, v_2v_1, w_1v_8, y_{j+1}z, y_{j+2}y_{j+3}, y_{j+4}y_{j+5}$ are 6 independent edges, a contradiction.);

(iv) $N_G(y_{j+1}) \subseteq \{v_2, v_8, y_{j+2}, y_{j+4}\}$ (since (i), (ii) and (iii)).

Now we prove that all the longest paths of $G$ contain $v_8$. If there exists a longest path $Q_1$ not containing $v_8$, then by Claim 2.30, $Q_1$ contains $w_1$, $y_{j+1}$, $v_1$. Now $w_1$, $y_{j+1}$, $v_1$ are not end-vertices of $Q_1$, since otherwise adding $v_8$ to $Q_1$ results in a longer path, a contradiction. Thus $v_2w_1y_{j+2}$ or $v_2w_1y_{j+4}$ or $y_{j+2}w_1y_{j+4}$ is a segment of $Q_1$. Without loss of generality, suppose that $v_2w_1y_{j+2}$ is a segment of $Q_1$. Now $v_2y_{j+1}y_{j+4}$ or $y_{j+2}y_{j+1}y_{j+4}$ is a segment of $Q_1$. Without loss of generality, suppose that $v_2y_{j+1}y_{j+4}$ is a segment of $Q_1$. Now we assume that $y_{j+2}v_1y_{j+4}$ is a segment of $Q_1$, since otherwise $v_2v_1y_{j+4}y_{j+1}v_2$ or $v_2v_1y_{j+2}w_1v_2$ is a segment of $Q_1$, a contradiction. But now $y_{j+4}y_{j+1}v_2w_1y_{j+2}v_1y_{j+4}$ is a segment of $Q_1$, a contradiction.

Since $G$ is a counterexample, $v_1y_{j+2}, w_1y_{j+2} \notin E(G)$. Now if there exists a longest path $Q_2$ not containing $v_8$, then by Claim 2.30, $Q_2$ contains $v_1$, $w_1$. We say that $v_1$, $w_1$ are not end-vertices of $Q_2$, since otherwise adding $v_8$ to $Q_2$ results in a longer path, a contradiction. But now $v_2v_1y_{j+4}w_1v_2$ is a segment of $Q_2$, a contradiction. Thus all the longest paths of $G$ contain $v_8$. Since $G$ is a counterexample, $v_1y_{j+4} \notin E(G)$. $\square$

Similarly, we could prove that $v_1y_{k-4} \notin E(G)$.

**Claim 2.34.** $v_1y_{j+3} \notin E(G)$.

P r o o f. If $v_1y_{j+3} \in E(G)$, then as $r = 8$, $v_1y_{k-2}, v_1y_{k-3} \notin E(G)$. By Claim 2.3, $v_1y_{j+2} \notin E(G)$. We assume that $d_G(v_1) = 3$, since if there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1v_1' \in E(G)$, then both $Q[y_0, v_2]$ and $Q[y_{j+3}, y_s]$ have lengths at least 2. But now $s \geqslant 9$ and $y_0y_1, y_2y_3, y_4y_5, y_6y_7, y_8y_9, v_1v_1'$ are 6 independent edges, a contradiction.

Now we can check that for the vertex $w_1$ the following assertions hold:

(i) $w_1$ is not adjacent to any vertex in $Q[y_0, y_{k-2}] \cup Q[y_{j+5}, y_s]$ (since if there exists a vertex $z \in Q[y_0, y_{k-2}] \cup Q[y_{j+5}, y_s]$ such that $w_1z \in E(G)$, then $zQv_2v_1y_{j+3}Qv_8w_1z$ or $zQv_8v_1v_2w_1z$ is a cycle of length at least 9);

(ii) $w_1$ is not adjacent to any vertex in $\{y_{k-1}, y_{j+4}\}$. Since otherwise

$$y_0Qy_{k-1}w_1v_2v_1v_8Qy_s \quad \text{or} \quad y_0Qv_2v_1y_{j+3}Qw_1y_{j+4}Qy_s$$

is a path longer than $Q$;

(iii) $w_1$ is not adjacent to any vertex in $\{y_{j+1}, y_{j+2}\}$. Since otherwise

$$y_0 Q v_2 v_1 v_8 w_1 y_{j+1} Q y_s \quad \text{or} \quad y_0 Q w_1 y_{j+2} y_{j+1} v_8 v_1 y_{j+3} Q y_s$$

is a path longer than $Q$;

(iv) $w_1$ is not adjacent to any vertex in $V(G) \setminus V(Q)$ (If there exists a vertex $z \in V(G) \setminus V(Q)$ such that $w_1 z \in E(G)$, then both $Q[y_0, v_2]$ and $Q[y_{j+3}, y_s]$ have lengths at least 2, for otherwise $z w_1 v_2 v_1 v_8 Q y_s$ or $z w_1 Q y_{j+3} v_1 v_2 Q y_0$ is a path longer than $Q$. But now $s \geqslant 9$ and $y_0 y_1$, $v_2 v_1$, $w_1 z$, $v_8 y_{j+1}$, $y_{j+2} y_{j+3}$, $y_{j+4} y_{j+5}$ are 6 independent edges, a contradiction.);

(v) $N_G(w_1) \subseteq \{v_2, v_8, y_{j+3}\}$ (since (i), (ii), (iii) and (iv)).

Now we prove that all the longest paths of $G$ contain $v_8$. If there exists a longest path $Q_3$ not containing $v_8$, then by Claim 2.30, $v_1, v_7 \in V(Q_3)$. If $w_1 \in V(Q_3)$, then $w_1$, $v_1$ are not end-vertices of $Q_3$, since otherwise adding $v_8$ to $Q_3$ results in a longer path, a contradiction. But now $v_2 v_1 y_{j+3} w_1 v_2$ is a subpath of $Q_3$, a contradiction. Thus $w_1 \notin V(Q_3)$. By the proof of Claim 2.29, $v_2, y_{j+1} \in V(Q_3)$. We could check that $N_{Q_3}(v_8) \subseteq \{v_2, y_{j+1}\}$, a contradiction to $v_1, v_7 \in V(Q_3)$.

