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ON OSCILLATORY NONLINEAR FOURTH-ORDER DIFFERENCE EQUATIONS WITH DELAYS

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Abstract. In this work, oscillatory behaviour of solutions of a class of fourth-order neutral functional difference equations of the form

$$\Delta^2(r(n)\Delta^2(y(n)+p(n)y(n-m)))+q(n)G(y(n-k))=0$$

is studied under the assumption

$$\sum_{n=0}^{\infty}\frac{n}{r(n)}<\infty.$$

New oscillation criteria have been established which generalize some of the existing results in the literature.

Keywords: oscillation; nonlinear; delay; neutral functional difference equation

MSC 2010: 39A10, 39A12

1. Introduction

The study of the behaviour of solutions of functional difference equations is a major area of research and is fast growing due to the development of time scales and the time-scale calculus (see, e.g., [3], [4]). Most papers on higher-order nonlinear neutral equations deal with the existence of positive solutions and the asymptotic behaviour of solutions. However, not much attention has been given to oscillation results. We refer the reader to some of the works [2], [5], [6], [9], [11], [12], [13] and the references cited therein.

In [13], the present author has studied the oscillatory and asymptotic behaviour of solutions of

$$\Delta^2(r(n)\Delta^2(y(n)+p(n)y(n-\tau)))+q(n)G(y(n-\sigma))=0 \tag{1.1}$$

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and

$$\Delta^2(r(n)\Delta^2(y(n)+p(n)y(n-\tau)))+q(n)G(y(n-\sigma))=f(n), \tag{1.2}$$

where Δ is the forward difference operator defined by $\Delta y(n)=y(n+1)-y(n)$, r, p, q and f are real-valued functions defined on $N(n_0)=\{n_0,n_0+1,\ldots\}$, $n_0\geqslant 0$, such that $r(n)>0$, $q(n)>0$ for $n\geqslant n_0$, $G\in C(\mathbb{R},\mathbb{R})$ is nondecreasing and τ, σ are positive constants, under the assumption that

$$\sum_{n=0}^{\infty}\frac{n}{r(n)}<\infty. \tag{A$_0$}$$

In [8], Migda has discussed the asymptotic properties of nonoscillatory solutions of neutral difference equations of the form

$$\Delta^m(x_n+p_nx_{n-\tau})+f(n,x_{\sigma(n)})=h_n \tag{1.3}$$

and has shown that any nonoscillatory solution x_n has the property $x_n=cn^{m-1}+o(n^{m-1})$ for some $c\in\mathbb{R}$. For $m=4$, $f(n,x_{\sigma(n)})=q(n)G(x_{\sigma(n)})$ and $\sigma(n)=n-\sigma$, (1.3) reduces to (1.2) for $r(n)\equiv 1$ and hence the papers [13] and [8] are comparable. But, more emphasis may be given to [13], which deals with the oscillatory, nonoscillatory and asymptotic characters. It has been established that (1.2) is oscillatory under a suitable forcing function $f(n)$, whereas (1.1) is oscillatory only when $p(n)\geqslant 0$. In the case $p(n)\leqslant 0$, the solution of (1.1) either oscillates or converges to zero as $n\to\infty$. The objective of this work is to study the oscillatory behaviour of solutions of functional difference equations (1.1) under the assumption (A$_0$) with different ranges of $p(n)$. Some oscillation criteria have been established by applying the discrete Taylor series [1].

The motivation of the present work has come from two directions. First is due to [13] and [14], and the second is due to [10]. Indeed, Parhi and Tripathy in [10] have discussed the oscillatory and asymptotic behaviour of solutions of

$$\Delta^m(y(n)+p(n)y(n-\tau))+q(n)G(y(n-\sigma))=0,\quad m\geqslant 2$$

and they have used the criterion that any higher-order difference equation can be converted into a first-order difference inequality.

By a solution of (1.1) we mean a real-valued sequence defined for $n\geqslant n_0-\varrho$ which satisfies (1.1) for $n\geqslant n_0$, where $\varrho=\max\{\tau,\sigma\}$. If

$$y(n)=\varphi(n),\quad n=n_0-\varrho,n_0-\varrho+1,\ldots,0,1,2,\ldots \tag{1.4}$$

are given, then (1.1) admits a unique solution satisfying the initial conditions (1.4). A solution $y(n)$ of (1.1) is said to be oscillatory if for every integer $N > 0$, there exists an $n \geqslant N$ such that $y(n)y(n+1) \leqslant 0$. Otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory, if all its solutions are oscillatory.

We need the following lemma for our use in the next discussion.

Lemma 1.1 ([7]). *If $p(n) > 0$ for all $n \geqslant n_0 \geqslant 0$ and*

$$\liminf_{n\to\infty} \sum_{j=n-k}^{n-1} p(j) > \frac{k^{k+1}}{(k+1)^{k+1}},$$

then $\Delta x(n) + p(n)x(n-k) \leqslant 0$ ($\geqslant 0$), $n \geqslant n_0 \geqslant 0$ cannot have an eventually positive (negative) solution.

