Sudhananda Maharana; Jugal K. Prajapat; Deepak Bansal
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GEOMETRIC PROPERTIES OF WRIGHT FUNCTION

Sudhananda Maharana, Jugal K. Prajapat, Kishangarh,
Deepak Bansal, Bikaner

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Abstract. In the present paper, we investigate certain geometric properties and inequalities for the Wright function and mention a few important consequences of our main results. A nonlinear differential equation involving the Wright function is also investigated.

Keywords: analytic function; univalent function; starlike function; strongly starlike function; convex function; close-to-convex function; Wright function; Bessel function; subordination of functions

MSC 2010: 30C45, 33C10

1. Introduction

The entire function (of z)

$$W_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma(\lambda n+\mu)},\quad \lambda>-1,\ \mu\in\mathbb{C}, \tag{1.1}$$

called the Wright function, has appeared for the first time in connection with the partitions of natural numbers, see [28]. Later on, it has been used in the asymptotic theory of partitions, Mikusinski operational calculus, integral transforms and in fractional differential equations (see [10], [13]). The Wright function can be represented in terms of familiar hypergeometric functions (see [10], page 389) and in terms of the Bessel functions J_ν (see [23], page 204).

Also, the Wright function generalizes various functions like array function, Whittaker function, (Wright-type) entire auxiliary functions, etc. The reader is referred to [10], [12] for details and many interesting results on the Wright function.

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Let $\mathcal{A}$ denote the class of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$ having the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.2}$$

By $\mathcal{S}$, we denote the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{D}$. For two analytic functions f and F in $\mathbb{D}$, we say that f is subordinated to F, and express this symbolically by $f(z) \prec F(z)$, if $f(z) = F(w(z))$ in $\mathbb{D}$, for some analytic function w in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$. In particular, if $F \in \mathcal{S}$, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$.

A function $f \in \mathcal{A}$ is called starlike, if $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0,1]$. The class of starlike functions in $\mathcal{A}$ is denoted by $\mathcal{S}^*$. Analytically, a function $f \in \mathcal{A}$ is called starlike if and only if it satisfies $\Re\{zf'(z)/f(z)\} > 0$, $z \in \mathbb{D}$. A function $f \in \mathcal{A}$ which maps $\mathbb{D}$ onto a convex domain is called a convex function and the class of such functions is denoted by $\mathcal{K}$. A function $f \in \mathcal{A}$ is called convex if and only if it satisfies $1 + \Re\{zf''(z)/f'(z)\} > 0$, $z \in \mathbb{D}$. Let $\widetilde{\mathcal{S}}^*(\alpha)$, $0 < \alpha \leqslant 1$ be the class of strongly starlike functions of order α in $\mathbb{D}$, which is defined by

$$\widetilde{\mathcal{S}}^*(\alpha) = \Big\{f \in \mathcal{A}\colon \Big|\arg\Big(\frac{zf'(z)}{f(z)}\Big)\Big| < \frac{\alpha\pi}{2},\ z \in \mathbb{D}\Big\}. \tag{1.3}$$

Note that $\widetilde{\mathcal{S}}^*(1) \equiv \mathcal{S}^*$. Further, a function $f \in \mathcal{A}$ is called close-to-convex in $\mathbb{D}$ if the complement of $f(\mathbb{D})$ can be written as the union of non-intersecting half-lines. A function $f \in \mathcal{A}$ is close-to-convex with respect to a starlike function g, denoted by $\mathcal{C}_g$, if it satisfies $\Re\{zf'(z)/g(z)\} > 0$, $z \in \mathbb{D}$. For more details about these classes one can refer to [7], [9].

