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ON SHORT CYCLES IN TRIANGLE-FREE ORIENTED GRAPHS

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Abstract. An orientation of a simple graph is referred to as an oriented graph. Caccetta and Häggkvist conjectured that any digraph on n vertices with minimum outdegree d contains a directed cycle of length at most $\lceil n/d \rceil$. In this paper, we consider short cycles in oriented graphs without directed triangles. Suppose that α_0 is the smallest real such that every n-vertex digraph with minimum outdegree at least $\alpha_0 n$ contains a directed triangle. Let $\varepsilon < (3-2\alpha_0)/(4-2\alpha_0)$ be a positive real. We show that if D is an oriented graph without directed triangles and has minimum outdegree and minimum indegree at least $(1/(4-2\alpha_0)+\varepsilon)|D|$, then each vertex of D is contained in a directed cycle of length l for each $4 \leqslant l < (4-2\alpha_0)\varepsilon|D|/(3-2\alpha_0)+2$.

Keywords: oriented graph; cycle; minimum semidegree

MSC 2010: 05C20, 05C38

1. Introduction

The notation follows that of [1]. We consider only digraphs without loops and parallel arcs. For a digraph D, we denote by $V(D)$ the vertex set of D, by $A(D)$ the arc set of D. We write $|D|$ for the *order* of D, that is, the number of vertices in D, and $e(D)$ for the number of arcs. We write $N_D^+(x)$ for the *outneighbourhood* of a vertex $x \in V(D)$ and $d_D^+(x) = |N_D^+(x)|$ for its *outdegree*. Similarly, we write $N_D^-(x)$ for the *inneighbourhood* of x and $d_D^-(x) = |N_D^-(x)|$ for its *indegree*. We write $N_D(x) = N_D^+(x) \cup N_D^-(x)$ for the *neighbourhood* of x and $d_D(x) = d_D^+(x) + d_D^-(x)$ for its *degree*. Let $\delta^+(D) = \min\{d_D^+(x)\colon x \in V(D)\}$ be the *minimum outdegree* and $\delta^-(D) = \min\{d_D^-(x)\colon x \in V(D)\}$ be the *minimum indegree* of D. Define the *minimum semidegree* $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$. Given $X \subseteq V(D)$ we define

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$d^+_X(x) = |N^+_D(x) \cap X|$ and $d^-_X(x) = |N^-_D(x) \cap X|$. Let $N^+_D(X) = \bigcup_{a \in X} N^+_D(a)$. Note that $N^+_D(X)$ may include also vertices contained in X. Similarly, we can define $N^-_D(X)$. The subgraph of D induced by X is denoted by $D[X]$ and we write $e(X)$ for the number of its arcs. $D-X$ denotes the digraph obtained from D by deleting X and all arcs incident with X. We say that X is *independent* if $D[X]$ contains no arcs. Given disjoint vertex sets S and T in D, an *S-T arc* is an arc ab where $a \in S$ and $b \in T$. We write $A(S,T)$ (or $e(S,T)$) for the set (or number) of all S-T arcs in D. *When referring to paths and cycles in digraphs we always mean that they are directed without mentioning this explicitly.* A *path* of length k of D is a list $x_1, \ldots, x_k$ of distinct vertices such that $x_i x_{i+1} \in A(D)$ for $1 \leqslant i \leqslant k-1$. For two vertices x, y in D, an *x-y path* is a path which joins x to y. A *cycle* of length $l \geqslant 2$ is a list $x_1, \ldots, x_l, x_1$ of vertices with $x_1, \ldots, x_l$ distinct, $x_i x_{i+1} \in A(D)$ for $1 \leqslant i \leqslant l-1$ and $x_l x_1 \in A(D)$. We refer to cycles of length l as l-cycles and a 3-cycle is also called a *triangle*. A digraph D is an *oriented graph* if it does not contain 2-cycles, that is to say it is obtained from a simple graph where each edge is given an orientation. So, an oriented graph is called *triangle-free* if it does not contain directed triangles. Throughout the paper, we omit all floor and ceiling signs whenever these are not crucial, to simplify the presentation.

The problem of deciding whether a given digraph D contains a cycle of length l is one of the most natural and easily stated problems in graph theory. Especially when $l = |D|$ it becomes the well-known Hamilton problem. In recent years, lots of results (see [4], [7], [8], [11]) were obtained for Hamilton cycles in digraphs (or oriented graphs) by using the celebrated "robust expansion" technique. We direct the interested reader to the survey paper of Kühn and Osthus, see [10]. For cycles of other lengths, Caccetta and Häggkvist in [3] posed the following conjecture in 1978.