Since $G$ is a counterexample, $v_1 y_{j+3} \notin E(G)$. $\qquad\square$

Similarly, we could prove that $v_1 y_{k-3} \notin E(G)$.

**Claim 2.35.** $v_1 y_{j+2} \notin E(G)$.

P r o o f.   If $v_1 y_{j+2} \in E(G)$, then by the proof of the case $v_1 y_{j+4} \in E(G)$, $v_1 y_{k-2} \notin E(G)$.

Now we can check that for the vertex $w_1$, the following assertions hold:

(i) $w_1$ is not adjacent to any vertex in $Q[y_0, y_{k-3}] \cup Q[y_{j+5}, y_s]$ (since $r = 8$);

(ii) $w_1$ is not adjacent to any vertex in $\{y_{j+1}, y_{k-1}, y_{j+3}\}$ (since otherwise $y_0 Q v_2 v_1 v_8 w_1 y_{j+1} Q y_s$ or $y_0 Q y_{k-1} w_1 v_2 v_1 v_8 Q y_s$ or $y_0 Q v_2 v_1 y_{j+2} Q w_1 y_{j+3} Q y_s$ is a path longer than $Q$);

(iii) $N_Q(w_1) \subseteq \{v_2, v_8, y_{j+2}, y_{j+4}, y_{k-2}\}$ (since (i) and (ii)).

By symmetry, we could prove that $N_Q(y_{j+1}) \subseteq \{v_2, v_8, y_{j+2}, y_{j+4}, y_{k-2}\}$.

If $w_1 y_{k-2} \in E(G)$, then $w_1 y_{j+4}, y_{j+1} y_{j+4} \notin E(G)$, for otherwise

$$y_{k-2} y_{k-1} v_2 v_1 v_8 Q y_{j+4} w_1 y_{k-2} \quad \text{or} \quad y_{k-2} y_{k-1} v_2 v_1 y_{j+2} y_{j+3} y_{j+4} y_{j+1} v_8 w_1 y_{k-2}$$

is a cycle of length at least 10, a contradiction. Now $Q[y_0, v_8]$ has length at least 5, otherwise $y_{k-1} y_{k-2} w_1 v_2 v_1 v_8 Q y_s$ is a path longer than $Q$, a contradiction. Further, if there exists a vertex $w_1' \in V(G) \setminus V(Q)$ such that $w_1 w_1' \in E(G)$, then $Q[y_{j+2}, y_s]$ has length at least 2, otherwise $w_1' w_1 v_8 y_{j+1} y_{j+2} v_1 v_2 Q y_0$ is a path longer than $Q$,

a contradiction. But now $y_0y_1$, $y_{k-1}v_2$, $w_1w_1'$, $v_8v_1$, $y_{j+1}y_{j+2}$, $y_{j+3}y_{j+4}$ are 6 independent edges, a contradiction. Thus $N_G(w_1) = N_Q(w_1) \subseteq \{v_2, v_8, y_{j+2}, y_{k-2}\}$. Similarly, we could obtain that $N_G(y_{j+1}) \subseteq \{v_2, v_8, y_{j+2}, y_{k-2}\}$. Now if there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1'v_1 \in E(G)$, then $Q[y_0, v_8]$ has length at least 6, otherwise $v_1'v_1v_2y_{k-1}y_{k-2}w_1v_8Qy_s$ is a path longer than $Q$, a contradiction. But now $s \geqslant 9$ and we could find 6 independent edges, a contradiction. Thus $N_G(v_1) = N_Q(v_1) \subseteq \{v_2, v_8, y_{j+2}\}$.

Now we prove that all the longest paths of $G$ contain $v_8$. If there is a longest path $Q_4$ not containing $v_8$, then by Claim 2.30, $v_1, w_1, y_{j+1} \in V(Q_4)$. We assume that $v_1$, $w_1$, $y_{j+1}$ are not end-vertices of $Q_4$, since otherwise adding $v_8$ to $Q_4$ results in a longer path, a contradiction. Since $N_G(v_1) \subseteq \{v_2, v_8, y_{j+2}\}$, $v_2v_1y_{j+2}$ is a subpath of $Q_4$. Now either $v_2w_1y_{k-2}$ and $y_{j+2}y_{j+1}y_{k-2}$ or $y_{j+2}w_1y_{k-2}$ and $v_2y_{j+1}y_{k-2}$ are two subpaths of $Q_4$. But now $y_{k-2}w_1v_2v_1y_{j+2}y_{j+1}y_{k-2}$ or $y_{k-2}w_1y_{j+2}v_1v_2y_{j+1}y_{k-2}$ is a subpath of $Q_4$, a contradiction. Since $G$ is a counterexample, $w_1y_{k-2} \notin E(G)$. Similarly, we could prove that $w_1y_{j+4}, y_{j+1}y_{k-2}, y_{j+1}y_{j+4} \notin E(G)$. Thus $N_Q(v) \subseteq \{v_2, v_8, y_{j+2}\}$, $v \in \{w_1, y_{j+1}\}$.

If there exists a vertex $w_1' \in V(G) \setminus V(Q)$ such that $w_1w_1' \in E(G)$, then $N_G(w_1') \subseteq \{w_1, y_{j+2}\}$. Since if there exists a vertex $w_1'' \in V(G) \setminus V(Q)$, then both $Q[y_0, v_2]$ and $Q[y_{j+2}, y_s]$ have lengths at least 3, for otherwise $w_1''w_1'w_1v_2v_1v_8Qy_s$ or $w_1''w_1'w_1v_8y_{j+1}y_{j+2}v_1v_2Qy_0$ is a path longer than $Q$, a contradiction. But now $y_0y_1$, $y_{k-1}v_2$, $w_1w_1'$, $v_8v_1$, $y_{j+1}y_{j+2}$, $y_{j+3}y_{j+4}$ are 6 independent edges, a contradiction. Thus $N_G(w_1') = N_Q(w_1')$. Since $N_Q(w_1) \subseteq \{v_2, v_8, y_{j+2}\}$, $N_Q(w_1') \subseteq \{w_1, v_2, v_8, y_{j+2}\}$. By Claim 2.3, $w_1'v_2, w_1'v_8 \notin E(G)$. Thus $N_G(w_1') \subseteq \{w_1, y_{j+2}\}$. Similarly, we could prove that for any vertex $y_{j+1}' \in V(G) \setminus V(Q)$ such that $y_{j+1}y_{j+1}' \in E(G)$, $N_G(y_{j+1}') \subseteq \{y_{j+1}, v_2\}$.