In this paper, we consider the following normalized form of the Wright function:

$$\mathbb{W}_{\lambda,\mu}(z) = z\Gamma(\mu)W_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{\Gamma(\mu)z^{n+1}}{n!\,\Gamma(\lambda n + \mu)}, \quad \lambda > -1,\ \mu > 0,\ z \in \mathbb{D}. \tag{1.4}$$

The normalized Wright function $\mathbb{W}_{\lambda,\mu}$ was studied recently by the present author in [23] (see also [17]). Note that

$$\mathbb{W}_{1,\nu+1}(-z) = \mathbb{J}_\nu(z) = \Gamma(\nu+1)z^{1-\nu/2}J_\nu(2\sqrt{z}). \tag{1.5}$$

Here, $\mathbb{J}_\nu(z)$ denotes the normalized Bessel function, investigated recently for the geometric properties in [2], [22], [25]. The function $J_\nu(z)$ is the well known Bessel function, defined by

$$J_\nu(z) = \Big(\frac{z}{2}\Big)^{\nu} W_{1,\nu+1}\Big(\frac{-z^2}{4}\Big) = \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2}z)^{2n+\nu}}{n!\,\Gamma(n+\nu+1)}. \tag{1.6}$$

The special functions play an important role in function theory, especially the hypergeometric function, which appeared in De-Branges' solution of the famous Bieberbach conjecture (see [6]). Several researchers studied classes of analytic functions involving special functions $\mathcal{F} \subset \mathcal{A}$, to find different conditions such that the members of $\mathcal{F}$ to have certain geometric properties such as univalency, starlikeness or convexity in $\mathbb{D}$. In this context many results are available in the literature regarding the hypergeometric functions (see [14], [24], [21], [20]), normalized Bessel functions (see [2], [4], [22], [25]), generalized Bessel functions (see [3], [16]), generalized Struve functions (see [30], [31]), Lommel functions (see [29]), Wright functions (see [23]) and Mittag-Leffler function (see [1]). In this paper, our main aim is to examine the geometric properties and inequalities of the Wright function $\mathbb{W}_{\lambda,\mu}$. We also investigate an initial value problem involving the Wright function.

2. Close-to-convexity and starlikeness of $\mathbb{W}_{\lambda,\mu}$

In this section we obtain certain sufficient conditions for close-to-convexity and starlikeness of $\mathbb{W}_{\lambda,\mu}$ in $\mathbb{D}$. To prove our results, we shall need the following known results.

Lemma 2.1 (Fejér [8])**.** *Let $f \in \mathcal{A}$ be of the form* (1.2) *with $a_n \geqslant 0$. If the sequences $\{na_n\}$ and $\{na_n - (n+1)a_{n+1}\}$ are non-increasing, then f is starlike in $\mathbb{D}$.*

Lemma 2.2 (Ozaki [18])**.** *Let $f \in \mathcal{A}$ be of the form* (1.2)*. If*

$$1 \geqslant 2a_2 \geqslant \ldots \geqslant na_n \geqslant (n+1)a_{n+1} \ldots \geqslant 0$$

or

$$1 \leqslant 2a_2 \leqslant \ldots \leqslant na_n \leqslant (n+1)a_{n+1} \ldots \leqslant 2,$$

then f is close-to-convex with respect to $g(z) = z/(1-z)$.

Lemma 2.3 (Halenbeck and Ruscheweyh [11])**.** *Let $G(z)$ be convex and univalent in $\mathbb{D}$ and $F(z)$ be analytic in $\mathbb{D}$ with $G(0) = F(0) = 1$. If $F(z) \prec G(z)$ in $\mathbb{D}$, then*

$$(n+1)z^{-n-1}\int_0^z t^n F(t)\,\mathrm{d}t \prec (n+1)z^{-n-1}\int_0^z t^n G(t)\,\mathrm{d}t, \quad n \in \mathbb{N} \cup \{0\}.$$

Our first result is given below by Theorem 2.1:

Theorem 2.1. *Let $\lambda \geqslant 1$ and $\mu \geqslant 1$. If $\Gamma(\lambda+\mu) \geqslant 2\Gamma(\mu)$, then $\mathbb{W}_{\lambda,\mu}$ is close-to-convex with respect to $g(z) = z/(1-z)$.*

P r o o f. By using Lemma 2.2, it is sufficient to show that

$$1 \geqslant 2a_2 \geqslant \ldots \geqslant na_n \geqslant (n+1)a_{n+1} \ldots \geqslant 0. \tag{2.1}$$

