Kybernetika

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Kybernetika, Vol. 53 (2017), No. 5, 892–910

Persistent URL: http://dml.cz/dmlcz/147100

KYBERNETIKA — VOLUME *53* (2017), NUMBER 5, PAGES 892–910

CONGRUENCES AND HOMOMORPHISMS ON Ω-ALGEBRAS

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The topic of the paper are Ω-algebras, where Ω is a complete lattice. In this research we deal with congruences and homomorphisms. An Ω-algebra is a classical algebra which is not assumed to satisfy particular identities and it is equipped with an Ω-valued equality instead of the ordinary one. Identities are satisfied as lattice theoretic formulas. We introduce Ω-valued congruences, corresponding quotient Ω-algebras and Ω-homomorphisms and we investigate connections among these notions. We prove that there is an Ω-homomorphism from an Ω-algebra to the corresponding quotient Ω-algebra. The kernel of an Ω-homomorphism is an Ω-valued congruence. When dealing with cut structures, we prove that an Ω-homomorphism determines classical homomorphisms among the corresponding quotient structures over cut subalgebras. In addition, an Ω-congruence determines a closure system of classical congruences on cut subalgebras. Finally, identities are preserved under Ω-homomorphisms.

Keywords: lattice-valued algebra, congruence, homomorphism

Classification: 06D72, 08A72

1. INTRODUCTION

In this research we investigate some universal algebraic aspects of Ω-valued algebraic structures, where Ω is a complete lattice.

Our research originates in fuzzy structures and in Ω-sets. As it is well known, the fuzzy set theory was found in 1965. by L. Zadeh, and has become a highly developed theory since then. Ω-sets, as an intention for modeling intuitionistic logic, appeared 1979, in the paper [16] by Fourman and Scott. An Ω-set is a nonempty set A equipped with an Ω-valued equality E, with truth values in a complete Heyting algebra Ω. E is a symmetric and transitive function from A^2 to Ω. Ω-sets have been further applied to non-classical predicate logics, and also to theoretical foundations of the fuzzy set theory ([18, 20]).

We use Ω-sets, but not strictly. In our approach Ω *is a complete lattice* (not necessarily a Heyting algebra). The main reason for this membership values structure is that it allows the use of cut-sets as a tool appearing in the fuzzy set theory. In this setting, main algebraic and set-theoretic notions and their properties can be generalized

DOI: 10.14736/kyb-2017-5-0892

from their classical origin to the lattice-valued framework ([22]). So we deal also with lattice-valued structures, using an ordinary complete lattice.

Therefore, we combine the two approaches in a specific way: Ω-sets as basic objects, and lattice-valued structures generalizing the notion of a subset, together with cut-sets techniques as a bridge from functions to classical sets.

Lattice-valued structures were developed within the fuzzy set theory in which the unit interval has been replaced by a complete lattice (firstly by Goguen [17]). This approach is widely used for dealing with algebraic topics (see e. g., [14], then also [23]), and with the lattice-valued topology (starting with [21] and many others). In the recent decades, along with the development of the fuzzy logic, a complete lattice as a membership (truth values) structure is often replaced by a complete residuated lattice (see e. g., [2]). But then the cut structures do not keep algebraic properties satisfied on the basic fuzzy structure.

A lattice-valued equality generalizing the classical one has been introduced in the fuzzy mathematics by Höhle in [19], and then it was used in investigations of fuzzy functions and fuzzy algebraic structures by many authors, in particular by Demirci ([11]), Bělohlávek and V. Vychodil ([3]) and others.

Identities for lattice-valued structures with a fuzzy equality were firstly investigated in [3], see also book [4]. Identities as used here were introduced in [24], and then developed in [5, 6, 7, 8]. In this framework, an identity holds if the corresponding lattice-theoretic formula is fulfilled. What is new in this approach is that an identity may hold on a lattice-valued algebra, while the underlying classical algebra does not satisfy the analogue classical identity.

This paper is organized as follows. The preliminary section contains relevant notions and their properties from the topics we use: lattice theory, universal algebra, lattice valued functions and relations. There are also some notions about Ω-structures, published previously. Section 3 contains the results of the paper. Some general notions concerning Ω-structures are introduced: Ω-set-map and Ω-homomorphism, Ω-congruence, kernel... Some of these notions, like an Ω-quotient structure are essentially different from the classical counterpart. Starting with Ω-algebras, we prove that the natural Ω-set-map is an Ω-homomorphism, and conversely, that the kernel of an Ω-homomorphism is an Ω-congruence on the domain Ω-algebra. In addition, we prove that identities are preserved under Ω-homomorphism. Finally, a representation theorem for an Ω-homomorphism in terms of classical homomorphisms over the corresponding quotient cut structures is proved.

There are essential differences among our approach and other known investigations of lattice-valued algebraic structures, like [3, 11, 14] and others mentioned in references. Namely, since Ω-algebras are based on Ω-sets, the classical equality is replaced by a lattice-valued one, but with weakened reflexivity (in [3] and [11], reflexivity is classical: $E(x, x) = 1$ for every x). Therefore, in our case all algebraic properties are fulfilled on the lattice-valued domain, not necessarily on the crisp basic algebra. Next, algebraic identities hold on Ω-algebras via particular lattice-valued formulas (in [3] identities are fulfilled up to some value, in other papers they hold in the classical way). As a consequence (since Ω is a complete lattice), basic algebraic properties like identities, connection of homomorphisms and congruences,... are preserved in the classical sense on quotients of cut-substructures over cuts of Ω-equality. E.g., an Ω-semigroup does not necessarily fulfill the crisp associativity, an Ω-lattice may not possess an order in the classical way, an Ω-homomorphism need not be a homomorphism of the basic structure. Still, particular quotients constructed by cuts of the Ω-equality are classical semigroups, in the case of Ω-lattices these quotients are lattice ordered, an Ω-homomorphism induces a classical homomorphisms on the mentioned quotients etc. Finally, the closest approach that we adopt here, the one by Höhle ([20]) differs from ours in some definitions (as indicated throughout the text), due to our usage of a complete lattice, cut-structures and weak reflexivity.