Now we prove that all the longest paths of $G$ contain $v_8$. If there exists a longest path $Q_5$ not containing $v_8$, then by Claim 2.30, $v_1, v_7 \in V(Q_5)$. If $w_1 \notin V(Q_5)$, then by the proof of Claim 2.29, $v_2, y_{j+1} \in V(Q_5)$. We could check that $N_{Q_5}(v_8) \subseteq \{v_2, y_{j+1}\}$, a contradiction to $v_1, v_7 \in V(Q_5)$. Thus $w_1 \in V(Q_5)$. Similarly, we could prove that $y_{j+1} \in V(Q_5)$. Now if $v_2v_1y_{j+2}$ is a subpath of $Q_5$, then there are two vertices $u_1, u_2 \in V(G) \setminus V(Q)$ such that $u_1w_1v_2$, $u_2y_{j+1}y_{j+2}$ or $u_1w_1y_{j+2}$, $u_2y_{j+1}v_2$ are two segments of $Q_5$. But now $Q_5 = u_1w_1v_2v_1y_{j+2}y_{j+1}u_2$ or $Q_5 = u_2y_{j+1}v_2v_1y_{j+2}w_1u_1$, a contradiction. Thus, there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1'v_1v_2$ or $v_1'v_1y_{j+2}$ is a segment of $Q_5$. Now $Q = y_0y_1v_2w_1v_8y_{j+1}y_{j+2}y_7y_8$, otherwise we could find 6 independent edges, a contradiction. We assume that $w_1'y_{j+2} \notin E(G)$, otherwise $y_0y_1v_2v_1v_8w_1w_1'y_{j+2}y_7y_8$ is a path longer than $Q$, a contradiction. Thus $d_G(w_1') = 1$. Similarly, $d_G(y_{j+1}') = 1$. Without loss of generality, suppose that $v_1'v_1v_2$ is a segment of $Q_5$. Since $s = 8$, $N_G(v_1') = N_Q(v_1') \cup \{v_1\}$. Since $N_Q(v_1) \subseteq \{v_2, v_8, y_{j+2}\}$, $N_Q(v_1') \subseteq \{v_2, v_8, y_{j+2}\}$. By Claim 2.3, $v_1'v_2, v_1'v_8, v_1'y_{j+2} \notin$

361

$E(G)$. Thus $d_G(v_1') = 1$. If $v_2 w_1 y_{j+2}$ or $v_2 y_{j+1} y_{j+2}$ is a segment of $Q_5$, then there exists a vertex $u \in V(G) \setminus V(Q)$ such that $y_{j+2} y_{j+1} u$ or $y_{j+2} w_1 u$ is a segment of $Q_5$. But now $Q_5 = v_1' v_1 v_2 w_1 y_{j+2} y_{j+1} u$ or $Q_5 = v_1' v_1 v_2 y_{j+1} y_{j+2} w_1 u$, a contradiction. Thus, there are two vertices $u_1, u_2 \in V(G) \setminus V(Q)$ such that $u_1 w_1 v_2$, $u_2 y_{j+1} y_{j+2}$ or $u_1 w_1 y_{j+2}$, $u_2 y_{j+1} v_2$ are two segments of $Q_5$. Now either $Q_5 = u_1 w_1 v_2 v_1 v_1'$ or $Q_5 = u_2 y_{j+1} v_2 v_1 v_1'$, a contradiction.

Thus, all the longest paths of $G$ contain $v_8$. Since $G$ is a counterexample, $v_1 y_{j+2} \notin E(G)$. $\qquad\square$

Similarly, we could prove that $v_1 y_{k-2} \notin E(G)$. Thus $d_Q(v_1) = 2$.

**Case 2.6.2.** $v_2 w_1 w_2 v_8$ **is a subpath of** $Q$. In this case, we can check that for vertex $v_1 \in V(G) \setminus V(Q)$ the following assertions hold:

(i) $v_1$ is not adjacent to any vertex in $\{w_1, w_2, y_{k-1}, y_{j+1}\}$ (by Claim 2.3);

(ii) $v_1$ is not adjacent to any vertex in $Q[y_0, y_{k-4}] \cup Q[y_{j+4}, y_s]$ (since $r = 8$);

(iii) $N_Q(v_1) \subseteq \{v_2, v_8, y_{j+2}, y_{j+3}, y_{k-2}, y_{k-3}\}$ (since (i) and (ii)).

**Claim 2.36.** $v_1 y_{j+2} \notin E(G)$ and $v_1 y_{k-2} \notin E(G)$.

P r o o f.   We could obtain the result similarly as in the proof of Case 2.6.1. $\qquad\square$

**Claim 2.37.** $v_1 y_{j+3} \notin E(G)$.

P r o o f.   If $v_1 y_{j+3} \in E(G)$, then as $r = 8$, $v_1 y_{k-3}, v_1 y_{k-2} \notin E(G)$. By Claim 2.3, $v_1 y_{j+2} \notin E(G)$. If there exists a vertex $v_1' \in V(G) \setminus V(Q)$ such that $v_1 v_1' \in E(G)$, then both $Q[y_0, v_2]$ and $Q[y_{j+3}, y_s]$ have lengths at least 2. But now $y_0 y_1$, $v_2 w_1$, $w_2 v_8$, $y_{j+1} y_{j+2}$, $y_{j+3} y_{j+4}$, $v_1 v_1'$ are 6 independent edges, a contradiction. Thus $d_G(v_1) = 3$.