From (1.4), we have

$$\begin{aligned} na_n - (n+1)a_{n+1} &= na_n - \frac{(n+1)\Gamma(\lambda(n-1)+\mu)}{n\Gamma(\lambda n+\mu)} a_n \\ &= \frac{a_n}{n\Gamma(\lambda n+\mu)} (n^2\Gamma(\lambda n+\mu) - (n+1)\Gamma(\lambda(n-1)+\mu)) \\ &= \frac{a_n}{n\Gamma(\lambda n+\mu)} X(n), \end{aligned}$$

where

$$X(n) = n^2\Gamma(\lambda n+\mu) - (n+1)\Gamma(\lambda(n-1)+\mu).$$

Under the hypothesis, it is clear that

$$\begin{aligned} n^2\Gamma(\lambda n+\mu) = n^2\Gamma(\lambda(n-1)+\lambda+\mu) &\geqslant n^2\Gamma(\lambda(n-1)+1+\mu) \\ &= n^2(\lambda(n-1)+\mu)\Gamma(\lambda(n-1)+\mu) \\ &\geqslant (n+1)\Gamma(\lambda(n-1)+\mu), \quad n \in \mathbb{N} \setminus \{1\}. \end{aligned}$$

Also, $X(1) \geqslant 0$ and $\Gamma(\lambda n+\mu) \geqslant \Gamma(\lambda(n-1)+\mu)$, $n \geqslant 2$. Hence $X(n) \geqslant 0$ for all $n \geqslant 1$. This shows that the inequality (2.1) holds. This completes the proof. □

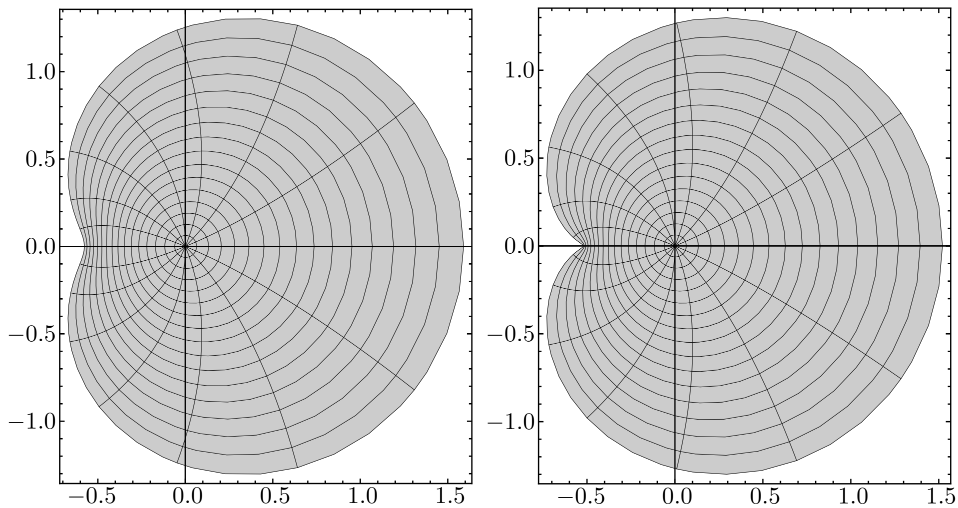
Taking $\lambda = 1$, $\mu = \nu + 1$ $(\nu > -1)$ and replacing z by $-z$ in Theorem 2.1, we get the following result:

Corollary 2.1. *If $\nu \geqslant 1$, then $\mathbb{J}_\nu$ is close-to-convex in $\mathbb{D}$ with respect to $g(z) = z/(1-z)$.*

E x a m p l e 2.1. Taking $\lambda = 1$ in Theorem 2.1, we obtain that the function $\mathbb{W}_{1,\mu}$ is close-to-convex for $\mu \geqslant 2$. Also, we obtain that the function $\mathbb{W}_{2,\mu}$ is close-to-convex for $\mu \geqslant 1$. In particular, functions $\mathbb{W}_{1,2}$ and $\mathbb{W}_{2,1}$ are close to convex and their image domains under $\mathbb{D}$ are given below in Figures (a) and (b).