2. PRELIMINARIES

2.1. Lattices, algebras

A partially ordered set $(\Omega, \leqslant)$, where every subset M has both a meet $\bigwedge M$ and a join $\bigvee M$ is a **complete lattice**. A complete lattice possesses the least and the greatest elements 0 and 1, respectively. A meet and a join of a two-element subset $\{a, b\}$ of Ω are binary operations, denoted by $a \wedge b$ and $a \vee b$, respectively.

A **language** (or a **type**) $\mathcal{L}$ is a set $\mathcal{F}$ of functional symbols, together with a set of natural numbers (arities) associated to these symbols. An **algebra** of type $\mathcal{L}$ is a structure $\mathcal{A} = (A, F^A)$, where A is a nonempty set and F^A is a set of (fundamental) operations on A. An n-ary operation in F^A corresponds to an n-ary symbol in the language. A **subalgebra** of $\mathcal{A}$ is an algebra of the same type, defined on a subset of A, closed under the operations in F. **Terms** in a language are regular expressions constructed by the variables and operational symbols (see [9]). A term $t(x_1, \dots, x_n)$ in the language of an algebra $\mathcal{A}$ is here denoted in the same way as the corresponding term-operation $A^n \to A$. An **identity** in a language is a formula $t_1 \approx t_2$, where t_1, t_2 are terms in the same language. An identity $t_1(x_1, \dots, x_n) \approx t_2(x_1, \dots, x_n)$ is said to be **valid** on an algebra $\mathcal{A} = (A, F^A)$, or is **satisfied** by $\mathcal{A}$, if for all $a_1, \dots, a_n \in A$, the equality $t_1(a_1, \dots, a_n) = t_2(a_1, \dots, a_n)$ holds. An equivalence relation ρ on A which is compatible with all fundamental operations, meaning that $x_i \rho y_i, i = 1, \dots, n$ implies $f(x_1, \dots, x_n) \rho f(y_1, \dots, y_n)$, is a **congruence** relation on $\mathcal{A}$. If ρ is a congruence on $\mathcal{A}$, then for $a \in A$, the **congruence class** of a, $[a]_\rho$, and the quotient algebra $\mathcal{A}/\rho$ are defined respectively by $[a]_\rho := \{x \in A \mid (a, x) \in \rho\}$; $\mathcal{A}/\rho := (A/\rho, F^{A/\rho})$, where $A/\rho = \{[a]_\rho \mid a \in A\}$, and the operations in $F^{A/\rho}$ are defined by the representatives of the congruence classes.

2.2. Ω-valued functions and relations

Throughout the paper, $(\Omega, \wedge, \vee, \leqslant)$ is a complete lattice with the top and the bottom elements 1 and 0 respectively.

An Ω**-valued function** μ on a nonempty set A is a mapping $\mu : A \to \Omega$.

For $p \in L$, a **cut set** or a p**-cut** of an Ω-valued function $\mu : A \to \Omega$ is a subset μ_p of A which is the inverse image of the principal filter in Ω, generated by p:

$$\mu_p = \mu^{-1}(\uparrow(p)) = \{x \in X \mid \mu(x) \geqslant p\}.$$

An Ω**-valued** (binary) **relation** R **on** A is an Ω-valued function on A^2, i. e., it is a mapping $R : A^2 \to \Omega$.

$$R \text{ is } \textbf{symmetric} \text{ if } \ R(x,y) = R(y,x) \ \text{ for all } \ x,y \in A; \tag{1}$$

$$R \text{ is } \textbf{transitive} \text{ if } \ R(x,y) \geqslant R(x,z) \wedge R(z,y) \text{ for all } \ x,y,z \in A. \tag{2}$$

An Ω-valued symmetric and transitive relation R on A fulfills ([20]):

$$R(x,y) \leqslant R(x,x) \wedge R(y,y) \quad (\textbf{strictness}) \ . \tag{3}$$

Let $\mu : A \to \Omega$ and $R : A^2 \to \Omega$ be an Ω-valued function and an Ω-valued relation on A, respectively. Then we say that R is an Ω**-valued relation on** μ if for all $x,y \in A$

$$R(x,y) \leqslant \mu(x) \wedge \mu(y). \tag{4}$$

An Ω-valued relation R on $\mu : A \to \Omega$ is said to be **reflexive on** μ or μ**-reflexive** if

$$R(x,x) = \mu(x) \text{ for every } x \in A. \tag{5}$$

A symmetric and transitive Ω-valued relation R on A, which is reflexive on $\mu : A \to \Omega$ is an Ω**-valued equivalence on** μ.

Clearly, an Ω-valued equivalence R on μ fulfills the strictness property (3).

For an algebra $\mathcal{A} = (A, F^A)$, an Ω-valued function $\mu : A \to \Omega$ is said to be **compatible over** $\mathcal{A}$ if it fulfils the following: For any operation f from F^A with arity greater than 0, $f : A^n \to A, n \in \mathbb{N}$, for all $a_1,\dots,a_n \in A$, and a constant $c \in F^A$, we have

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(f(a_1,\dots,a_n)), \quad \text{and} \quad \mu(c) = 1. \tag{6}$$

The following is known (see [8]).