Conjecture 1.1 (Caccetta, Häggkvist [3]). *Any digraph on n vertices with minimum outdegree d contains a cycle of length at most $\lceil n/d \rceil$.*

The special case of Conjecture 1.1 that has attracted most interest is that of $d = \lceil n/3 \rceil$ (see [2], [3], [5], [14]). Throughout this paper, we use α_0 to denote the smallest real such that every n-vertex digraph with minimum outdegree at least $\alpha_0 n$ contains a triangle. Considering the blow-up of a 4-cycle, it's easy to see that $\alpha_0 \geqslant 1/4$. In fact, we can define a sequence of graphs $D^{(k)}$, for all integers $k \geqslant 0$, such that (1) $D^{(0)} = C_4$, (2) $D^{(i+1)}$ is obtained by taking four copies of $D^{(i)}$, denoted by D_0, D_1, D_2 and D_3, and adding all arcs from D_j to D_{j+1}, where $0 \leqslant j \leqslant 3$ and the subscripts modulo 4. Clearly, $D^{(k)}$ does not contain any triangle and when $k \to \infty$ we have $\alpha_0 \geqslant 1/3$. If Conjecture 1.1 is true, then it implies that $\alpha_0 = 1/3$. Using the theory of flag algebras developed by Razborov in [13], Hladký et al. in [6] proved the following up-to-now best known bound. For other details on the Caccetta-Häggkvist conjecture, see [15].

Theorem 1.2 (Hladký, Kráľ, Norin [6]). *Every n-vertex digraph with minimum outdegree at least $0.3465n$ contains a triangle.*

Note that a triangle will be preserved if we change the direction of all arcs in a digraph, thus Hladký et al.'s result implies that any digraph with minimum indegree at least $0.3465n$ contains a triangle. Therefore, each triangle-free oriented graph must have both minimum outdegree and minimum indegree less than $0.3465n$. In fact, it is unknown whether any digraph of order n with minimum semidegree at least $\lceil n/3 \rceil$ contains a cycle of length at most 3. Hamburger et al. in [5] proved that for $\gamma \geqslant 0.34564$, any n-vertex digraph D with $\delta^0(D) \geqslant \gamma n$ contains a cycle of length at most 3. Though using Theorem 1.2, this bound was improved to $\gamma \geqslant 0.343545$ by Lichiardopol in [12]. For related questions of which minimum semidegree forces cycles of length exactly $l \geqslant 4$ in an oriented graph, Kelly et al. in [9] proved the following:

Theorem 1.3 (Kelly, Kühn and Osthus [9]). *Let $l \geqslant 4$ be a positive integer. If D is an oriented graph on $n \geqslant 10^{10}l$ vertices with $\delta^0(D) \geqslant \lfloor n/3 \rfloor + 1$, then D contains an l-cycle. Moreover for any vertex $u \in V(D)$ there is an l-cycle containing u.*

As noted in [9], the minimum semidegree condition is best possible for $l \geqslant 4$ and $l \neq 0 \pmod 3$ (considering the blow-up of a directed triangle). It is also noted in [9] that when $l \geqslant 4$ and $3 \mid l$, the minimum semidegree condition is also best possible for the moreover part. All the extremal graphs for Theorem 1.3 have many directed triangles. In this paper, we consider short cycles in triangle-free oriented graphs and prove the following theorem.

Theorem 1.4. *Let $\varepsilon < (3-2\alpha_0)/(4-2\alpha_0)$ be a positive real and D be a triangle-free oriented graph on n vertices. If $\delta^0(D) \geqslant (1/(4-2\alpha_0)+\varepsilon)n$, then for any vertex $v \in V(D)$, D contains an l-cycle through v for each $4 \leqslant l < (4-2\alpha_0)\varepsilon n/(3-2\alpha_0)+2$.*

By Theorem 1.2, we know that $\alpha_0 \leqslant 0.3465$. Thus we have the following immediate corollary.