2. Oscillation criteria

In this section, new oscillation criteria for (1.1) will be established. We define the quasi-difference operators as follows:

$L_1u(n) = \Delta L_0 u(n) = \Delta u(n)$,
$L_2u(n) = r(n)\Delta L_1 u(n)$,
$L_3u(n) = \Delta L_2 u(n)$,
$L_4u(n) = \Delta L_3 u(n)$.

We need the following lemmas for our use in the sequel.

Lemma 2.1 ([13]). *Let* (A_0) *hold. Let u be a real-valued function such that $L_4u(n) \leqslant 0$ for large n. If $u(n) > 0$ ultimately, then one of cases* (a)–(d) *holds for large n, and if $u(n) < 0$ ultimately, then one of cases* (b)–(f) *holds for large n, where*

(a) $L_1u(n) > 0$, $L_2u(n) > 0$ *and* $L_3u(n) > 0$,
(b) $L_1u(n) > 0$, $L_2u(n) < 0$ *and* $L_3u(n) > 0$,
(c) $L_1u(n) > 0$, $L_2u(n) < 0$ *and* $L_3u(n) < 0$,
(d) $L_1u(n) < 0$, $L_2u(n) > 0$ *and* $L_3u(n) > 0$,
(e) $L_1u(n) < 0$, $L_2u(n) < 0$ *and* $L_3u(n) > 0$,
(f) $L_1u(n) < 0$, $L_2u(n) < 0$ *and* $L_3u(n) < 0$.

Lemma 2.2 ([13]). *Let the conditions of Lemma* 2.1 *hold. If $u(n) > 0$ ultimately, then there exist constants $C_1 > 0$ and $C_2 > 0$ such that $C_1R(n) \leqslant u(n) \leqslant nC_2$ for large n, where*

$$R(n) = \sum_{s=n}^{\infty} \frac{s-n}{r(s)}.$$

Before stating our main results, we have the following notations:

$$D[k,m] = \sum_{l=m}^{k-2} \frac{(k-l-1)(l-m)}{r(l)},$$
$$E[k,m] = \sum_{l=m}^{k-1} \frac{(k-l-1)(l+1-m)}{r(l)},$$
$$F[k,m] = \sum_{l=m}^{k-1} \frac{k-l-1}{r(l)},$$
$$M[k,m] = \sum_{l=m}^{k-1} \frac{(l+1-m)(l-m)}{r(l)}.$$

Theorem 2.3. *Let* $0 \leqslant p(n) \leqslant d < \infty$ *and* $\sigma \geqslant 2\tau$. *If* (A_0) *and*

(A_1) $$\frac{G(u)}{u} \geqslant \beta > 0, \quad u \neq 0,\ u \in \mathcal{R},$$

(A_2) $$G(uv) \geqslant G(u)G(v), \quad G(-u) = -G(u), \quad u, v \in \mathcal{R},\ u, v > 0,$$

(A_3) *there exists* $\lambda > 0$ *such that* $G(u) + G(v) \geqslant \lambda G(u+v)$ *for* $u, v \in \mathcal{R}$ *and* $u, v > 0$,

(A_4) $$Q(n) = \min\{q(n), q(n-\tau)\}, \quad n \geqslant \tau,$$

(A_5) $$\limsup_{k\to\infty} \sum_{j=k-\tau}^{k} Q(j)G(D[j-\sigma, k-\sigma]) > \frac{1+G(d)}{\lambda\beta},$$

(A_6) $$\limsup_{k\to\infty} \sum_{j=k-\tau}^{k} Q(j)G(E[j-\sigma, k-\sigma]) > \frac{1+G(d)}{\lambda\beta},$$

(A_7) $$\limsup_{m\to\infty} \sum_{j=m-\sigma-2}^{m-\sigma-1} Q(j)G(M[j-\sigma, k-\sigma]) > \frac{1+G(d)}{\lambda\beta}$$

(A_8) $$\limsup_{j\to\infty} \sum_{k=j+\tau-\sigma-2}^{j+\tau-\sigma} Q(k)G(M[j-\sigma, k-\sigma]) > \frac{1+G(d)}{\lambda\beta}$$

hold, then (1.1) *is oscillatory.*

Proof. Let $y(n)$ be a nonoscillatory solution of (1.1) such that $y(n) > 0$ for $n \geqslant n_0 > \varrho$. If we set

$$z(n) = y(n) + p(n)y(n-\tau), \tag{2.1}$$

then (1.1) becomes

$$L_4 z(n) = -q(n)G(y(n-\sigma)) < 0. \tag{2.2}$$

Hence, we can find $n_1 > n_0$ such that $L_i z(n)$, $i = 1, 2, 3$ are eventually of one sign on $[n_1, \infty)$. In what follows, we consider the possible cases (a)–(d) of Lemma 2.1.