Proposition 2.1. Let $\mu : A \to \Omega$ be a compatible function over an algebra $\mathcal{A}$ and let $t(x_1,\dots,x_n)$ be a term in the language of $\mathcal{A}$. If $a_1,\dots,a_n \in A$, then

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(t(a_1,\dots,a_n)). \tag{7}$$

Similarly, an Ω-valued relation $R : A^2 \to \Omega$ on an algebra $\mathcal{A} = (A, F^A)$ is **compatible** with the operations in F if the following hold: for every n-ary operation $f \in F^A$, for all $a_1,\dots,a_n,b_1,\dots,b_n \in A$, and for every constant $c \in F^A$

$$\bigwedge_{i=1}^{n} R(a_i,b_i) \leqslant R(f(a_1,\dots,a_n), f(b_1,\dots,b_n)), \quad \text{and} \quad R(c,c) = 1. \tag{8}$$

Proposition 2.2. Let $R : A^2 \to \Omega$ be a compatible Ω-valued relation on an algebra $\mathcal{A}$ and let $t(x_1,\dots,x_n)$ be a term in the language of $\mathcal{A}$. If $a_1,\dots,a_n,b_1,\dots,b_n \in A$, then

$$\bigwedge_{i=1}^{n} R(a_i,b_i) \leqslant R(t(a_1,\dots,a_n), t(b_1,\dots,b_n)). \tag{9}$$

□

2.3. Ω-set

According to [16], an **Ω-set** is a pair (A, E), where A is a nonempty set, and E is a symmetric and transitive Ω-valued relation on A. In this framework, E is said to be an Ω-valued equality on A. This notation is used throughout the text.

Generalizing a notion of a subset, for an Ω-set (A, E), we denote by μ^E the Ω-valued function on A, defined by

$$\mu^E(x) := E(x, x). \tag{10}$$

We say that μ^E is *determined by* E. Clearly, by the strictness property (3), E is an Ω-valued relation on μ^E, namely, it is an Ω-valued equivalence on μ^E.

Lemma 2.3. If (A, E) is an Ω-set and $p \in \Omega$, then the cut E_p is an equivalence relation on the corresponding cut μ^E_p of μ^E.

2.4. Ω-algebra; identities

Here we equip Ω-sets with algebraic structures and investigate them in the framework of lattice-valued functions. Notions and claims in this section were presented and proved in [8].

Let $\mathcal{A} = (A, F^A)$ be an algebra and $E : A^2 \to \Omega$ an Ω-valued equality on A, which is compatible with the operations in F^A. Then we say ([8]) that $(\mathcal{A}, E)$ is an **Ω-algebra**. Algebra $\mathcal{A}$ is the **underlying algebra** of $(\mathcal{A}, E)$.

Proposition 2.4. (Budimirović et al. [8]) Let $(\mathcal{A}, E)$ be an Ω-algebra. Then the following hold:

(i) The function $\mu^E : A \to \Omega$ determined by E, is compatible over $\mathcal{A}$.

(ii) For every $p \in \Omega$, the cut μ^E_p of μ^E is a subalgebra of $\mathcal{A}$.

(iii) For every $p \in \Omega$, the cut E_p of E is a congruence relation on μ^E_p.

Remark 2.5. Let us explain what we are generalizing by introducing Ω-algebras and related notions. In the classical case, when Ω is a two-element chain, an Ω-set (A, E) identifies a *subset* B of A such that E is an *equivalence relation* on B. Consequently, an Ω-algebra $(\mathcal{A}, E)$ in this case determines a *subalgebra* $\mathcal{B}$ of $\mathcal{A}$ so that E is a congruence relation on $\mathcal{B}$ (or a *week congruence* on $\mathcal{A}$, see e. g., [10]). Clearly, the subset (subalgebra) B is determined by the maximal diagonal sub-relation of E. We deal mostly with cases in which neither the algebra $\mathcal{A}$ nor the subalgebra $\mathcal{B}$ possesses some algebraic property (e. g., does not fulfill some identities), but the quotient structure $\mathcal{B}/E$ does.

In our generalization by which we combine Ω-sets and lattice-valued structures, a subalgebra $\mathcal{B}$ becomes a compatible function μ^E, so that E is a compatible Ω-valued equivalence on μ^E. Consequently, what we generalize here are classical algebraic notions related not to algebras themselves, but to their *quotient structures on subalgebras.*

We shall use the following property of Ω-equalities (which is a direct consequence of Proposition 2.2).

Lemma 2.6. (Budimirović et al. [8]) Let $(\mathcal{A}, E)$ be an Ω-algebra and $t(x_1, \ldots, x_n)$ a term in the language of the algebra $\mathcal{A}$. Then for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$

$$\bigwedge_{i=1}^{n} E(a_i, b_i) \leqslant E(t(a_1, \ldots, a_n), t(b_1, \ldots, b_n)).$$

Next we define how identities hold on Ω-algebras, according to the approach in [24].

Let and $u(x_1, \ldots, x_n) \approx v(x_1, \ldots, x_n)$ (briefly $u \approx v$) be an identity in the type of an Ω-algebra $(\mathcal{A}, E)$. We assume, as usual, that variables appearing in terms u and v are from $x_1, \ldots, x_n$. Then, $(\mathcal{A}, E)$ **satisfies identity** $u \approx v$ (i. e., this identity **holds** on $(\mathcal{A}, E)$) if for all $a_1, \ldots, a_n \in A$

$$\bigwedge_{i=1}^{n} \mu^E(a_i) \leqslant E(u(a_1, \ldots, a_n), v(a_1, \ldots, a_n)). \tag{11}$$

If Ω-algebra $(\mathcal{A}, E)$ satisfies an identity, then this identity need not hold on $\mathcal{A}$. On the other hand, if the supporting algebra fulfills an identity then also the corresponding Ω-algebra does (see [8]).

Theorem 2.7. (Budimirović et al. [8]) Let $(\mathcal{A}, E)$ be an Ω-algebra, and $\mathcal{F}$ a set of identities in the language of $\mathcal{A}$. Then, $(\mathcal{A}, E)$ satisfies all identities in $\mathcal{F}$ if and only if for every $p \in L$ the quotient algebra μ_p^E / E_p satisfies the same identities.