Theorem 2.2. *Let $\lambda \geqslant 1$ and $\mu \geqslant 1$. If $\Gamma(\lambda+\mu) \geqslant 4\Gamma(\mu)$, then $\mathbb{W}_{\lambda,\mu}$ is starlike in $\mathbb{D}$.*

P r o o f. In view of Lemma 2.1, it is sufficient to prove that $\{na_n\}$ and $\{na_n -$



(a) Mapping of $\mathbb{W}_{1,2}(z)$ over $\mathbb{D}$ (b) Mapping of $\mathbb{W}_{2,1}(z)$ over $\mathbb{D}$

$(n+1)a_{n+1}\}$ are non-increasing sequences for all $n \geqslant 1$. Clearly, the sequence $\{na_n\}$ is non-increasing by Theorem 2.1. Therefore, it suffices to show that

$$na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2} \geqslant 0 \quad \forall n \geqslant 1. \tag{2.2}$$

Under the hypothesis, we have

$$\begin{aligned} n^2\Gamma(\lambda n + \mu) \geqslant n^2\Gamma(\lambda(n-1)+\mu+1) &= n^2(\lambda(n-1)+\mu)\Gamma(\lambda(n-1)+\mu) \\ &\geqslant 2(n+1)\Gamma(\lambda(n-1)+\mu), \quad n \in \mathbb{N}. \end{aligned}$$

Hence

$$\begin{aligned} &na_n - 2(n+1)a_{n+1} + (n+2)a_{n+2} \\ &\quad = \frac{n\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)(n-1)!} - \frac{2(n+1)\Gamma(\mu)}{\Gamma(\lambda n+\mu)n!} + \frac{(n+2)\Gamma(\mu)}{\Gamma(\lambda(n+1)+\mu)(n+1)!} \\ &\quad = \frac{\Gamma(\mu)}{(n-1)!}\left(\frac{n}{\Gamma(\lambda(n-1)+\mu)} - \frac{2(n+1)}{\Gamma(\lambda n+\mu)n} + \frac{(n+2)}{\Gamma(\lambda(n+1)+\mu)(n+1)n}\right) \geqslant 0. \end{aligned}$$

This shows that the inequality (2.2) holds, hence $\mathbb{W}_{\lambda,\mu}(z)$ is starlike in $\mathbb{D}$. □

Taking $\lambda = 1$, $\mu = \nu + 1$, $\nu > -1$ and replacing z by $-z$ in Theorem 2.2, we get the following result:

Corollary 2.2. *If $\nu \geqslant 3$, then $\mathbb{J}_\nu$ is starlike in $\mathbb{D}$.*

E x a m p l e 2.2. Taking $\lambda = 1$ in Theorem 2.2, we obtain that the function $\mathbb{W}_{1,\mu}$ is starlike for $\mu \geqslant 4$. Also, we obtain that the function $\mathbb{W}_{2,\mu}$ is starlike for $\mu \geqslant \frac{1}{2}(-1+\sqrt{17})$. Further, we observe that, as λ increases, μ decreases to preserve the starlikeness of the function $\mathbb{W}_{\lambda,\mu}$.

Theorem 2.3. *If $\lambda \geqslant 1$ and $\mu \geqslant 1+\sqrt{3}$, then $\mathbb{W}_{\lambda,\mu} \in \widetilde{\mathcal{S}}^*(\alpha)$. Here α is given by*

$$\alpha = \frac{2}{\pi} \arcsin\Big(\eta\sqrt{1-\tfrac{1}{4}\eta^2} + \tfrac{1}{2}\eta\sqrt{1-\eta^2}\Big), \tag{2.3}$$

where $\eta = 2(\mu+1)/\mu^2$.

P r o o f. Under the hypothesis, the inequality $\Gamma(\mu+n) \leqslant \Gamma(\lambda n+\mu)$, $n \in \mathbb{N}$ holds and is equivalent to

$$\frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)} \leqslant \frac{\Gamma(\mu)}{\Gamma(n+\mu)} = \frac{1}{(\mu)_n}, \quad n \in \mathbb{N}, \tag{2.4}$$

where $(x)_n$ is the well known Pochhammer symbol defined by

$$(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)\dots(x+n-1), & n \in \mathbb{N}. \end{cases}$$