Corollary 1.5. *Let $\varepsilon < 0.6976$ be a positive real and D be a triangle-free oriented graph on n vertices. If $\delta^0(D) \geqslant (0.3024+\varepsilon)n$, then for any vertex $v \in V(D)$, D contains an l-cycle through v for each $4 \leqslant l \leqslant 1.4334\varepsilon n+2$.*

2. Proof of the main result

In this section we give a proof of Theorem 1.4. Before that we give some lemmas. Recall that α_0 is the smallest real such that every n-vertex digraph with minimum outdegree at least $\alpha_0 n$ contains a triangle. Throughout this section, let

$$\alpha = \frac{1}{4-2\alpha_0} \quad \text{and} \quad \beta = \frac{2-\alpha_0}{7-4\alpha_0}.$$

Since $\alpha_0 \geqslant 1/3$, we have $\alpha_0 > \alpha > \beta$.

Given two vertices x, y of a digraph D, the *distance* $\operatorname{dist}(x,y)$ from x to y is the length of the shortest x-y path. The *diameter* of D is the maximum distance between any ordered pair of vertices. We use the following lemma to control the diameter of D.

Lemma 2.1. *If D is a triangle-free oriented graph on n vertices with $\delta^0(D) \geqslant \alpha n$, then the diameter of D is at most 4.*

Proof. For any vertex $x \in V(D)$ define $X_1 = N^+(x)$ and $X_2 = N^+(X_1) \cup X_1$. If $\delta^+(D[X_1]) \geqslant \alpha_0|X_1|$, then $D[X_1]$ contains a triangle by the definition of α_0, a contradiction. So there exists a vertex $x_1 \in X_1$ with $|N^+(x_1) \cap X_1| < \alpha_0|X_1|$. Hence,

$$\begin{aligned} |X_2| &= |X_1| + |N^+(X_1) \setminus X_1| \\ &\geqslant |X_1| + \bigl(d^+(x_1) - |N^+(x_1) \cap X_1|\bigr) \\ &> |X_1| + \bigl(\delta^0(D) - \alpha_0|X_1|\bigr) \\ &\geqslant (2-\alpha_0)\delta^0(D) \geqslant \frac{n}{2}, \end{aligned}$$

here we use the fact $|X_1| \geqslant \delta^0(D)$ and $\delta^0(D) \geqslant \alpha n = n/(4-2\alpha_0)$.

Similarly, for any vertex y with $y \neq x$, by considering the indegrees we have

$$|\{v \in V(D)\colon \operatorname{dist}(v,y) \leqslant 2\}| > \frac{n}{2}.$$

This implies that there exists a directed x-y path of length at most 4. So the diameter of D is at most 4. $\square$

A *transitive triangle* is obtained by orienting the edges of an undirected 3-cycle such that it does not form a directed triangle.

Lemma 2.2. *If D is a triangle-free oriented graph on n vertices with $\delta^0(D) \geqslant \beta n$, then for any vertex x, $D[\{x\} \cup N^+(x)]$ contains at least one transitive triangle.*

P r o o f. By contradiction, suppose that there exists $x \in V(D)$ such that $D[\{x\} \cup N^+(x)]$ does not contain any transitive triangle. Choose $X \subseteq N^+(x)$ with $|X| = \beta n$. Clearly, X is independent. Let Y be a set of βn inneighbours of x. Arbitrarily choose an $x' \in X$. Let X' be a set of βn outneighbours of x' and Y' be the set of inneighbours of x'. Denote $Z = V(D) \setminus (\{x\} \cup X \cup X' \cup Y \cup Y')$ and $T = Y' \cap Y$. In the following we are going to prove

$$|N^-(T) \setminus T| > |(Y \cup Y' \cup Z) \setminus T|. \tag{2.1}$$

This implies that $A(X \cup X', T) \neq \emptyset$. It follows that

$$A(X, T) \neq \emptyset \quad \text{or} \quad A(X', T) \neq \emptyset.$$

Choosing an arc from $A(X, T)$ (or $A(X', T)$) arbitrarily, together with x (or x') and its incident arcs it will yield a triangle in D. This contradiction completes the proof.