We can check that for vertex $w_2$ the following assertions hold:

(i) $w_2$ is not adjacent to any vertex in $\{y_{k-1}, y_{j+2}\}$. Since otherwise

$$y_0 Q y_{k-1} w_2 w_1 v_2 v_1 v_8 Q y_s \quad \text{or} \quad y_0 Q w_2 y_{j+2} y_{j+1} v_8 v_1 y_{j+3} Q y_s$$

is a path longer than $Q$;

(ii) $w_2$ is not adjacent to any vertex in $Q[y_0, y_{k-2}] \cup Q[y_{j+4}, y_s]$ (since $r = 8$);

(iii) $w_2$ is not adjacent to any vertex in $V(G) \setminus V(Q)$ (Since if there exists a vertex $z \in V(G) \setminus V(Q)$ such that $w_2 z \in E(G)$, then $s \geqslant 9$, for otherwise $z w_2 w_1 v_2 v_1 v_8 Q y_s$ is a path longer than $Q$. But now $y_0 y_1$, $v_2 w_1$, $w_2 z$, $v_8 v_1$, $y_{j+1} y_{j+2}$, $y_{j+3} y_{j+4}$ are 6 independent edges, a contradiction.);

(iv) $N_G(w_2) \subseteq \{v_2, v_8, y_{j+3}, w_1, y_{j+1}\}$ (since (i), (ii) and (iii)).

Similarly, we could prove that $N_G(y_{j+1}) \subseteq \{v_2, v_8, y_{j+2}, y_{j+3}, w_2\}$.

We can check that for vertex $w_1$ the following assertions hold:

(i) $w_1$ is not adjacent to $y_{j+1}$, $y_{j+4}$ (since otherwise $y_0Qv_2v_1v_8w_2w_1y_{j+1}Qy_s$ or $y_0Qv_2v_1y_{j+3}Qw_1y_{j+4}Qy_s$ is a path longer than $Q$);

(ii) $w_1$ is not adjacent to any vertex in $Q[y_0, y_{k-1}] \cup Q[y_{j+5}, y_s]$ (since $r = 8$);

(iii) $w_1$ is not adjacent to any vertex in $V(G) \backslash V(Q)$ (Since if there exists a vertex $z \in V(G) \backslash V(Q)$ such that $w_1z \in E(G)$, then $s \geqslant 9$, for otherwise $zw_1Qy_{j+3}v_1v_2Qy_0$ is a path longer than $Q$. But now $y_{k-1}v_2$, $w_1z$, $w_2v_8$, $y_{j+1}y_{j+2}$, $y_{j+3}v_1$, $y_{j+4}y_{j+5}$ are 6 independent edges, a contradiction.);

(iv) $N_G(w_1) \subseteq \{v_2, v_8, y_{j+3}, w_2, y_{j+2}\}$ (since (i), (ii) and (iii)).

Similarly, we could prove that $N_G(y_{j+2}) \subseteq \{v_2, v_8, y_{j+3}, y_{j+1}, w_1\}$.

Now we prove that all the longest paths of $G$ contain $v_8$. If there exists a longest path $Q_6$ not containing $v_8$, then by Claim 2.30, $v_1, w_2, y_{j+1} \in V(Q_6)$. Now $v_1$, $w_2$ and $y_{j+1}$ are not end-vertices of $Q_6$, since otherwise adding $v_8$ to $Q_6$ results in a longer path, a contradiction. Since $d_G(v_1) = 3$, $v_2v_1y_{j+3}$ is a segment of $Q_6$. If $v_2w_2w_1$ is a segment of $Q_6$, then since $N_G(y_{j+1}) \subseteq \{v_2, v_8, y_{j+2}, y_{j+3}, w_2\}$, $y_{j+3}y_{j+1}y_{j+2}$ is a segment of $Q_6$. Since $N_G(w_1) \subseteq \{v_2, v_8, y_{j+3}, w_2, y_{j+2}\}$, $N_G(y_{j+2}) \subseteq \{v_2, v_8, y_{j+3}, w_1, y_{j+1}\}$, $Q_6 = w_1w_2v_2v_1y_{j+3}y_{j+1}y_{j+2}$, a contradiction. If $v_2w_2y_{j+1}$ is a segment of $Q_6$, then since $N_G(y_{j+1}) \subseteq \{v_2, v_8, y_{j+2}, y_{j+3}, w_2\}$, $y_{j+1}y_{j+2} \in E(Q_6)$. Now $y_{j+2}w_1 \notin E(Q_6)$, otherwise $y_0Qv_2w_1y_{j+2}y_{j+1}w_2v_8v_1y_{j+3}Qy_s$ is a path longer than $Q$, a contradiction. If $w_1 \notin V(Q_6)$, then $(Q_6 - v_2w_2) \cup v_2w_1w_2$ is a path longer than $Q_6$, a contradiction. Thus $w_1 \in V(Q_6)$. Since $N_G(w_1) \subseteq \{v_2, v_8, y_{j+3}, w_2, y_{j+2}\}$, $y_{j+3}w_1 \in E(Q_6)$. But now $Q_6 = y_{j+2}y_{j+1}w_2v_2v_1y_{j+3}w_1$, a contradiction. If $v_2w_2y_{j+3}$ is a segment of $Q_6$, then $y_{j+3}w_2v_2v_1y_{j+3}$ is a segment of $Q_6$, a contradiction. Thus $w_2v_2 \notin E(Q_6)$. Similarly, $y_{j+1}v_2$, $w_2y_{j+3}$, $y_{j+1}y_{j+3} \notin E(Q_6)$. Therefore $w_1w_2y_{j+1}$ and $w_2y_{j+1}y_{j+2}$ are two segments of $Q_6$. Since $N_G(w_1) \subseteq \{v_2, v_8, y_{j+3}, w_2, y_{j+2}\}$, $N_G(y_{j+2}) \subseteq \{v_2, v_8, y_{j+3}, w_1, y_{j+1}\}$, $w_1v_2, y_{j+2}y_{j+3} \in E(Q_6)$ or $w_1y_{j+3}, y_{j+2}v_2 \in E(Q_6)$. But now $v_2v_1y_{j+3}y_{j+2}y_{j+1}w_2w_1v_2$ or $v_2v_1y_{j+3}w_1w_2y_{j+1}y_{j+2}v_2$ is a segment of $Q_6$, a contradiction. Therefore all the longest paths of $G$ contain $v_8$. Since $G$ is a counterexample, $v_1y_{j+3} \notin E(G)$.                                                $\square$