Case (c): For $k - 1 \geqslant m \geqslant n_1$, it follows from the discrete Taylor series that

$$-z(k) = -z(m) - (k-m)\Delta z(k) + \sum_{l=m}^{k-1}(l+1-m)\Delta^2 z(l) \leqslant \sum_{l=m}^{k-1}(l+1-m)\Delta^2 z(l) \tag{2.3}$$

and

$$L_2 z(l) - L_2 z(m) = \sum_{s=m}^{l-1} L_3 z(s) \leqslant (l-m) L_3 z(m)$$

implies that $L_2 z(l) \leqslant (l-m)L_3 z(m)$, that is, $\Delta^2 z(l) \leqslant L_3 z(m)(l-m)/r(l)$. Consequently,

$$z(k) \geqslant -\sum_{l=m}^{k-1}(l+1-m)\frac{l-m}{r(l)} L_3 z(m) = -L_3 z(m) M[k, m].$$

For $j - \sigma \geqslant k - \sigma + 2 \geqslant n_1 + 2$, the above inequality can be written as

$$z(j-\sigma) \geqslant -L_3 z(m-\sigma) M[j-\sigma, k-\sigma].$$

Using (1.1), it is easy to verify that

$$0 = L_4 z(n) + q(n)G(y(n-\sigma)) + G(d)L_4 z(n-\tau) + G(d)q(n-\tau)G(y(n-\sigma-\tau)). \tag{2.4}$$

Due to (A_2) and (A_3), (2.4) yields that

$$\begin{aligned} 0 &\geqslant L_4 z(j) + G(d)L_4 z(j-\tau) + \lambda Q(j)G(z(j-\sigma)) \\ &\geqslant L_4 z(j) + G(d)L_4 z(j-\tau) + \lambda Q(j)G(M[j-\sigma, k-\sigma])G(-L_3 z(m-\sigma)), \end{aligned}$$

that is,

$$\begin{aligned} \lambda \sum_{j=m-\sigma-2}^{m-\sigma-1} & Q(j)G(M[j-\sigma, k-\sigma])G(-L_3 z(m-\sigma)) \\ &\leqslant -\sum_{j=m-\sigma-2}^{m-\sigma-1}(L_4 z(j) + G(d)L_4 z(j-\tau)) \\ &\leqslant -L_3 z(m-\sigma) - G(d)L_3 z(m-\sigma-\tau) \\ &\leqslant -(1+G(d))L_3 z(m-\sigma). \end{aligned}$$

As a result,

$$\sum_{j=m-\sigma-2}^{m-\sigma-1} Q(j)G(M[j-\sigma,k-\sigma]) \leqslant \frac{1+G(d)}{\lambda}\frac{-L_3z(m-\sigma)}{G(-L_3z(m-\sigma))} \leqslant \frac{1+G(d)}{\lambda\beta},$$

which is a contradiction to (A_7) due to (A_1).

Case (d): For $k-1 \geqslant m \geqslant n_1$, it follows from (2.3) that

$$z(m) = z(k) - (k-m)\Delta z(k) + \sum_{l=m}^{k-1}(l+1-m)\Delta^2 z(l) \geqslant \sum_{l=m}^{k-1}(l+1-m)\Delta^2 z(l).$$

Since

$$L_2z(l) - L_2z(m) = \sum_{s=m}^{l-1} L_3z(s) \geqslant (l-m)L_3z(l-1),$$

we have $L_2z(l) \geqslant (l-m)L_3z(l-1)$, that is, $\Delta^2 z(l) \leqslant L_3z(m)(l-m)/r(l)$. Consequently,

$$z(m) \geqslant \sum_{l=m}^{k-1}(l+1-m)\frac{l-m}{r(l)}L_3z(l-1) = L_3z(k-2)M[k,m].$$

For $j-\sigma \geqslant k-\sigma+2 \geqslant n_1+2$, we write the preceding inequality as

$$z(k-\sigma) \geqslant L_3z(j-\sigma-2)M[j-\sigma,k-\sigma]. \tag{2.5}$$

Applying (A_2) and (A_3) in (2.4), it follows that

$$0 \geqslant L_4z(k) + G(d)L_4z(k-\tau) + \lambda Q(k)G(z(k-\sigma)).$$

Due to (2.5) the above inequality becomes

$$\begin{aligned}\lambda \sum_{k=j+\tau-\sigma-2}^{j+\tau-\sigma} Q(k)G(M[j-\sigma,k-\sigma])G(L_3z(j-\sigma-2))&\\ \leqslant -\sum_{k=j+\tau-\sigma-2}^{j+\tau-\sigma}&(L_4z(k)+G(d)L_4z(k-\tau))\\ \leqslant\ & L_3z(j+\tau+\sigma-2)+G(d)L_3z(j-\sigma-2)\\ \leqslant\ &(1+G(d))L_3z(j-\sigma-2).\end{aligned}$$

Therefore,

$$\sum_{k=j+\tau-\sigma-2}^{j+\tau-\sigma} Q(k)G(M[j-\sigma,k-\sigma]) \leqslant \frac{1+G(d)}{\lambda}\frac{L_3z(j-\sigma-2)}{G(L_3z(j-\sigma-2))} \leqslant \frac{1+G(d)}{\lambda\beta}$$

contradicts (A_8) due to (A_1). Cases (a) and (b) can be dealt with similarly as the above cases. Also, these two cases follow from [14].