To see (2.1), since X is independent, $X \cap Y' = \emptyset$. Note that D is an oriented graph, so we have $X' \cap Y' = \emptyset$. Therefore,

$$(X \cup X') \cap Y' = \emptyset. \tag{2.2}$$

Thus we have

$$\begin{aligned} |T| = |Y' \cap Y| &= d^-(x') - |Y' \setminus Y| \\ &\overset{(2.2)}{\geqslant} \delta^0(D) - |V \setminus (X \cup X' \cup Y)| \\ &\geqslant \delta^0(D) - (1 - 3\beta)n \geqslant (4\beta - 1)n. \end{aligned} \tag{2.3}$$

Note that since D is triangle-free, it follows from the definition of α_0 that there exists $x' \in T$ such that $d_T^-(x') < \alpha_0 |T|$; so

$$|N^-(T) \setminus T| \geqslant d_{D \setminus T}^-(x') \geqslant \delta^0(D) - d_T^-(x') > \beta n - \alpha_0 |T|. \tag{2.4}$$

Since D is oriented, we have $X \cap Y = \emptyset$. Again the fact that D is triangle-free implies $X' \cap Y = \emptyset$. Thus, $(X \cup X') \cap Y = \emptyset$. So we have

$$\begin{aligned} |(Y \cup Y' \cup Z) \setminus T\}| &= |V \setminus (\{x\} \cup X \cup X' \cup T)| \\ &\leqslant (1 - 2\beta)n - |T| \\ &\overset{(2.3)}{\leqslant} \beta n - \alpha_0 |T| \\ &\overset{(2.4)}{<} |N^-(T) \setminus T|. \end{aligned}$$

The second inequality is equivalent to showing $|T| \geqslant (1 - 3\beta)n/(1 - \alpha_0)$. By (2.3) and $\beta = (2 - \alpha_0)/(7 - 4\alpha_0)$, we have $|T| \geqslant (4\beta - 1)n = (1 - 3\beta)n/(1 - \alpha_0)$. This completes the proof of (2.1). $\square$

In order to prove the cases $l \geqslant 7$ of Theorem 1.4 we need a new notation. An *xy-butterfly* is an oriented graph with vertices x, y, a, b, c, d, e and with arcs xa, xb, ab, bc, bd, cd, de, dy and ey (see Figure 1). The crucial fact about a butterfly is that it contains x-y paths of lengths $3, 4, 5$ and 6, and is thus a useful tool in finding cycles of prescribed length: any y-x path of length $l-3$, $l-4$, $l-5$ or $l-6$ whose interior avoids the xy-butterfly can yield an l-cycle containing x.

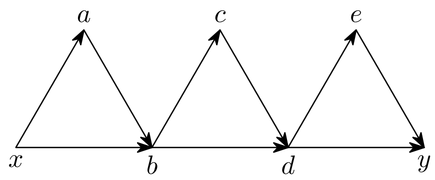

Figure 1. An xy-butterfly.

The following lemma tells us that in a triangle-free oriented graph, minimum semidegree βn is enough to guarantee the existence of a butterfly.

Lemma 2.3. *If D is a triangle-free oriented graph on n vertices with $\delta^0(D) \geqslant \beta n$, then for any vertex $x \in V(D)$ there exists a vertex y such that D contains an xy-butterfly.*

Proof. Using Lemma 2.2 repeatedly, we can choose an arc ab in the outneighbourhood of x, an arc cd in the outneighbourhood of b and an arc ey in the outneighbourhood of d. Since D is triangle-free, all the vertices we have chosen are distinct. $\square$

Now we are ready to prove Theorem 1.4. In the following, we suppose that D is an oriented graph satisfying the assumptions of Theorem 1.4. At first, we show that each vertex of D is contained in an l-cycle for each $7 \leqslant l < (4-2\alpha_0)\varepsilon n/(3-2\alpha_0)+2$. Then we show that each vertex of D can be contained in cycles of length $4, 5, 6$, respectively.

Lemma 2.4. *Each vertex of D is contained in an l-cycle for each $7 \leqslant l < \varepsilon n/(1-\alpha)+2 = (4-2\alpha_0)\varepsilon n/(3-2\alpha_0)+2$.*

Proof. Note that $\delta^0(D) \geqslant (\alpha+\varepsilon)n > \beta n$. For any $x \in V(D)$, by Lemma 2.3 we can find an xy-butterfly with some vertex $y \in V(D)$, and a, b, c, d, e as in the definition of an xy-butterfly. Since $\varepsilon < (3-2\alpha_0)/(4-2\alpha_0) = 1-\alpha$ we have

$$\delta^0(D) \geqslant (\alpha+\varepsilon)n > \frac{\varepsilon n}{1-\alpha} > l-2.$$

We may greedily pick a path P of length $l-7$ from y to some vertex v such that P avoids a, b, c, d, e.