Similarly, we could obtain that $v_1y_{k-3} \notin E(G)$. Therefore $d_Q(v_1) = 2$.

**Case 2.6.3.** $v_2w_1w_2w_3v_8$ **is a subpath of** $Q$**.** In this case, similar to the proof of Case 2.6.1, $d_Q(v_1) = 2$.

**Case 2.6.4.** $v_2w_1w_2w_3w_4v_8$ **is a subpath of** $Q$**.** In this case, similar to the proof of Case 2.6.1, $d_Q(v_1) = 2$.

**Case 2.6.5.** $v_2w_1w_2w_3w_4w_5v_8$ **is a subpath of** $Q$**.** In this case, similar to the proofs of Case 2.6.1 and Case 2.6.2, $d_Q(v_1) = 2$.                                                $\square$

**Claim 2.38.** *If there is a longest path $Q = y_0 y_1 \ldots y_s$ of $G$ such that $|V(Q) \cap V(C)| \leqslant 7$, then all the longest paths of $G$ have a common vertex.*

P r o o f.    Without loss of generality, suppose that $v_1 \in V(C) \setminus V(Q)$. By Claim 2.32, $d_Q(v_1) = 2$. If $d_G(v_1) = 2$, then by Claim 2.31, all the longest paths of $G$ share a common vertex.

If $d_G(v_1) \geqslant 3$, then similar to the proof of Claim 2.15 in the third, forth, fifth paragraphs we could obtain that for any vertex $w \in V(G) \setminus V(Q)$ such that $v_1 w \in E(G)$, $N_G(w) \cap (V(G) \setminus V(Q)) = \{v_1\}$.

Now we prove that all the longest paths of $G$ contain $v_2$. If there exists a longest path $Q_2$ not containing $v_2$, then by Claim 2.30, $v_1, v_3 \in V(Q_2)$. If $w_1 \notin V(Q_2)$, then there exists a vertex $u \in Q[w_1, v_8]$ such that $u \in V(Q_2)$, for otherwise by Claim 2.30, $v_7 \in V(Q_2)$. By the proof of Claim 2.29, $Q_2$ is a path of length at least 10, a contradiction. Thus $w_1 u \in E(Q)$ and $u \in V(Q_2)$. Now we could check that $N_{Q_2}(v_2) \subseteq \{v_1, u\}$. Thus $u = v_3$. By Claim 2.32, $y_{k-1} \notin V(Q_2)$. By the above, $y_{k-2} \in V(Q_2)$. But now $Q_2$ is a path of length at least 10, a contradiction. Thus $w_1 \in V(Q_2)$. By Claim 2.32, $w_1 = v_3$ and $y_{k-1} \notin V(Q_2)$. By the above, $y_{k-2} \in V(Q_2)$. But now we could check that $N_{Q_2}(v_2) \subseteq \{v_3, y_{k-2}\}$, a contradiction. $\square$

Since $G$ is a counterexample, by Claim 2.38, for any longest path $Q$ of $G$, $V(C) \subset V(Q)$. But now all the longest paths of $G$ contain $V(C)$, a contradiction.

**2.7. Proof of the case $r = 9$.** If $r = 9$, then by Claim 2.1 we have that $s \geqslant 9$. Theorem 2.1 is very important for the following proof.

**Theorem 2.1** (Petersen's theorem, [8])**.** *Every bridgeless cubic graph has a perfect matching.*

**Claim 2.39.** *If $Q = y_0 y_1 \ldots y_s$ is a longest path of $G$, then there is no edge in $G \setminus V(Q)$.*

P r o o f.    If there is an edge $e \in G \setminus V(Q)$, then the 5 independent edges in $Q$ together with $e$ are 6 independent edges, a contradiction. $\square$

**Claim 2.40.** *For any $v \in V(G) \setminus V(Q)$, $d_G(v) = 3$.*

P r o o f.    If $d(v) \leqslant 2$, then suppose that $vx \in E(G)$. Now we assume that all the longest paths of $G$ contain $x$, since if there is a longest path $Q_1$ not containing $x$, then by Claim 2.39, $v \in V(Q_1)$. Now $v$ is not the end-vertex of $Q_1$, since otherwise adding $x$ to $Q_1$ results in a longer path, a contradiction. But now $d_G(v) \geqslant 3$, a contradiction.

If $d_G(v) \geqslant 4$, then by Claim 2.3 and $r = 9$, $d_G(v) = 4$. Suppose that $N_G(v) = \{u_1, u_2, u_3, u_4\}$ and $u_1 = y_{j_1}$, $u_2 = y_{j_2}$, $u_3 = y_{j_3}$, $u_4 = y_{j_4}$, $1 \leqslant j_1 < j_2 < j_3 < j_4 \leqslant s - 1$.

If $vu_1Q[u_1, u_4]u_4v$ is a cycle of length 9, then either $u_1w_1w_2u_2w_3u_3w_4u_4$ or $u_1w_1u_2w_2w_3u_3w_4u_4$ or $u_1w_1u_2w_2u_3w_3w_4u_4$ is a subpath of $Q$. If $u_1w_1w_2u_2w_3u_3w_4u_4$ is a subpath of $Q$, then $Q = y_0u_1w_1w_2u_2w_3u_3w_4u_4y_9$, for otherwise we could find 6 independent edges, a contradiction.