For $n \in \mathbb{N}$, we have

$$(x)_n = x(x+1)_{n-1}, \quad x^n \leqslant (x)_n. \tag{2.5}$$

Using (2.5) in (2.4), we have

$$\begin{aligned} |\mathbb{W}'_{\lambda,\mu}(z) - 1| &\leqslant \sum_{n=1}^{\infty} \frac{(n+1)\Gamma(\mu)}{n!\,\Gamma(\lambda n+\mu)}|z|^n < \frac{1}{\mu}\sum_{n=1}^{\infty} \frac{n+1}{n!}\frac{1}{(\mu)_{n-1}} \\ &= \frac{1}{\mu}\Big(\sum_{n=0}^{\infty}\frac{n}{n!}\frac{1}{(\mu)_n} + \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{(\mu)_n}\Big) \leqslant \frac{1}{\mu}\Big(\sum_{n=0}^{\infty}\frac{1}{(\mu)_n} + \sum_{n=0}^{\infty}\frac{1}{(\mu)_n}\Big) \\ &< \frac{2}{\mu}\sum_{n=0}^{\infty}\frac{1}{(\mu+1)^n} = \frac{2(\mu+1)}{\mu^2} = \eta. \end{aligned} \tag{2.6}$$

Note that under the hypothesis $0 < \eta \leqslant 1$. From (2.6), we conclude that $\mathbb{W}'_{\lambda,\mu}(z) \prec 1+\eta z$, $z \in \mathbb{D}$, which implies that

$$|\arg(\mathbb{W}'_{\lambda,\mu}(z))| < \arcsin\eta, \quad z \in \mathbb{D}. \tag{2.7}$$

Using Lemma 2.3, for $F(z) = \mathbb{W}'_{\lambda,\mu}(z)$, $G(z) = 1 + \eta z$ and $n = 0$, we obtain $\mathbb{W}_{\lambda,\mu}(z)/z \prec 1 + \frac{1}{2}\eta z$, $z \in \mathbb{D}$, and consequently

$$\left|\arg\left(\frac{\mathbb{W}_{\lambda,\mu}(z)}{z}\right)\right| < \arcsin\frac{\eta}{2}, \quad z \in \mathbb{D}. \tag{2.8}$$

Now from (2.7) and (2.8), we conclude that

$$\begin{aligned}\left|\arg\left(\frac{z\mathbb{W}'_{\lambda,\mu}(z)}{\mathbb{W}_{\lambda,\mu}(z)}\right)\right| &= \left|\arg\left(\frac{z}{\mathbb{W}_{\lambda,\mu}(z)}\right) + \arg(\mathbb{W}'_{\lambda,\mu}(z))\right| \\ &\leqslant \left|\arg\left(\frac{z}{\mathbb{W}_{\lambda,\mu}(z)}\right)\right| + |\arg(\mathbb{W}'_{\lambda,\mu}(z))| \\ &< \arcsin\frac{\eta}{2} + \arcsin\eta \\ &= \arcsin\left(\eta\sqrt{1-\tfrac{1}{4}\eta^2} + \tfrac{1}{2}\eta\sqrt{1-\eta^2}\right),\end{aligned}$$

i.e., $\mathbb{W}_{\lambda,\mu}(z) \in \widetilde{\mathcal{S}}^*(\alpha)$ for α given by (2.3). □

Corollary 2.3. *Let* $\lambda \geqslant 1$ *and* $\mu \geqslant 1 + \sqrt{3}$. *If* $0 < \alpha \leqslant 1$ *and*

$$\eta = \frac{2(\mu+1)}{\mu^2} = 2\nu\sqrt{\frac{5-4\sqrt{1-\nu^2}}{16\nu^2+9}}, \tag{2.9}$$

where $\nu = \sin\left(\frac{1}{2}\alpha\pi\right)$, *then* $\mathbb{W}_{\lambda,\mu} \in \widetilde{\mathcal{S}}^*(\alpha)$.

P r o o f. If we substitute the value of η from (2.9) to (2.3), we obtain the required α. Hence the result. □

Taking $\alpha = 1$ in Corollary 2.3, we get

$$\nu = 1 \Rightarrow \eta = \frac{2(\mu+1)}{\mu^2} = \frac{2}{\sqrt{5}} \Rightarrow \frac{\mu+1}{\mu^2} = \frac{1}{\sqrt{5}}.$$

Hence, we get the following result:

Corollary 2.4. *Let* $\lambda \geqslant 1$ *and* $\mu = \mu^*$, *where* μ^* *is the positive root of* $\mu^2 - \sqrt{5}\mu - \sqrt{5} = 0$. *Then* $\mathbb{W}_{\lambda,\mu}$ *is starlike in* $\mathbb{D}$.