Finally, we suppose that $y(n) < 0$ for $n \geqslant n_0$. Using (A_2) and putting $x(n) = -y(n)$ in (1.1), we obtain $x(n) > 0$ and hence

$$\Delta^2(r(n)\Delta^2(x(n) + p(n)x(n-\tau))) + q(n)G(x(n-\sigma)) = 0. \tag{2.6}$$

Proceeding as above, we can show that every solution of (2.6) oscillates. This completes the proof of the theorem. $\square$

Theorem 2.4. *Let $-1 \leqslant p(n) \leqslant 0$ and $\sigma \geqslant 2\tau$. If* (A_0)–(A_2), *and*

$$\limsup_{k\to\infty} \sum_{j=k-\tau}^{k} q(j)G(D[j-\sigma, k-\sigma]) > \frac{1}{\beta}, \tag{A_9}$$

$$\limsup_{k\to\infty} \sum_{j=k-\tau}^{k} q(j)G(E[j-\sigma, k-\sigma]) > \frac{1}{\beta}, \tag{A_{10}}$$

$$\limsup_{m\to\infty} \sum_{j=m-\sigma-2}^{m-\sigma-1} q(j)G(M[j-\sigma, k-\sigma]) > \frac{1}{\beta}, \tag{A_{11}}$$

$$\limsup_{j\to\infty} \sum_{k=j+\tau-\sigma-2}^{j+\tau-\sigma} q(k)G(M[j-\sigma, k-\sigma]) > \frac{1}{\beta}, \tag{A_{12}}$$

$$\limsup_{k\to\infty} \sum_{j=k+\tau-\sigma}^{k+\sigma-\tau} q(j)G(F[k+\tau-\sigma, j+\tau-\sigma]) > \frac{1}{\beta}, \tag{A_{13}}$$

$$\limsup_{l\to\infty} \sum_{j=l+\tau-\sigma}^{l+\tau-\sigma+1} q(j)G(M[k+\tau-\sigma, j+\tau-\sigma]) > \frac{1}{\beta}, \tag{A_{14}}$$

$$\limsup_{k\to\infty} \sum_{j=k+\tau-\sigma-2}^{k} q(j)G(M[j+\tau-\sigma, k+\tau-\sigma]) > \frac{1}{\beta} \tag{A_{15}}$$

hold, then every solution of (1.1) *oscillates.*

P r o o f. Suppose on the contrary that $y(n)$ is a nonoscillatory solution of (1.1) such that $y(n) > 0$ for $n \geqslant n_0 > \varrho$. The case $y(n) < 0$ for $n \geqslant n_0 > \varrho$ can be dealt with similarly. Setting $z(n)$ as in (2.1), we get (2.2). Consequently, we can find $n_1 > n_0$ such that $z(n)$ and $L_i z(n)$, $i = 1, 2, 3$ are eventually of one sign on $[n_1, \infty)$. Let $z(n) > 0$. Then there exists $n_2 > n_1$ such that for $n \geqslant n_2$, $z(n) \leqslant y(n)$ and (2.2) becomes

$$L_4 z(n) + q(n)G(z(n-\sigma)) \leqslant 0. \tag{2.7}$$

Upon applying Lemma 2.1 to the inequality (2.7) and then proceeding as in the proof of Theorem 2.3, we get contradictions to (A_9)–(A_{12}) due to $\sigma > 2\tau > \tau$.

Next, we suppose that $z(n) < 0$ for $n \geqslant n_1$. There exists $n_2 > n_1$ such that for $n \geqslant n_2$, $z(n) \geqslant -y(n-\tau)$ implies that $y(n-\sigma) \geqslant -z(n+\tau-\sigma)$. By Lemma 2.1, any of cases (b)–(f) holds for $n \geqslant n_2$. Consider case (b). Since

$$-L_2z(l) \geqslant L_2z(k-1) - L_2z(l) = \sum_{s=l}^{k-2} L_3z(s) \geqslant (k-l-1)L_3z(k-2),$$

it follows that

$$-\Delta^2 z(l) \geqslant \frac{k-l-1}{r(l)} L_3z(k-2)$$

for $k \geqslant l+2 > n_1$. Summing the above inequality from $l = m$ to $k-1$, we obtain

$$\Delta z(m) \geqslant L_3z(k-2) \sum_{l=m}^{k-1} \frac{k-l-1}{r(l)},$$

that is,

$$z(m+1) - z(m) \geqslant L_3z(k-2) \sum_{l=m}^{k-1} \frac{k-l-1}{r(l)},$$

which implies that

$$-z(m) \geqslant L_3z(k-2) \sum_{l=m}^{k-1} \frac{k-l-1}{r(l)} = L_3z(k-2)F[k,m] \geqslant L_3z(k)F[k,m]$$

for $k \geqslant l+2 > n_2$. Therefore, for $k+\tau-\sigma \geqslant j+\tau-\sigma \geqslant l+\tau-\sigma+2 > n_2$,

$$-z(j+\tau-\sigma) \geqslant L_3z(k+\tau-\sigma)F[k+\tau-\sigma, j+\tau-\sigma]. \tag{2.8}$$