3. RESULTS: Ω-SET-MAPS, CONGRUENCES AND HOMOMORPHISMS

3.1. Ω-subset; Ω-set-map

Let (A, E) be an Ω-set, and $E1 : A^2 \to \Omega$ a symmetric and transitive Ω-relation on A fulfilling $E1 \leqslant E$, so that the following holds: for all $x, y \in A$

$$E1(x, y) = E(x, y) \wedge E1(x, x) \wedge E1(y, y). \tag{12}$$

Obviously, $(A, E1)$ is an Ω-set and we say that it is an **Ω-subset** of (A, E).

Remark 3.1. A straightforward generalization of a subset to an Ω-subset (as it is done in [16] and generally in category theory) would be to take $B \subseteq A$, and a restriction $E|_B$ of E to B. However, originating in fuzzy framework, we consider E to be an Ω-relation on a function μ^E determined by the diagonal of E, which generalizes a characteristic function; in this terminology μ^E is already a fuzzy subset and E is an Ω-equivalence on it. Consequently, formula (12) does define a restriction of E to a fuzzy subset $\mu^{E1} : A \to \Omega$, $\mu^{E1}(x) := E1(x, x)$. Needless to say, a classical restriction of E to a subset B of A also fulfils condition (12), as it is in formula (16) in the sequel.

Let (M, E) and (N, G) be two Ω-sets. Then a mapping $\varphi : M \to N$, such that for all $a, b \in M$

$$E(a, b) \leqslant G(\varphi(a), \varphi(b)), \quad \text{and} \tag{13}$$

$$\mu^E(a) = \mu^G(\varphi(a)) \text{ i.e., } \quad E(a,a) = G(\varphi(a), \varphi(a)) \tag{14}$$

is called an **Ω-set-morphism**, or an **Ω-set-map**. Symbolically, we denote an Ω-set-map as $\varphi : (M, E) \to (N, G)$.

The following is straightforward.

Lemma 3.2. If φ is a bijection from M to N, then the inverse function $\varphi^{-1} : N \to M$ is an Ω-set-map from (N, G) to (M, E) if and only if

$$E(a, b) = G(\varphi(a), \varphi(b)). \tag{15}$$

An Ω-set-map $\varphi : (M, E) \to (N, G)$ is injective or surjective if $\varphi : M \to N$ fulfils the same property in the classical sense. Due to Lemma 3.2, we define an Ω-set-map $\varphi : (M, E) \to (N, G)$ to be an **Ω-bijection** if it is injective and surjective and fulfils the property (15).

Let $\varphi : (M, E) \to (N, G)$ be an Ω-set-map. Let $\varphi E : N \times N \to \Omega$ be defined by

$$\varphi E(u, v) := \begin{cases} G(u, v) & \text{if } u, v \in \varphi(M) \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

The following is straightforward.

Lemma 3.3. If $\varphi : (M, E) \to (N, G)$ is an Ω-set-map, then $(N, \varphi E)$ is an Ω-subset of (N, G).

Analogously to the classical case we define the kernel of an Ω-set-map.

Let φ be an Ω-set-map from (M, E) to (N, G). Then a binary Ω-valued relation $\ker_\Omega \varphi : M^2 \to \Omega$ given by

$$\ker_\Omega \varphi(a, b) = G(\varphi(a), \varphi(b)), \quad \text{for all } a, b \in M, \tag{17}$$

is called an **Ω-valued kernel** of φ. To simplify notation, we denote $\ker_\Omega \varphi$ by K^φ. Obviously, the kernel K^φ of an Ω-set-map $\varphi : (M, E) \to (N, G)$ is an Ω-valued equivalence relation on μ^E; namely, it is a μ^E-reflexive, symmetric and transitive Ω-valued relation on M.

Proposition 3.4. Let (M, E) and (N, G) be Ω-sets, $\varphi : M \to N$ a function and $K^\varphi : M^2 \to \Omega$ defined by

$$K^\varphi(a, b) = G(\varphi(a), \varphi(b)).$$

Then φ is an Ω-set-map from (M, E) to (N, G) if and only if for all $a, b \in M$, $E(a, b) \leqslant K^\varphi(a, b)$ and $E(a, a) = K^\varphi(a, a)$.

Proof. Assuming $E(a, b) \leqslant K^\varphi(a, b)$ for all $a, b \in M$, we have

$$E(a, b) \leqslant K^\varphi(a, b) = G(\varphi(a), \varphi(b)), \quad \text{hence} \quad E(a, b) \leqslant G(\varphi(a), \varphi(b)).$$

In addition,

$$G(\varphi(a), \varphi(a)) = K^\varphi(a, a) = E(a, a),$$

and φ is an Ω-set-map from (M, E) to (N, G).

On the other hand, if φ is an Ω-set-map from (M, E) to (N, G), then for all $a, b \in M$ $E(a, b) \leqslant E(a, a) \wedge E(b, b)$ and $E(a, b) \leqslant G(\varphi(a), \varphi(b))$ hence $E(a, b) \leqslant K^{\varphi}(a, b)$; further, by (14) it is straightforward that also $E(a, a) = K^{\varphi}(a, a)$ holds and the conditions required by the proposition are fulfilled. □

3.2. Ω-algebra and Ω-subalgebra

Let $(\mathcal{A}, E)$ be an Ω-algebra, and (A, E_1) an Ω-subset of (A, E). By (12), E_1 is a symmetric and transitive Ω-relation on A, fulfilling for all $x, y \in A$

$$E_1(x, y) = E(x, y) \wedge E_1(x, x) \wedge E_1(y, y).$$

Let also E_1 be compatible with the operations in $\mathcal{A}$. Obviously, $(\mathcal{A}, E_1)$ is an Ω-algebra and we say that it is an **Ω-subalgebra** of $(\mathcal{A}, E)$.