3. A nonlinear differential equation

In this section, we aim to study a nonlinear differential equation involving the Wright function. For this, we shall need the following lemmas:

Lemma 3.1 (Miller and Mocanu [15]). *Let $\Omega \subset \mathbb{C}$. Suppose that the function $\psi(z)\colon \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ satisfies the condition $\psi(M\mathrm{e}^{\mathrm{i}\theta}, K\mathrm{e}^{\mathrm{i}\theta}; z) \notin \Omega$ for all $K \geqslant M(M-|a|)/(M+|a|)$, $\theta \in \mathbb{R}$ and $z \in \mathbb{D}$. Let $p(z)$ be an analytic function of the form*

$$p(z) = a + a_1 z + a_2 z^2 + \ldots, \quad z \in \mathbb{D}, \tag{3.1}$$

such that $\psi(p(z), zp'(z); z) \in \Omega$ for all $z \in \mathbb{D}$. Then $|p(z)| < M$, where $0 \leqslant |a| < M$.

Lemma 3.2 (Tuneski [26]). *If $f \in \mathcal{A}$ and $|f''(z)| \leqslant 1$, $z \in \mathbb{D}$, then f is starlike in $\mathbb{D}$.*

Theorem 3.1. *For all $\lambda > -1$ and $\mu > 0$, let $\mathbb{W}_{\lambda,\mu}(z)$ satisfy the inequality*

$$|z\mathbb{W}_{\lambda,\mu}(z)| < \frac{M(M-|a|)}{(M+1)(M+|a|)}, \quad 0 \leqslant |a| < M \leqslant 1;\ z \in \mathbb{D}. \tag{3.2}$$

Let φ be the (unique) solution of the initial value problem

$$\varphi^{(n+1)}(z) + \mathbb{W}_{\lambda,\mu}(z)\varphi^{(n)}(z) = \mathbb{W}_{\lambda,\mu}(z), \quad z \in \mathbb{D}$$
$$(n \in \mathbb{N} \cup \{0\},\ \varphi(0) = 0,\ \varphi'(0) = 1,\ \varphi^{(k)}(0) = 0,\ k = 2, \ldots, n-1,\ \varphi^{(n)}(0) = a),$$

where $\varphi^{(n)}$ denotes the nth derivative with respect to z. Then the inequality $|\varphi^{(n)}(z)| < M$, $z \in \mathbb{D}$ holds.

P r o o f. Let the function $p(z)$ be defined by $p(z) = \varphi^{(n)}(z)$, $z \in \mathbb{D}$. Note that $p(z)$ has the form (3.1), and then it follows that

$$\frac{zp'(z)}{1+p(z)} = \frac{z\varphi^{(n+1)}(z)}{1+\varphi^{(n)}(z)} = z\mathbb{W}_{\lambda,\mu}(z), \quad z \in \mathbb{D};\ \varphi^{(n)}(z) \neq -1.$$

We denote $\psi(r, s; z)$ and Ω by

$$\psi(r, s; z) := \frac{s}{1+r}, \quad r \neq -1$$

and

$$\Omega := \Big\{w \in \mathbb{C}\colon\ |w| < \frac{M(M-|a|)}{(M+1)(M+|a|)},\ 0 \leqslant |a| < M \leqslant 1\Big\},$$

respectively. Then clearly

$$\psi(p(z), zp'(z); z) = \frac{zp'(z)}{1+p(z)} = \frac{z\varphi^{(n+1)}(z)}{1+\varphi^{(n)}(z)} \in \Omega, \quad z \in \mathbb{D}.$$

Further, for any $\theta \in \mathbb{R}$, $K \geqslant M(M-|a|)/(M+|a|)$ and $z \in \mathbb{D}$, we have

$$|\psi(Me^{i\theta}, Ke^{i\theta}; z)| = \left|\frac{Ke^{i\theta}}{1+Me^{i\theta}}\right| \geqslant \frac{M(M-|a|)}{(M+1)(M+|a|)},$$

which gives that

$$\psi(Me^{i\theta}, Ke^{i\theta}; z) \notin \Omega.$$

Therefore, in view of Lemma 3.1 it follows that

$$|p(z)| = |\varphi^{(n)}(z)| < M, \quad z \in \mathbb{D};\ 0 \leqslant |a| < M \leqslant 1.$$