Since (1.1) can be viewed as

$$L_4z(j) + q(j)G(-z(j+\tau-\sigma)) \leqslant 0, \tag{2.9}$$

using (2.8) and (A_2), (2.9) yields

$$L_4z(j) + q(j)G(L_3z(k+\tau-\sigma))G(F[k+\tau-\sigma, j+\tau-\sigma]) \leqslant 0.$$

Summing the last inequality from $j = k+\tau-\sigma$ to $k+\sigma-\tau$, it follows that

$$G(L_3z(k+\tau-\sigma)) \sum_{j=k+\tau-\sigma}^{k+\sigma-\tau} q(j)G(F[k+\tau-\sigma, j+\tau-\sigma]) \leqslant L_3z(k+\tau-\sigma).$$

Hence,

$$\limsup_{k\to\infty} \sum_{j=k+\tau-\sigma}^{k+\sigma-\tau} q(j)G(F[k+\tau-\sigma, j+\tau-\sigma]) \leqslant \frac{1}{\beta},$$

which gives a contradiction to (A_{13}).

Consider case (c). From (2.3), it follows that

$$-z(m) = -z(k) + (k-m)\Delta z(k) - \sum_{l=m}^{k-1}(l+1-m)\Delta^2 z(l) \geqslant -\sum_{l=m}^{k-1}(l+1-m)\Delta^2 z(l)$$

for $k-1 \geqslant m \geqslant n_1$ and

$$L_2 z(l) - L_2 z(m) = \sum_{s=m}^{l-1} L_3 z(s) \leqslant (l-m)L_3 z(m)$$

implies that $L_2 z(l) \leqslant (l-m)L_3 z(m)$, that is,

$$-\Delta^2 z(l) \geqslant \frac{l-m}{r(l)} L_3 z(m).$$

Consequently,

$$-z(m) \geqslant -\sum_{l=m}^{k-1}(l+1-m)\frac{l-m}{r(l)} L_3 z(m) = -L_3 z(m) M[k,m]$$

and hence

$$\begin{aligned} -z(j+\tau-\sigma) &\geqslant -L_3 z(j+\tau-\sigma) M[k+\tau-\sigma, j+\tau-\sigma] \\ &\geqslant -L_3 z(l+\tau-\sigma+2) M[k+\tau-\sigma, j+\tau-\sigma] \end{aligned} \tag{2.10}$$

holds for $k+\tau-\sigma \geqslant j+\tau-\sigma \geqslant l+\tau-\sigma+2 > n_2$. Using (A_2) and (2.10) in (2.9), and then summing from $l+\tau-\sigma$ to $l+\tau-\sigma+1$, we obtain

$$\begin{aligned} G(-L_3 z(l+\tau-\sigma+2)) \sum_{j=l+\tau-\sigma}^{l+\tau-\sigma+1} q(j) G(M[k+\tau-\sigma, j+\tau-\sigma]) \\ \leqslant -L_3 z(l+\tau-\sigma+2). \end{aligned}$$

Therefore,

$$\sum_{j=l+\tau-\sigma}^{l+\tau-\sigma+1} q(j) G(M[k+\tau-\sigma, j+\tau-\sigma]) \leqslant \frac{-L_3 z(l+\tau-\sigma+2)}{G(-L_3 z(l+\tau-\sigma+2))} \leqslant \frac{1}{\beta},$$

which gives a contradiction to (A_{14}).

In case (d), we use (2.3) and it follows that

$$-z(k) = -z(m) - (k-m)\Delta z(k) + \sum_{l=m}^{k-1}(l+1-m)\Delta^2 z(l) \geqslant \sum_{l=m}^{k-1}(l+1-m)\Delta^2 z(l)$$

for $k-1 \geqslant m \geqslant n_1$. Since

$$L_2 z(l) \geqslant L_2 z(l) - L_2 z(m) = \sum_{s=m}^{l-1} L_3 z(s) \geqslant (l-m)L_3 z(l-1),$$

we have $\Delta^2 z(l) \geqslant L_3 z(l-1)(l-m)/r(l)$, which implies that

$$-z(k) \geqslant \sum_{l=m}^{k-1}(l+1-m)\frac{l-m}{r(l)}L_3 z(l-1) = L_3 z(k-2)M[k,m].$$

Hence for $j+\tau-\sigma \geqslant k+\tau-\sigma+2 > n_1+2$, it follows that

$$-z(j+\tau-\sigma) \geqslant L_3 z(j+\tau-\sigma-2)M[j+\tau-\sigma, k+\tau-\sigma].$$