We can check that for each vertex $v \in \{w_3, w_4\}$ the following assertions hold:

(i) $v$ is not adjacent to $y_0, y_9$ (since $r = 9$);

(ii) $v$ is not adjacent to $w_1, w_2$ (since otherwise we could obtain a path longer than $Q$);

(iii) $v$ is not adjacent to any vertex in $V(G) \setminus V(Q)$ (since if there exists a vertex $z \in V(G) \setminus V(Q)$ such that $vz \in E(G)$, then $y_0u_1$, $w_1w_2$, $u_2v$, $w_3z$, $u_3w_4$, $u_4y_9$ or $y_0u_1$, $w_1w_2$, $u_2v$, $w_3u_3$, $w_4z$, $u_4y_9$ are 6 independent edges, a contradiction);

(iv) $w_3w_4 \notin E(G)$ (since otherwise $y_0Qw_3w_4u_3vu_4Qy_s$ is a path longer than $Q$, a contradiction);

(v) $N_G(v) \subseteq \{u_1, u_2, u_3, u_4\}$ (since (i), (ii), (iii) and (iv)).

Now we prove that all the longest paths of $G$ contain $u_3$. If there exists a longest path $Q_1$ not containing $u_3$, then by Claim 2.39, $w_3, w_4, v \in V(Q_1)$. Now $w_3, w_4, v$ are not end-vertices of $Q_1$, since otherwise adding $u_3$ to $Q_1$ results in a longer path, a contradiction. Suppose that $u_1vu_2$ is a segment of $Q_1$, then $u_1w_3u_4$, $u_2w_4u_4$ or $u_2w_3u_4$, $u_1w_4u_4$ are two segments of $Q_1$. But now $u_4w_3u_1vu_2w_4u_4$ or $u_4w_4u_1vu_2w_3u_4$ is a segment of $Q_1$, a contradiction. Thus, all the longest paths of $G$ contain $u_3$. Since $G$ is a counterexample, $u_1w_1w_2u_2w_3u_3w_4u_4$ is not a subpath of $Q$. Similarly, $u_1w_1u_2w_2u_3w_3w_4u_4$ is not a subpath of $Q$.

If $u_1w_1u_2w_2w_3u_3w_4u_4$ is a subpath of $Q$, then $Q = y_0u_1w_1u_2w_2w_3u_3w_4u_4y_9$, for otherwise we could find 6 independent edges, a contradiction.

We can check that for each vertex $v \in \{w_1, w_2\}$ the following assertions hold:

(i) $v$ is not adjacent to $y_0$, $y_9$, $w_4$ (since otherwise we could find a path longer than $Q$);

(ii) $v$ is not adjacent to any vertex in $V(G) \setminus V(Q)$ (since if there exists a vertex $z \in V(G) \setminus V(Q)$ such that $vz \in E(G)$, then $zw_1Qu_4vu_1y_0$ or $zw_2Qu_4vu_2Qy_0$ is a path longer than $Q$);

(iii) $w_1$ is not adjacent to $\{w_2, w_3\}$. Since otherwise

$$y_0u_1vu_2w_1w_2Qy_9 \quad \text{or} \quad y_0Qw_1w_3w_2u_2vu_3Qy_9$$

is a path longer than $Q$, a contradiction;

(iv) $N_G(w_1) \subseteq \{u_1, u_2, u_3, u_4\}$ and $N_G(w_2) \subseteq \{w_3, u_1, u_2, u_3, u_4\}$ (since (i), (ii) and (iii)).

Similarly, we could prove that $N_G(w_3) \subseteq \{w_2, u_1, u_2, u_3, u_4\}$ and $N_G(w_4) \subseteq \{u_1, u_2, u_3, u_4\}$.

Now we prove that all the longest paths of $G$ contain $u_2$. If there exists a longest path $Q_2$ not containing $u_2$, then by Claim 2.39, $w_1, w_2, v \in V(Q_2)$. Now $w_1, w_2, v$ are not end-vertices of $Q_2$, since otherwise adding $u_2$ to $Q_2$ results in a longer path, a contradiction. If $u_1 v u_3$ is a segment of $Q_2$, then $u_1 w_1 u_4$ or $u_3 w_1 u_4$ is a segment of $Q_2$. Without loss of generality, suppose that $u_1 w_1 u_4$ is a segment of $Q_2$. Now $u_4 w_2 w_3$ or $u_3 w_2 w_3$ is a segment of $Q_2$. Without loss of generality, suppose that $u_4 w_2 w_3$ is a segment of $Q_1$. Since $N_G(w_3) \subseteq \{w_2, u_1, u_2, u_3, u_4\}$, $w_3$ is an end-vertex of $Q_2$. If $w_4 \in V(Q_2)$, then since $N_G(w_4) \subseteq \{u_1, u_2, u_3, u_4\}$, $Q_2 = w_3 w_2 u_4 w_1 u_1 v u_3 w_4$, a contradiction. Thus $w_4 \notin V(Q_2)$. Now $u_3$ is not an end-vertex of $Q_2$. If $u_3 y_0$ or $u_3 y_s \in E(Q_2)$, then $Q_2 = w_3 w_2 u_4 w_1 u_1 v u_3 y_0$ or $Q_2 = w_3 w_2 u_4 w_1 u_1 v u_3 y_s$, a contradiction. Thus, there exists a vertex $z \in V(G) \setminus V(Q)$ such that $u_3 z \in E(Q_2)$. But now $Q_2 = w_3 w_2 u_4 w_1 u_1 v u_3 z$, a contradiction. Thus $u_1 v u_3$ is not a segment of $Q_2$. Similarly, we could prove that $u_1 v u_4$, $u_3 v u_4$ are not segments of $Q_2$, a contradiction. Thus, all the longest paths of $G$ contain $u_2$. Since $G$ is a counterexample, $v u_1 Q[u_1, u_4] u_4 v$ is a cycle of length 8.