The following is obvious.

Proposition 3.5. If $(\mathcal{A}, E_1)$ is an Ω-subalgebra of an Ω-algebra $(\mathcal{A}, E)$, and $\mu_1 : A \to \Omega$ is the Ω-valued function on A defined by $\mu_1(x) := E_1(x, x)$, then μ_1 is compatible over $\mathcal{A}$, i. e., it fulfills (6).

An Ω-subalgebra $(\mathcal{A}, E_1)$ of $(\mathcal{A}, E)$ fulfills all the identities that the latter does, as follows.

Theorem 3.6. Let $(\mathcal{A}, E_1)$ be an Ω-subalgebra of an Ω-algebra $(\mathcal{A}, E)$. If $(\mathcal{A}, E)$ satisfies the set Σ of identities, then also $(\mathcal{A}, E_1)$ satisfies all the identities in Σ.

Proof. Let $u \approx v$ be an identity from Σ, with variables $x_1, \dots, x_n$. Then, since $u \approx v$ holds in $(\mathcal{A}, E)$, by the definition of E_1 and the fact that it is compatible with operations on $\mathcal{A}$, by the definition of μ_1, and by Proposition 2.1, for all $a_1, \dots, a_n \in A$, we have

$$\begin{aligned}
& \bigwedge_{i=1}^{n} \mu_1(a_i) = \bigwedge_{i=1}^{n} E_1(a_i, a_i) \\
= \;& \bigwedge_{i=1}^{n} E(a_i, a_i) \wedge \bigwedge_{i=1}^{n} E_1(a_i, a_i) = \bigwedge_{i=1}^{n} \mu^E(a_i) \wedge \bigwedge_{i=1}^{n} E_1(a_i, a_i) \\
\leqslant \;& E(u(a_1, \dots, a_n), v(a_1, \dots, a_n)) \wedge \bigwedge_{i=1}^{n} E_1(a_i, a_i) \\
= \;& E(u(a_1, \dots, a_n), v(a_1, \dots, a_n)) \wedge \bigwedge_{i=1}^{n} \mu_1(a_i) \\
\leqslant \;& E(u(a_1, \dots, a_n), v(a_1, \dots, a_n)) \wedge \mu_1(u(a_1, \dots, a_n)) \wedge \mu_1(v(a_1, \dots, a_n)) \\
= \;& E(u(a_1, \dots, a_n), v(a_1, \dots, a_n)) \wedge E_1(u(a_1, \dots, a_n), u(a_1, \dots, a_n)) \\
\wedge \;& E_1(v(a_1, \dots, a_n), v(a_1, \dots, a_n)) \\
= \;& E_1(u(a_1, \dots, a_n), v(a_1, \dots, a_n)).
\end{aligned}$$

□

3.3. Ω-valued congruences and homomorphism

Let $\mathcal{M} = (M, F^M)$ be an algebra and $\overline{\mathcal{M}} = (\mathcal{M}, E)$ a corresponding Ω-algebra. An **Ω-valued congruence** on $\overline{\mathcal{M}}$ is an Ω-valued equivalence Θ on μ^E, compatible with the operations in $\mathcal{M}$ in the sense of (8), satisfying also for all $x, y \in M$

$$E(x,y) \leqslant \Theta(x,y). \tag{18}$$

Obviously, E is also a congruence on $\overline{\mathcal{M}}$, the smallest one with respect to componentwise order (18). Observe also that by the definition of a congruence Θ, for every $x \in M$,

$$\Theta(x,x) = E(x,x) = \mu^E(x).$$

For an element $a \in M$, the **a-block** (block) of an Ω-congruence Θ on $\overline{\mathcal{M}}$ is a mapping $\Theta[a] : M \to \Omega$, such that for $x \in M$, $\Theta[a](x) := \Theta(a,x)$. The collection of all blocks, the **quotient set** of M over Θ, is denoted by M/Θ. Operations on M/Θ are defined as induced by the operations on $\mathcal{M}$: for an n-ary operation f,

$$f(\Theta[a_1], \dots, \Theta[a_n]) := \Theta[f(a_1, \dots, a_n)]. \tag{19}$$

Remark 3.7. Observe that these blocks are functions and they can be equal. In our approach, we consider them to be different, since each of these functions is denoted (indexed) by the element to which it is associated: $\Theta[a]$ is a function denoted by a, and suppose that for $b \in M$, $b \neq a$, $\Theta[a](x) = \Theta[b](x)$, for all $x \in M$. In our approach these functions are denoted by different elements (a and b), hence they are distinct elements (blocks, functions) in M/Θ.

By the above definition of operations, it is clear that $\mathcal{M}/\Theta = (M/\Theta, F)$ is an algebra isomorphic with $\mathcal{M}$ under $\Theta[x] \mapsto x$. It can be endowed with an Ω-valued equality E^Θ as follows:

$$E^\Theta(\Theta[a], \Theta[b]) := \Theta(a,b).$$

Then $\overline{\mathcal{M}/\Theta} = (\mathcal{M}/\Theta, E^\Theta)$ is an Ω-algebra, **the quotient Ω-algebra of** M, where

$$\mu/\Theta : \mathcal{M}/\Theta \to \Omega \text{ is defined by } \mu/\Theta(\Theta[x]) := E^\Theta(\Theta[x], \Theta[x]). \tag{20}$$

Obviously, for every $x \in M$, $\mu/\Theta(\Theta[x]) = \mu^E(x)$.