Consequently, (2.9) becomes

$$\sum_{j=k+\tau-\sigma-2}^{k} q(j)G(M[j+\tau-\sigma, k+\tau-\sigma])G(L_3 z(j+\tau-\sigma-2)) \leqslant L_3 z(k+\tau-\sigma-2).$$

As a result,

$$\sum_{j=k+\tau-\sigma-2}^{k} q(j)G(M[j+\tau-\sigma, k+\tau-\sigma]) \leqslant \frac{L_3 z(k+\tau-\sigma-2)}{G(L_3 z(k+\tau-\sigma-2))} \leqslant \frac{1}{\beta},$$

which contradicts to our assumption (A_{15}).

In both cases (e) and (f), $\lim_{n\to\infty} z(n) = -\infty$. On the other hand, $z(n) < 0$ for $n \geqslant n_1$ implies that $y(n) \leqslant y(n-\tau)$ for $n \geqslant n_1$, that is,

$$y(n) \leqslant y(n-\tau) \leqslant y(n-2\tau) \leqslant \ldots \leqslant y(n_1),$$

$y(n)$ is bounded and hence $z(n)$ is bounded, a contradiction. This completes the proof of the theorem. $\square$

Theorem 2.5. *Let* $-\infty < -d \leqslant p(n) \leqslant -1$, $d > 0$ *and* $2\tau \leqslant \sigma$. *Assume that* (A_0)–(A_2) *hold. Furthermore, if*

$$(A_{16})\qquad \sum_{n=0}^{\infty} q(n) = \infty,$$

$$(A_{17})\qquad \limsup_{m\to\infty} \sum_{j=m+\tau-\sigma-2}^{m+\tau-\sigma-1} q(j)G(D[j+\tau-\sigma, m+\tau-\sigma]) > \frac{1}{\beta G(d^{-1})},$$

$$(A_{18})\qquad \limsup_{k\to\infty} \sum_{j=k-\tau}^{k} q(j)G(D[j-\sigma, k-\sigma]) > \frac{1}{\beta G(d^{-1})},$$

$$(A_{19})\qquad \limsup_{k\to\infty} \sum_{j=k-\tau}^{k} q(j)G(E[j-\sigma, k-\sigma]) > \frac{1}{\beta G(d^{-1})},$$

$$(A_{20})\qquad \limsup_{m\to\infty} \sum_{j=m-\sigma-2}^{m-\sigma-1} q(j)G(M[j-\sigma, k-\sigma]) > \frac{1}{\beta G(d^{-1})},$$

$$(A_{21})\qquad \limsup_{j\to\infty} \sum_{k=j+\tau-\sigma-2}^{j+\tau-\sigma} q(k)G(M[j-\sigma, k-\sigma]) > \frac{1}{\beta G(d^{-1})},$$

$$(A_{22})\qquad \limsup_{k\to\infty} \sum_{j=k+\tau-\sigma}^{k+\sigma-\tau} q(j)G(F[k+\tau-\sigma, j+\tau-\sigma]) > \frac{1}{\beta G(d^{-1})},$$

$$(A_{23})\qquad \limsup_{l\to\infty} \sum_{j=l+\tau-\sigma}^{l+\tau-\sigma+1} q(j)G(M[k+\tau-\sigma, j+\tau-\sigma]) > \frac{1}{\beta G(d^{-1})}$$

$$(A_{24})\qquad \limsup_{k\to\infty} \sum_{j=k+\tau-\sigma-2}^{k} q(j)G(M[j+\tau-\sigma, k+\tau-\sigma]) > \frac{1}{\beta G(d^{-1})}$$

hold, then every solution of (1.1) *oscillates.*

Proof. The proof of the theorem follows from the proof of Theorem 2.4. We consider cases (e) and (f) of Lemma 2.1 only when $z(n) < 0$ for $n \geqslant n_1$. There exists $n_2 > n_1$ such that for $n \geqslant n_2$, $y(n-\sigma) \geqslant -z(n+\tau-\sigma)/d$. Consequently, (1.1) becomes

$$L_4 z(n) - q(n)G\Big(\frac{1}{d}\Big)G(z(n+\tau-\sigma)) \leqslant 0 \tag{2.11}$$

for $n \geqslant n_2$ due to (A_2). In case (e), $z(n)$ is nonincreasing. So, we can find $n_3 > n_2$ and $L > 0$ such that $z(n) \leqslant -L$ for $n \geqslant n_3$. Therefore, (2.11) yields

$$L_4 z(n) + G\Big(\frac{1}{d}\Big)G(L)q(n) \leqslant 0$$

for $n \geqslant n_3$. Summing the above inequality from n_3 to ∞, we obtain a contradiction to (A_{16}).