As above, we could prove that $N_G(w_i) \subseteq \{u_1, u_2, u_3, u_4\}$ $(i = 1, 2, 3)$, and all the longest paths of $G$ have a common vertex, a contradiction. Thus $d_G(v) = 3$. $\qquad \square$

**Claim 2.41.** *$G$ has an independent set of 6 edges.*

P r o o f.   Since $G$ is a counterexample, for any vertex $v \in V(G)$ there exists a longest path not containing it. By Claim 2.40, $d_G(v) = 3$. Since $s \geqslant 9$, $G$ has at least 12 vertices. Suppose that $X$ is a connected component in $G \setminus V(C)$. If $|X| \geqslant 4$, then there is a path of length at least 11, and therefore $G$ has 6 independent edges. If $|X| = 3$, then since $G[X]$ is connected, there is a spanning path in $G[X]$. Since $G$ is cubic, there is a path of length at least 11 and $G$ has 6 independent edges. If $|X| \leqslant 2$, then since $G$ is a connected cubic graph, the edges connecting $X$ and $C$ are not cut edges. Thus $G$ is a bridgeless cubic graph. By Theorem 2.1, $G$ has 6 independent edges. $\qquad \square$

By Claim 2.41, $G$ has 6 independent edges, a contradiction.

**2.8. Proof of the case $r = 10$.** If $r = 10$, then by Claim 2.1 we have that $s = 10$.

**Claim 2.42.** *If $Q = y_0 y_1 \ldots y_{10}$ is a longest path of $G$, then there is no edge in $G \setminus V(Q)$.*

P r o o f.   If there is an edge $a \in G \setminus V(Q)$, then 5 independent edges in $Q$ together with $a$ are 6 independent edges, a contradiction. $\qquad \square$

**Claim 2.43.** *For any $v \in V(G) \setminus V(Q)$, $d_G(v) = 3$.*

P r o o f.   Since $s = 10$, $N_G(v) \subseteq \{y_1, y_3, y_5, y_7, y_9\}$. If $d_G(v) \leqslant 2$, then suppose that $vx \in E(G)$. Now we claim that all the longest paths of $G$ contain $x$. Since if there is a longest path $Q_1$ not containing $x$, then by Claim 2.42, $v \in Q_1$. We see that $v$ is not the end-vertex of $Q_1$, since otherwise adding $x$ to $Q_1$ results in a longer path, a contradiction. But now $d_G(v) \geqslant 3$, a contradiction.

If $d_G(v) = 5$, then $y_i y_j \notin E(G)$, $i, j \in \{0, 2, 4, 6, 8, 10\}$ (without loss of generality, suppose that $i < j$), for otherwise $y_0 Q y_i y_j Q y_{i+1} v y_{j+1} Q y_s$ is a path longer than $Q$, a contradiction. Now for the cycle $C_1 = vy_1 y_2 y_3 y_4 y_5 y_6 y_7 y_8 y_9 v$ there is no edge in $G \setminus V(C_1)$ and for each vertex $x \in V(G) \setminus V(C_1)$, $N_G(x) \subseteq \{y_1, y_3, y_5, y_7, y_9\}$. If there is a longest path $Q_2$ of $G$ not containing $y_1$, then $Q_2$ contains at most 9 vertices, a contradiction. Thus, every longest path of $G$ contains $y_1$. Since $G$ is a counterexample, $d_G(v) \neq 5$.

If $d_G(v) = 4$ and $vy_1, vy_9 \in E(G)$, then $vy_1 y_2 y_3 y_4 y_5 y_6 y_7 y_8 y_9 v$ is a cycle of length 10. If $vy_3 \notin E(G)$, then $y_2 y_j \notin E(G)$, $j \in \{6, 8, 0, 10\}$, for otherwise $y_0 y_1 v y_{j-1} Q y_2 y_j Q y_s$ or $y_{10} Q y_2 y_0 y_1 v$ or $y_{10} y_2 Q y_9 v y_1 y_0$ is a path longer than $Q$, a contradiction. Furthermore, $y_i y_j \notin E(G)$, $i, j \in \{0, 4, 6, 8, 10\}$ (without loss of generality, suppose that $i < j$), for otherwise $y_0 Q y_i y_j Q y_{i+1} v y_{j+1} Q y_s$ is a path longer than $Q$, a contradiction. Now for the cycle $C_2 = vy_1 y_2 y_3 y_4 y_5 y_6 y_7 y_8 y_9 v$ there is no edge in $G \setminus V(C_2)$ and for each vertex $x \in V(G) \setminus V(C_2)$, $N_G(x) \subseteq \{y_1, y_3, y_5, y_7, y_9\}$. If $y_2 y_4 \notin E(G)$, then as above, every longest path of $G$ not containing $y_1$ contains at most 9 vertices, a contradiction. Since $G$ is a counterexample, $y_2 y_4 \in E(G)$. Now if there is a longest path $Q_3$ of $G$ not containing $y_1$, then $Q_3$ contains at most 10 vertices, a contradiction. Thus $vy_3 \in E(G)$. Similarly, we could prove that $vy_5, vy_7 \in E(G)$. But now $d_G(v) = 5$, a contradiction. Thus $vy_1 \notin E(G)$ or $vy_9 \notin E(G)$. Now we could obtain that $y_i y_j \notin E(G)$, $i, j \in \{2, 4, 6, 8\}$ (without loss of generality, suppose that $i < j$), for otherwise $y_0 Q y_i y_j Q y_{i+1} v y_{j+1} Q y_s$ is a path longer than $Q$, a contradiction. If $y_0 y_i \in E(G)$ or $y_{10} y_i \in E(G)$, $i \in \{0, 4, 6, 8, 10\}$, then $y_2 y_1 y_0 y_{10} Q y_3 v$ or $y_{10} Q y_i y_0 Q y_{i-1} v$ or $y_{10} y_i Q y_9 v y_{i-1} Q y_0$ is a path longer than $Q$, a contradiction. Thus $y_0 y_i, y_{10} y_i \notin E(G)$, $i \in \{0, 4, 6, 8, 10\}$. Furthermore, $y_2 y_{10} \notin E(G)$, for otherwise $y_0 y_1 y_2 y_{10} Q y_3 v$ is a path longer than $Q$, a contradiction. If $y_4 y_1 \in E(G)$, then $C_3 = y_1 y_4 Q y_9 v y_3 y_2 y_1$ is a cycle of length 10. Now $y_2 y_0 \notin E(G)$, for otherwise $y_0 y_2 y_1 y_4 y_3 v y_5 Q y_s$ is a path longer than $Q$, a contradiction. But now, as above, we could obtain that all the longest paths of $G$ contain $y_3, y_5, y_7, y_9$, a contradiction. Thus $y_4 y_1 \notin E(G)$. Similarly, we could obtain that $y_i y_1 \notin E(G)$, $i \in \{6, 8\}$. Now $N_G(y_i) \subseteq \{y_3, y_5, y_7, y_9\}$, $i \in \{4, 6, 8\}$. If there is a longest path $Q_3$ of $G$ not containing $y_7$, then by Claim 2.42, $y_6, y_8, v \in V(Q_3)$. Now $y_6, y_8, v$ are not end-vertices of $Q_3$. Since otherwise adding $y_7$ to $Q_3$ results in a longer path, a contradiction. If $y_3 v y_5$