Let $\mathcal{M} = (M, F^M)$ and $\mathcal{N} = (N, F^N)$ be two algebras of the same type, and $\overline{\mathcal{M}} = (\mathcal{M}, E)$, $\overline{\mathcal{N}} = (\mathcal{N}, G)$ two corresponding Ω-algebras. An Ω-set-map $\varphi : (M, E) \to (N, G)$ is said to be an **Ω-homomorphism** from $\overline{\mathcal{M}}$ to $\overline{\mathcal{N}}$ if for all $a, a_1, \dots, a_n \in M$, for every constant $c \in M$ and the corresponding constant $c_1 \in N$, and for every n-ary function $f \in F^M$, the following conditions hold:

$$\mu^E(a_1) \wedge \cdots \wedge \mu^E(a_n) \leqslant G(\varphi(f(a_1, \dots, a_n)), f(\varphi(a_1), \dots, \varphi(a_n))); \tag{21}$$

$$\mu^E(c) \leqslant G(\varphi(c), c_1). \tag{22}$$

Observe that an Ω-homomorphism need not be a homomorphism of the underlying algebras, see Example 2.

As usual, an injective Ω-homomorphism is an **Ω-monomorphism**, and an **Ω-epimorphism** is a surjective Ω-homomorphism. Finally, an **Ω-isomorphism** φ is an Ω-homomorphism which is an Ω-bijection, i. e., it fulfils also (15).

The next lemma describes the extension of Ω-homomorphisms to terms.

Lemma 3.8. Let u be an n-ary term over the set $\{x_1, \dots, x_n\}$ of variables, in the language of algebras $\overline{\mathcal{M}} = (\mathcal{M}, E)$ and $\overline{\mathcal{N}} = (\mathcal{N}, G)$. If $\varphi : M \to N$ is an Ω-homomorphism, then for all $a_1, \dots, a_n \in M$

$$\bigwedge_{i=1}^{n} \mu^E(a_i) \leqslant G(\varphi(u(a_1, \dots, a_n)), u(\varphi(a_1), \dots, \varphi(a_n))).$$

P r o o f. We give a proof by induction on the complexity, i. e., on the number of occurrences of n-ary operation symbols, length of u, denoted by $l(u)$. If $l(u) = 1$, then $u = f$, for a fundamental function symbol f, and hence the statement is true by the definition of an Ω-homomorphism.

Inductively, assume that $l(u) > 1$ and that for every term v, $l(v) < l(u)$ the assumption holds. Then,

$$u(x_1, \dots, x_n) = f(u_1(x_1, \dots, x_n), \dots, u_n(x_1, \dots, x_n))$$

and since $l(u_i) < l(u)$, for $i \in \{1, \dots, n\}$ we have that

$$\begin{aligned}
&\bigwedge_{i=1}^{n} \mu^E(a_i) \leqslant G(\varphi(u_i(a_1, \dots, a_n)), u_i(\varphi(a_1), \dots, \varphi(a_n))), \quad \text{and} \\
&\bigwedge_{i=1}^{n} \mu^E(a_i) \leqslant \bigwedge_{i=1}^{n} G(\varphi(u_i(a_1, \dots, a_n)), u_i(\varphi(a_1), \dots, \varphi(a_n))) \\
&\leqslant G(f(\varphi(u_1(a_1, \dots, a_n)), \dots, \varphi(u_n(a_1, \dots, a_n))), \\
&f(u_1(\varphi(a_1), \dots, \varphi(a_n)), \dots, u_n(\varphi(a_1), \dots, \varphi(a_n))).
\end{aligned}$$

In addition,

$$\bigwedge_{i=1}^{n} \mu^E(a_i) \quad \leqslant \quad \bigwedge_{i=1}^{n} \mu^E(u_i(a_1, \dots, a_n))$$

$$\leqslant G(\varphi(f(u_1(a_1, \dots, a_n), \dots, u_n(a_1, \dots, a_n))), f(\varphi(u_1(a_1, \dots, a_n) \dots, \varphi(u_n(a_1, \dots, a_n)))).$$

Therefore by transitivity,

$$\begin{aligned}
&\bigwedge_{i=1}^{n} \mu^E(a_i) \\
&\leqslant G(\varphi(f(u_1(a_1, \dots, a_n), \dots, u_n(a_1, \dots, a_n))), \\
&\qquad f(u_1(\varphi(a_1), \dots, \varphi(a_n)), \dots, u_n(\varphi(a_1), \dots, \varphi(a_n)))) \\
&= G(\varphi(u(a_1, \dots, a_n)), u(\varphi(a_1), \dots, \varphi(a_n))).
\end{aligned}$$

This completes the proof. □

Theorem 3.9. Let $\overline{\mathcal{M}} = (\mathcal{M}, E), \overline{\mathcal{N}} = (\mathcal{N}, G)$ be two Ω-algebras and φ an Ω-homomorphism from $\overline{\mathcal{M}}$ to $\overline{\mathcal{N}}$. Then the Ω-valued kernel of φ, K^φ, as defined by (17), is an Ω-valued congruence on $\overline{\mathcal{M}}$.

Proof. Clearly, K^φ is an Ω-valued equivalence relation on μ^E. Hence we have to show that K^φ is compatible with every function $f \in F^M$. Let $(a_1, b_1), \dots, (a_n, b_n) \in M^2$ and $f \in F^M$ an n-ary function. Then