Assume that case (f) of Lemma 2.1 holds. For $k \geqslant n_1 + 2$, it follows from the discrete Taylor series

$$z(k) = z(n_1) + (n - n_1)\Delta z(n_1) + \sum_{l=n_1}^{k-2} (k - l - 1)\Delta^2 z(l)$$

that $z(k) \leqslant \sum\limits_{l=n_1}^{k-2} (k - l - 1)\Delta^2 z(l)$. Since

$$L_2 z(l) \leqslant L_2 z(l) - L_2 z(m) = \sum_{s=m}^{l-1} L_3 z(s) \leqslant (l - m) L_3 z(m),$$

we have $\Delta^2 z(l) \leqslant L_3 z(m)(l - m)/r(l)$, which implies that

$$z(k) \leqslant \sum_{l=n_1}^{k-2} (k - l - 1)\frac{l - m}{r(l)} L_3 z(m) = L_3 z(m) D[k, m]$$

for $k \geqslant m + 2 \geqslant n_1 + 2$. Therefore, for $j + \tau - \sigma \geqslant m + \tau - \sigma + 2 > n_1 + 2$ it follows that

$$-z(j + \tau - \sigma) \geqslant L_3 z(m + \tau - \sigma) D[j + \tau - \sigma, m + \tau - \sigma]. \tag{2.12}$$

Using (2.12) and (A_2) in (2.11), and then summing the resultant inequality from $m + \tau - \sigma - 2$ to $m + \tau - \sigma - 1$, we obtain

$$\begin{aligned} G\Big(\frac{1}{d}\Big) G(-L_3 z(m + \tau - \sigma)) \sum_{j=m+\tau-\sigma-2}^{m+\tau-\sigma-1} q(j) G(D[j + \tau - \sigma,\ m + \tau - \sigma]) \\ \leqslant -L_3 z(m + \tau - \sigma), \end{aligned}$$

that is,

$$\begin{aligned} \sum_{j=m+\tau-\sigma-2}^{m+\tau-\sigma-1} q(j) G(D[j + \tau - \sigma, m + \tau - \sigma]) &\leqslant \frac{-L_3 z(m + \tau - \sigma)}{G(d^{-1}) G(-L_3 z(m + \tau - \sigma))} \\ &\leqslant \frac{1}{\beta G(d^{-1})}, \end{aligned}$$

which is a contradiction to (A_{17}). Hence the proof of the theorem is complete. $\square$

Theorem 2.6. *Let $0 \leqslant p(n) \leqslant d < \infty$. Assume that* (A_0) *and* (A_2)–(A_4) *hold. Furthermore, suppose that*

$$(\mathrm{A}_{25}) \qquad \sum_{n=N}^{\infty} Q(n)G(R(n-\sigma)) = \infty, \quad N > \varrho$$

and

$$(\mathrm{A}_{26}) \qquad \sum_{n=N}^{\infty} \frac{Q(n)}{r(n+2)} G(R(n-\sigma)) = \infty, \quad N > \varrho$$

hold. Then (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.4, we consider cases (a)–(d) of Lemma 2.1. For each case, (2.4) holds. Upon using (A_2) and (A_4), it follows that

$$L_4 z(n) + G(d) L_4 z(n-\tau) + \lambda Q(n) G(z(n-\sigma)) \leqslant 0$$

for $n \geqslant n_2 \geqslant n_1$. To the last inequality, we apply Lemma 2.2. For cases (a), (b) and (d) it is easy to see that

$$(2.13) \qquad L_4 z(n) + G(d) L_4 z(n-\tau) + \lambda G(C_1) Q(n) G(R(n-\sigma)) \leqslant 0$$

holds due to (A_2). Summing (2.13) from n_3 $(\geqslant n_2 + \sigma)$ to ∞, we get a contradiction to (A_{25}). For case (c), we use (2.13) to get

$$\begin{aligned} \lambda G(C_1) Q(n) G(R(n-\sigma)) &\leqslant -(L_4 z(n) + G(d) L_4 z(n-\tau)) \\ &\leqslant -L_3 z(n+1) - G(d) L_3 z(n+1-\tau) \\ &\leqslant -(1+G(d)) L_3 z(n+1) \\ &= -(1+G(d))(L_2 z(n+2) - L_2 z(n+1)) \\ &\leqslant -(1+G(d)) L_2 z(n+2) \end{aligned}$$

for $n \geqslant n_2$, that is,

$$\lambda G(C_1) \frac{Q(n)}{r(n+2)} G(R(n-\sigma)) \leqslant -(1+G(d)) \Delta^2 z(n+2).$$

Summing the last inequality from n_3 $(> n_2 + \sigma)$ to ∞, we get a contradiction to (A_{26}). Thus the proof of the theorem is complete. $\square$