This completes the proof. □

By taking $n = 2$ in the above theorem we get the following result:

Corollary 3.1. *For all $\lambda > -1$ and $\mu > 0$, let $\mathbb{W}_{\lambda,\mu}(z)$ satisfy the inequality* (3.2) *in $\mathbb{D}$. Let φ be the (unique) solution of the initial value problem given by*

$$\varphi'''(z) + \mathbb{W}_{\lambda,\mu}(z)\varphi''(z) = \mathbb{W}_{\lambda,\mu}(z), \quad z \in \mathbb{D} \tag{3.3}$$
$$(\varphi(0) = 0,\ \varphi'(0) = 1,\ \varphi''(0) = a).$$

Then the inequality $|\varphi''(z)| < M$, $0 \leqslant |a| < M \leqslant 1$ holds.

Corollary 3.2. *If $\mathbb{W}_{\lambda,\mu}(z)$ satisfies the inequality*

$$|z\mathbb{W}_{\lambda,\mu}(z)| < \frac{1-|a|}{2(1+|a|)}, \quad 0 \leqslant |a| < 1$$

and the function $\varphi(z)$ is the (unique) solution of the initial value problem given by (3.3), *then φ is starlike in $\mathbb{D}$.*

P r o o f. The proof can be obtained easily by taking $M = 1$ in Corollary 3.1 and then using Lemma 3.2. □

4. Inequalities

The following result by Fejér will be needed in this section.

Lemma 4.1 (Fejér [8]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_1 = 1$. If the sequence $\{a_n\}$ is convex decreasing, i.e., $0 \geqslant a_{n+2} - a_{n+1} \geqslant a_{n+1} - a_n$ for all $n \in \mathbb{N} \setminus \{1\}$, then*

$$\Re\Big(\sum_{n=1}^{\infty} a_n z^{n-1}\Big) > \frac{1}{2}, \quad z \in \mathbb{D}.$$

The convex hull of $\mathcal{K}$, denoted by $\overline{\mathrm{co}}\,\mathcal{K}$, is the set of all convex combinations of functions belonging to $\mathcal{K}$. We recall from [5] that the closure of the set $\overline{\mathrm{co}}\,\mathcal{K}$ is

$$\overline{\mathrm{co}}\,\mathcal{K} = \Big\{f \in \mathcal{A}\colon \Re\Big(\frac{f(z)}{z}\Big) > \frac{1}{2},\ z \in \mathbb{D}\Big\}. \tag{4.1}$$

It is well known (see [27]) that a sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence for the class $\mathcal{X} \subset \mathcal{A}$, whenever we have

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec \sum_{n=1}^{\infty} a_n z^n, \quad z \in \mathbb{D} \tag{4.2}$$

for all $\sum\limits_{n=1}^{\infty} a_n z^n \in \mathcal{X}$.

Lemma 4.2 (Piejko and Sokół [19]). *The function of the form* (1.2) *is in the set $\overline{\mathrm{co}}\,\mathcal{K}$ if and only if $a_2, a_3, \ldots$ is a subordinating factor sequence for the class $\mathcal{K}$.*

Theorem 4.1. *For each $\lambda \geqslant 1$ and $\mu \geqslant 1$, we have*

$$|\mathbb{W}_{\lambda,\mu}(z)| \leqslant r\,{}_0F_1(-;\mu;r),$$

where ${}_0F_1(-;\mu;r)$ is the well known hypergeometric function and $|z| = r < 1$.

Proof. Using (2.4), we get

$$\begin{aligned} |\mathbb{W}_{\lambda,\mu}(z)| &\leqslant |z| + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)|z|^n}{\Gamma((n-1)\lambda+\mu)(n-1)!} \\ &\leqslant |z| + \sum_{n=1}^{\infty} \frac{|z|^{n+1}}{(\mu)_n n!} = r\,{}_0F_1(-;\mu;r). \end{aligned} \tag{4.3}$$

This proves the result. □

By using (1.4) and Theorem 4.1, we get the following result:

Corollary 4.1. *For each* $\lambda \geqslant 1$ *and* $\mu \geqslant 1$, *we have* $|W_{\lambda,\mu}(z)| \leqslant {}_0F_1(-;\mu;r)/\Gamma(\mu)$, $|z| = r < 1$.