$$\begin{array}{cl}
 & K^\varphi(a_1, b_1) \wedge \cdots \wedge K^\varphi(a_n, b_n) \\
= & G(\varphi(a_1), \varphi(b_1)) \wedge \cdots \wedge G(\varphi(a_n), \varphi(b_n)) \\
= & (G(\varphi(a_1), \varphi(b_1)) \wedge \mu^E(a_1) \wedge \mu^E(b_1)) \wedge \cdots \wedge (G(\varphi(a_n), \varphi(b_n)) \wedge \mu^E(a_n) \wedge \mu^E(b_n)) \\
= & (G(\varphi(a_1), \varphi(b_1)) \wedge \cdots \wedge G(\varphi(a_n), \varphi(b_n))) \wedge \mu^E(a_1) \wedge \cdots \wedge \mu^E(a_n) \\
\wedge & \mu^E(b_1) \wedge \cdots \wedge \mu^E(b_n) \\
\leqslant & G(f(\varphi(a_1) \dots \varphi(a_n)), f(\varphi(b_1) \dots \varphi(b_n))) \\
\wedge & G(\varphi(f(a_1, \dots, a_n), f(\varphi(a_1), \dots, \varphi(a_n)) \\
\wedge & G(\varphi(f(b_1, \dots, b_n), f(\varphi(b_1), \dots, \varphi(b_n)) \wedge \mu^E(f(a_1, \dots, a_n)) \wedge \mu^E(f(b_1, \dots, b_n)) \\
\leqslant & G(\varphi(f(a_1, \dots, a_n)), \varphi(f(b_1, \dots, b_n))) \wedge \mu^E(f(a_1, \dots, a_n)) \wedge \mu^E(f(b_1, \dots, b_n)) \\
= & K^\varphi(f(a_1, \dots, a_n), f(b_1, \dots, b_n)).
\end{array}$$

Hence K^φ is an Ω-valued congruence relation on $\overline{\mathcal{M}}$. □

Theorem 3.10. Let Θ be an Ω-valued congruence on an Ω-algebra $\overline{\mathcal{M}} = (\mathcal{M}, E)$. Then the mapping $\varphi : M \to M/\Theta$, given by $\varphi(x) = \Theta[x]$ is an Ω-homomorphism from $\overline{\mathcal{M}}$ to $\overline{\mathcal{M}/\Theta}$.

Proof. φ is an Ω-set-map from $(\mathcal{M}, E)$ to $(M/\Theta, E^\Theta)$, since $E^\Theta(\Theta[x], \Theta[y]) = \Theta(x, y)$, and $E \leqslant \Theta$. In addition, for $a \in M$

$$E(a, a) = \Theta(a, a) = E^\Theta(\Theta[a], \Theta[a]) = E^\Theta(\varphi(a), \varphi(a)).$$

Properties (21) and (22) also hold: for $a, a_1, \dots, a_n \in M$, and an n-ary operational symbol f from the language,

$$\begin{array}{cl}
 & \mu^E(a_1) \wedge \cdots \wedge \mu^E(a_n) = E(a_1, a_1) \wedge \cdots \wedge E(a_n, a_n) \\
= & \Theta(a_1, a_1) \wedge \cdots \wedge \Theta(a_n, a_n) \\
= & E^\Theta(\Theta[a_1], \Theta[a_1]) \wedge \cdots \wedge E^\Theta(\Theta[a_n], \Theta[a_n]) \\
= & \Theta(\varphi(a_1), \varphi(a_1)) \wedge \cdots \wedge \Theta(\varphi(a_n), \varphi(a_n)) \\
\leqslant & \Theta(f(\varphi(a_1), \dots, \varphi(a_n)), f(\varphi(a_1), \dots, \varphi(a_n))) \\
= & \Theta(\varphi(f(a_1, \dots, a_n)), f(\varphi(a_1), \dots, \varphi(a_n))),
\end{array}$$

by the definition (19) of operations on classes. If c is a constant in $\mathcal{M}$, then $\varphi(c) = \Theta[c] = 1$, hence

$$E^\Theta(\varphi(\Theta[c]), \Theta[c]) = \Theta(c, c) = 1 = \mu^E(c) = E(c, c),$$

and (22) holds. □

Theorem 3.11. Let Θ be an Ω-valued equivalence on an Ω-algebra $\overline{\mathcal{M}} = (\mathcal{M}, E)$. Then Θ is an Ω-valued congruence on $\overline{\mathcal{M}}$ if and only if for every $p \in \Omega$ such that $\mu_p^E \neq \emptyset$, the mapping $\phi_p : \mu_p^E \to \mu_p^E/\Theta_p$ given by $\phi_p(x) = [x]_{\Theta_p}$, is a classical homomorphism.

Proof. Let $p \in \Omega$ and suppose $\mu_p^E \neq \emptyset$. Then clearly, ϕ_p is a well defined function. It is a homomorphism: Let f be an n-ary fundamental operation in the language of $\mathcal{M}$. Then for $a_1, \ldots, a_n \in \mu_p^E$,

$$\begin{aligned} & \phi_p(f(a_1,\ldots,a_n)) = [f(a_1,\ldots,a_n)]_{\Theta_p} = f([a_1]_{\Theta_p},\ldots,[a_1]_{\Theta_p} \\ = \quad & f(\phi_p(a_1),\ldots,\phi_p(a_n)), \end{aligned}$$

since μ_p^E/Θ_p is a quotient structure.

The converse holds by the same sequence of equalities, with the assumption that ϕ_p is a homomorphism. □

As it is known in the classical case, if $h : M \to N$ is a homomorphism from $\mathcal{M}$ to $\mathcal{N}$, then $h(M)$ is the underlying set of the homomorphic image of $\mathcal{M}$ under h, which of course is a subalgebra of $\mathcal{N}$. Analogously, a homomorphic image of an Ω-algebra $\overline{\mathcal{M}}$ with respect to φ is, under an additional condition, an Ω-subalgebra of $\overline{\mathcal{N}}$.

Therefore, in the following (proposition and two theorems), we assume that Ω-homomorphism φ from $\overline{\mathcal{M}}$ to $\overline{\mathcal{N}}$ fulfills the following property. If f is an n-ary fundamental operation on the underlying algebra in $\overline{\mathcal{N}}$, then

$$f(\varphi(a_1),\ldots,\varphi(a_n)) \in \varphi(M). \tag{23}$$

Proposition 3.12. Let $\overline{\mathcal{M}} = (\mathcal{M}, E)$, $\overline{\mathcal{N}} = (\mathcal{N}, G)$ be two Ω-algebras and φ an Ω-homomorphism from $\overline{\mathcal{M}}$ to $\overline{\mathcal{N}}$. Then, $\overline{\varphi(\mathcal{M})} = (\mathcal{N}, \varphi E)$ is an Ω-subalgebra of $\overline{\mathcal{N}}$.