367

is a segment of $Q_3$, then $y_3y_6y_9$, $y_5y_8y_9$ or $y_5y_6y_9$, $y_3y_8y_9$ are two segments of $Q_3$. But now $y_9y_6y_3vy_5y_8y_9$ or $y_9y_8y_3vy_5y_6y_9$ is a segment of $Q_3$, a contradiction. Thus, $y_3vy_5$ is not a segment of $Q_3$. Similarly, we could obtain that $y_3vy_9, y_5vy_9 \notin E(G)$, a contradiction. Thus $d_G(v) \neq 4$. Therefore, $d_G(v) = 3$. □

**Claim 2.44.** *G has an independent set of 6 edges.*

P r o o f.    Since $G$ is a counterexample, for any vertex $v \in V(G)$ there exists a longest path not containing it. By Claim 2.43, $d_G(v) = 3$. Since $s \geqslant 10$, $G$ has at least 12 vertices. Suppose that $X$ is a connected component in $G \setminus V(C)$. If $|X| \geqslant 2$, then the 5 independent edges and an edge in $X$ are 6 independent edges, a contradiction. Thus $|X| = 1$. Since $G$ is a connected cubic graph, there are three edges connecting $X$ and $C$. Thus, the edges connecting $X$ and $C$ are not cut edges. Now $G$ is a bridgeless cubic graph. By Theorem 2.1, $G$ has 6 independent edges.    □

By Claim 2.44, $G$ has 6 independent edges, a contradiction. Thus, we complete the proof of Theorem 1.1.    □

P r o o f of Conjecture 1.2.    By Theorem 1.1, Gallai's conjecture is true for every connected graph $G$ with $\alpha'(G) \leqslant 5$. Thus, a smallest counterexample to Gallai's conjecture must have at least 6 independent edges. As the graph in Figure 1 has 12 vertices, we complete the proof.    □

**Acknowledgement.** The author is grateful to the referees for their valuable suggestions in improving this paper.

*References*

[1] *P. Balister, E. Győri, J. Lehel, R. Schelp*: Longest paths in circular arc graphs. Comb. Probab. Comput. *13* (2004), 311–317.    zbl MR doi
[2] *G. Brinkmann, N. Van Cleemput*: Private communication.
[3] *F. Chen*: Nonempty intersection of longest paths in a graph with a small matching number. Czech. Math. J. *65* (2015), 545–553.    zbl MR doi
[4] *G. Chen, J. Ehrenmüller, C. G. Fernandes, C. G. Heise, S. Shan, P. Yang, A. N. Yates*: Nonempty intersection of longest paths in series-parallel graphs. Discrete Math. *340* (2017), 287–304.    zbl MR doi
[5] *S. F. de Rezende, C. G. Fernandes, D. M. Martin, Y. Wakabayashi*: Intersecting longest paths. Discrete Math. *313* (2013), 1401–1408.    zbl MR doi
[6] *T. Gallai*: Problem 4. Theory of Graphs. Proceedings of the colloquium held at Tihany, Hungary, September 1966 (P. Erdös, G. Katona, eds.). Academic Press, New York, 1968.    zbl MR
[7] *S. Klavžar, M. Petkovšek*: Graphs with nonempty intersection of longest paths. Ars Comb. *29* (1990), 43–52.    zbl MR
[8] *J. Petersen*: Die Theorie der regulären graphs. Acta Math. *15* (1891), 193–220. (In German.)    zbl MR doi
[9] *H. Walther*: Über die Nichtexistenz eines Knotenpunktes, durch den alle längsten Wege eines Graphen gehen. J. Comb. Theory *6* (1969), 1–6. (In German.)    zbl MR doi

[10] *H. Walther, H. J. Voss*: Über Kreise in Graphen. VEB Deutscher Verlag der Wissenschaften, Berlin, 1974. (In German.)   zbl

[11] *T. I. Zamfirescu*: L'histoire et l'etat présent des bornes connues pour $P_k^j$, $C_k^j$, $\overline{P}_k^j$ et $\overline{C}_k^j$. Cah. Cent. Étud. Rech. Opér. *17* (1975), 427–439. (In French.)   zbl MR

[12] *T. I. Zamfirescu*: On longest paths and circuits in graphs. Math. Scand. *38* (1976), 211–239.   zbl MR doi

*Author's address*:  F u y u a n  C h e n, Anhui University of Finance and Economics, No. 962 Caoshan Road, Bengbu City, P.R. China, e-mail: `chenfuyuan19871010@163.com`.