Theorem 4.2. *Let* $\lambda \geqslant 1$ *and* $\mu \geqslant 1$. *If* $\Gamma(\lambda+\mu) \geqslant 2\Gamma(\mu)$, *then*

$$\Re\Big(\frac{\mathbb{W}_{\lambda,\mu}(z)}{z}\Big) > \frac{1}{2}, \quad z \in \mathbb{D}. \tag{4.4}$$

Proof. Under the hypothesis, the inequality

$$n!\,\Gamma(\lambda n+\mu) \geqslant (n-1)!\,\Gamma(\lambda(n-1)+\mu)$$

holds, which is equivalent to

$$\frac{1}{\Gamma(\lambda(n-1)+\mu)(n-1)!} \geqslant \frac{1}{\Gamma(\lambda n+\mu)n!}. \tag{4.5}$$

Now we need to show that

$$\{a_n\}_{n=1}^{\infty} = \Big\{\frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)(n-1)!}\Big\}_{n=1}^{\infty}$$

is a convex decreasing sequence. We observe that

$$\begin{aligned} &a_{n+2} - 2a_{n+1} + a_n \\ &\quad= \frac{\Gamma(\mu)}{\Gamma(\lambda(n+1)+\mu)(n+1)!} - \frac{2\Gamma(\mu)}{\Gamma(\lambda n+\mu)n!} + \frac{\Gamma(\mu)}{\Gamma(\lambda(n-1)+\mu)(n-1)!} \geqslant 0, \end{aligned}$$

which shows that $\{a_n\}_{n=1}^{\infty}$ is a convex decreasing sequence. Now applying Lemma 4.1, we get

$$\Re\Big\{\sum_{n=1}^{\infty} a_n z^{n-1}\Big\} > \frac{1}{2}, \quad z \in \mathbb{D}$$

which is equivalent to (4.4). This proves the result. □

Proceeding similarly as in Theorem 4.2, we get the following result:

Theorem 4.3. *Let* $\lambda \geqslant 1$ *and* $\mu \geqslant 1$. *If* $\Gamma(\lambda+\mu) \geqslant 4\Gamma(\mu)$, *then*

$$\Re\{\mathbb{W}'_{\lambda,\mu}(z)\} > \frac{1}{2}, \quad z \in \mathbb{D}. \tag{4.6}$$

Corollary 4.2. *Let $\lambda \geqslant 1$ and $\mu \geqslant 1$. If $\Gamma(\lambda+\mu) \geqslant 2\Gamma(\mu)$, then the sequence*

$$\left\{\frac{\Gamma(\mu)}{\Gamma(\lambda n+\mu)n!}\right\}_{n=1}^{\infty} \tag{4.7}$$

is a subordinating sequence for the class $\mathcal{K}$.

Proof. By using (4.1) and (4.4), we have $\mathbb{W}_{\lambda,\mu}(z) \in \overline{\text{co}}\,\mathcal{K}$. □

Now applying Lemma 4.2, we get the desired result.

Corollary 4.3. *Let $\lambda \geqslant 1$ and $\mu \geqslant 1$. If $\Gamma(\lambda+\mu) \geqslant 4\Gamma(\mu)$, then the sequence*

$$\left\{\frac{(n+1)\Gamma(\mu)}{\Gamma(\lambda n+\mu)n!}\right\}_{n=1}^{\infty} \tag{4.8}$$

is a subordinating sequence for the class $\mathcal{K}$.

Proof. By using (4.1) and (4.6), we have $z\mathbb{W}'_{\lambda,\mu}(z) \in \overline{\text{co}}\,\mathcal{K}$. Now applying Lemma 4.2, we get the desired result. □

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Authors' addresses: *Sudhananda Maharana*, *Jugal K. Prajapat*, Department of Mathematics, Central University of Rajasthan, Bandarsindri, Kishangarh-305817, Dist.-Ajmer, Rajasthan, India, e-mail: `snmmath@gmail.com`, `jkprajapat@gmail.com`; *Deepak Bansal*, Department of Mathematics, College of Engineering and Technology, Karni Industrial Area, Pugal Road, Bikaner-334004, Rajasthan, India, e-mail: `deepakbansal_79@yahoo.com`.