Theorem 2.7. *Let* $-1 \leqslant p(n) \leqslant 0$ *and* $\sigma > \max\{\tau, \tau-2\}$. *Suppose that* (A_0)–(A_2) *and* (A_{16}) *hold. Furthermore, if*

$$(A_{27}) \qquad \sum_{n=N}^{\infty} q(n)G(R(n-\sigma)) = \infty, \quad N > \varrho,$$

$$(A_{28}) \qquad \sum_{n=N}^{\infty} \frac{q(n)}{r(n+2)} G(R(n-\sigma)) = \infty, \quad N > \varrho,$$

$$(A_{29}) \qquad \liminf_{n\to\infty} \sum_{j=n+\tau-\sigma}^{n-1} \frac{q(j)}{r(j)} > \frac{1}{\beta} \frac{(\sigma-\tau)^{\sigma-\tau+1}}{(\sigma-\tau+1)^{\sigma-\tau+1}}, \quad \tau < \sigma$$

$$(A_{30}) \qquad \liminf_{n\to\infty} \frac{q(n)}{r(n+2)} > \frac{1}{\beta}$$

hold, then (1.1) *is oscillatory.*

Proof. On the contrary, we proceed as in Theorem 2.4 to obtain (2.7) for $n \geqslant n_2$. The rest of this case follows from the proof of Theorem 2.6.

When $z(n) < 0$ for $n \geqslant n_1$, we consider cases (b), (c) and (d) of Lemma 2.1 only. For case (b), we use (2.9) and it follows that

$$\begin{aligned} q(n)G(-z(n+\tau-\sigma)) &\leqslant -\Delta L_3 z(n) = -L_3 z(n+1) + L_3 z(n) \\ &\leqslant L_3 z(n) = \Delta L_2 z(n) \leqslant -L_2 z(n) \end{aligned}$$

for $n \geqslant n_2$. Consequently,

$$\frac{q(n)}{r(n)} G(-z(n+\tau-\sigma)) \leqslant -\Delta^2 z(n) \leqslant \Delta z(n)$$

implies that

$$(2.14) \qquad \Delta z(n) + \beta \frac{q(n)}{r(n)} z(n+\tau-\sigma) \geqslant 0$$

due to (A_1), that is, (2.14) cannot have an eventually negative solution (because of Lemma 1.1) due to (A_{29}), a contradiction.

Using a similar type of argument as in case (b), we find the inequality

$$z(n+2) - \beta \frac{q(n)}{r(n+2)} z(n+\tau-\sigma) \leqslant 0$$

for case (c) due to (A_1). Since $\sigma \geqslant \tau - 2$, then the above inequality reduces to

$$\Big(1 - \beta \frac{q(n)}{r(n+2)}\Big) z(n+\tau-\sigma) \leqslant 0,$$

which is not possible because of (A_{30}). In case (d), $z(n)$ is nonincreasing. Hence, there exist $C > 0$ and $n_2 > n_1$ such that $z(n) \leqslant -C$ for $n \geqslant n_2$. Using this fact in (2.9) and then summing the resultant inequality from n_3 $(> n_2)$ to ∞, we get a contradiction to (A_{16}). This completes the proof of the theorem. $\square$

Theorem 2.8. *Let* $-\infty < -d \leqslant p(n) \leqslant -1$, $d > 0$ *and* $\sigma > \max\{\tau, \tau - 2\}$. *Assume that* (A_0)–(A_2), (A_{16}), (A_{27}) *and* (A_{28}) *hold. Furthermore, if*

$$(A_{31}) \qquad \liminf_{n\to\infty} \sum_{j=n+\tau-\sigma}^{n-1} \frac{q(j)}{r(j)} > \frac{1}{G(d^{-1})\beta} \frac{(\sigma-\tau)^{\sigma-\tau+1}}{(\sigma-\tau+1)^{\sigma-\tau+1}}, \qquad \tau < \sigma,$$

$$(A_{32}) \qquad \liminf_{n\to\infty} \frac{q(n)}{r(n+2)} > \frac{1}{G(d^{-1})\beta}$$

hold, then every bounded solution of (1.1) *is oscillatory.*

P r o o f. The proof of the theorem follows from the proof of Theorem 2.7. Cases (e) and (f) of Lemma 2.1 are not possible when $z(n) < 0$ for $n \geqslant n_1$, since $y(n)$ is bounded. Hence, the details are omitted. $\square$

E x a m p l e 2.9. Consider

$$(2.15) \qquad \Delta^2(ne^n\Delta^2(y(n) + p(n)y(n-1))) + q(n)G(y(n-3)) = 0,$$

where $n > 3$, $p(n) = \mathrm{e}^{-2} + \mathrm{e}^{-n}$, $q(n) = (\mathrm{e}^2-1)^2(\mathrm{e}+1)(2\mathrm{e}+n\mathrm{e}+n)\mathrm{e}^n - (\mathrm{e}+1)^2(n+1)$, $r(n) = n\mathrm{e}^n$ and $G(u) = 4u/\mathrm{e}^2 = \beta u$. Clearly, all conditions of Theorem 2.3 are satisfied. Hence (2.15) is oscillatory. Indeed, $y(n) = (-1)^n$ is one of the oscillatory solutions of (2.15).

R e m a r k 2.10. The existence of positive solutions of (1.1)/(1.2) is discussed in [13].

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