Let $D' = D - (\{a,b,c,d,e\} \cup (V(P) \setminus \{v\}))$. Again by the fact that $\delta^0(D) \geqslant (\alpha+\varepsilon)n$ and $l < \varepsilon n/(1-\alpha) + 2$, we have

$$\delta^0(D') \geqslant \delta^0(D) - (l-2) > \alpha(n-(l-2)).$$

Applying Lemma 2.1 to D', we can find a v-x path P' of length at most 4. Pick a path P'' from x to y in the xy-butterfly such that $|P' \cup P''| = 7$, then $C = P \cup P' \cup P''$ is a desired l-cycle containing x. □

The idea for the proof of cases $l = 4, 5, 6$ is the same: To find an l-cycle containing x, we use Theorem 1.2 and the minimum semidegree condition to show that the outneighbourhoods and inneighbourhoods of some fixed vertex sets have nonempty intersection. Now we deal with them respectively.

Lemma 2.5. *Each vertex of D is contained in a 4-cycle.*

Proof. For any $x \in V(D)$, let X be a set of αn outneighbours and Y be a set of αn inneighbours of x. Consider $D[X]$, since D is triangle-free, there exists $x' \in X$ such that $d^+_X(x') < \alpha_0 |X|$ by the definition of α_0. Similarly, there exists $y' \in Y$ such that $d^-_Y(y') < \alpha_0 |Y|$. Let

$$X' = N^+(x') \setminus X \quad \text{and} \quad Y' = N^-(y') \setminus Y.$$

We have

$$|X'| \geqslant \delta^0(D) - d^+_X(x') > (1-\alpha_0)\alpha n.$$

Analogously, $|Y'| > (1-\alpha_0)\alpha n$. Again, from the fact that D is triangle-free it follows that

$$X' \cap Y = X \cap Y' = \emptyset.$$

If $X' \cap Y' = \emptyset$, then

$$n \geqslant 1 + |X| + |X'| + |Y| + |Y'| > (4 - 2\alpha_0)\alpha n = n,$$

a contradiction. So $X' \cap Y' \neq \emptyset$, which yields a 4-cycle through x. □

Lemma 2.6. *Each vertex of D is contained in a 5-cycle.*

P r o o f. Note that $\delta^0(D) > \alpha n > \beta n$. For any $x \in V(D)$, $D[\{x\} \cup N^+(x)]$ contains a transitive triangle by Lemma 2.2. There are vertices $a, y \in N^+(x)$ such that $xa, xy, ay \in A(D)$. To prove that x is contained in a 5-cycle, it suffices to prove the existence of at least one of the following:

(i) a y-x path of length 4,

(ii) a y-x path of length 3 avoiding a.

By Lemma 2.1, the diameter of D is at most 4, so we must have a y-x path of length at most 4. Since D is triangle-free we cannot have a y-x path of length 2. Thus we have a y-x path of length 3 or 4, and if it is of length 3, it must avoid a because D is oriented. $\square$

Lemma 2.7. *Each vertex of D is contained in a 6-cycle.*

P r o o f. Note that $\delta^0(D) > \alpha n > \beta n$. For any vertex $x \in V(D)$, by Lemma 2.2, $D[\{x\} \cup N^+(x)]$ contains a transitive triangle. We can pick an arc az in the outneighbourhood of x. Again use Lemma 2.2 to find an arc by in the outneighbourhood of z. To complete the proof it suffices to prove the existence of at least one of the following:

(i) a y-x path of length 2,

(ii) a y-x path of length 3,

(iii) a y-x path of length 4 avoiding z.

Let X be a set of αn inneighbours of x and Y be a set of αn outneighbours of y. Suppose $X \cap Y = \emptyset$, otherwise (i) is satisfied. By the same arguments as before, there exist $x' \in X$ with $d^-_X(x') < \alpha_0|X|$ and $y' \in Y$ with $d^+_Y(y') < \alpha_0|Y|$. Let

$$X' = N^-(x') \setminus X \quad \text{and} \quad Y' = N^+(y') \setminus Y.$$

We have

$$|X'| \geqslant \delta^0(D) - d^-_X(x') > (1 - \alpha_0)\alpha n,$$

and $|Y'| > (1 - \alpha_0)\alpha n$ similarly. We may assume that

$$X \cap Y' = X' \cap Y = \emptyset,$$

since otherwise (ii) is satisfied. If $X' \cap Y' = \emptyset$, then

$$n \geqslant 1 + |X| + |X'| + |Y| + |Y'| > (4 - 2\alpha_0)\alpha n = n,$$

a contradiction. So $X' \cap Y' \neq \emptyset$. This implies (iii). $\square$

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