Proof. By Lemma 3.3, $(N, \varphi E)$ is an Ω-subset of (N, G). By (23), $\varphi(M)$ is a classical subalgebra of $\mathcal{N}$. In addition, φE is compatible with the operations in $\mathcal{N}$; if f is an n-ary operation on $\mathcal{N}$, then for $x_1, \ldots, x_n; y_1, \ldots, y_n \in N$, we have either

a) some of these elements are not images under φ,

b) all of them belong to $\varphi(M)$.

In case *a*), $\varphi E(x_1, y_1) \wedge \ldots \wedge \varphi E(x_n, y_n) = 0$, and compatibility holds trivially; in case *b*), we have that by (23), $\varphi(M)$ is a classical subalgebra of $\mathcal{N}$, hence

$$\begin{aligned} & \varphi E(x_1,y_1) \wedge \ldots \wedge \varphi E(x_n,y_n) = G(x_1,y_1) \wedge \ldots \wedge G(x_n,y_n) \\ \leqslant \quad & G(f(x_1,\ldots,x_n), f(y_1,\ldots,y_n)) = \varphi E(f(x_1,\ldots,x_n), f(y_1,\ldots,y_n)). \end{aligned}$$

□

Theorem 3.13. Let $(\mathcal{M}, E)$, $(\mathcal{N}, G)$ be Ω-valued algebras of the same type such that $\varphi : \mathcal{M} \to \mathcal{N}$ is an Ω-homomorphism and $(\mathcal{N}, \varphi E)$ an Ω-homomorphic image of $(\mathcal{M}, E)$ under φ. If $(\mathcal{M}, E)$ fulfills identity $u(x_1, \ldots, x_n) \approx v(x_1, \ldots, x_n)$, then also $(\mathcal{N}, \varphi E)$ satisfies the same identity.

Proof. We show that for $a_1, \ldots, a_n \in M$,

$$\bigwedge_{i=1}^{n} \mu^G(\varphi(a_i)) \leqslant \varphi E(u(\varphi(a_1), \ldots, \varphi(a_n)), v(\varphi(a_1), \ldots, \varphi(a_n)).$$

Since the identity $u(x_1, \ldots, x_n) \approx v(x_1, \ldots, x_n)$ holds in $(\mathcal{M}, E)$, we have

$$\begin{aligned}\bigwedge_{i=1}^{n} \mu^E(a_i) &\leqslant E(u(a_1, \ldots, a_n), v(a_1, \ldots, a_n)) \\ &\leqslant G(\varphi(u(a_1, \ldots, a_n)), \varphi(v(a_1, \ldots, a_n))),\end{aligned}$$

thus

$$\bigwedge_{i=1}^{n} \mu^E(a_i) \leqslant G(\varphi(u(a_1, \ldots, a_n)), \varphi(v(a_1, \ldots, a_n))). \tag{24}$$

By Lemma 3.8 it follows that

$$\bigwedge_{i=1}^{n} \mu^E(a_i) \leqslant G(\varphi(u(a_1, \ldots, a_n)), u(\varphi(a_1), \ldots, \varphi(a_n))) \tag{25}$$

and

$$\bigwedge_{i=1}^{n} \mu^E(a_i) \leqslant G(\varphi(v(a_1, \ldots, a_n)), v(\varphi(a_1), \ldots, \varphi(a_n))). \tag{26}$$

Hence by equations (24), (25) and (26) we have

$$\begin{aligned}\bigwedge_{i=1}^{n} \mu^E(a_i) &\leqslant G(\varphi(u(a_1, \ldots, a_n)), u(\varphi(a_1), \ldots, \varphi(a_n))) \\ &\wedge G(\varphi(v(a_1, \ldots, a_n)), v(\varphi(a_1), \ldots, \varphi(a_n))) \\ &\wedge G(\varphi(u(a_1, \ldots, a_n)), \varphi(v(a_1, \ldots, a_n))) \\ &\leqslant G(u(\varphi(a_1), \ldots, \varphi(a_n)), \varphi(v(a_1, \ldots, a_n))) \\ &\wedge G(\varphi(v(a_1, \ldots, a_n)), v(\varphi(a_1), \ldots, \varphi(a_n))) \\ &\leqslant G(u(\varphi(a_1), \ldots, \varphi(a_n)), v(\varphi(a_1), \ldots, \varphi(a_n))).\end{aligned}$$

Further, by the definition of an Ω-homomorphism, $\bigwedge_{i=1}^{n} \mu^E(a_i) = \bigwedge_{i=1}^{n} \mu^G(\varphi(a_i))$. Thus

$$\bigwedge_{i=1}^{n} \mu^G(\varphi(a_i)) \leqslant G(u(\varphi(a_1), \ldots, \varphi(a_n)), v(\varphi(a_1), \ldots, \varphi(a_n))).$$

Hence by Theorem 3.6 and Proposition 3.12,

$$\bigwedge_{i=1}^{n} \mu^G(\varphi(a_i)) \leqslant \varphi E(u(\varphi(a_1), \ldots, \varphi(a_n)), v(\varphi(a_1), \ldots, \varphi(a_n))).$$

□

Theorem 3.14. Let $\overline{\mathcal{M}} = (\mathcal{M}, E)$, $\overline{\mathcal{N}} = (\mathcal{N}, G)$ be two Ω-algebras, φ an Ω-homomorphism from $\overline{\mathcal{M}}$ to $\overline{\mathcal{N}}$, and K^φ the Ω-valued kernel of φ, as defined by (17). Let also $\overline{\varphi(\mathcal{M})} = (\mathcal{N}, \varphi E)$ be the Ω-subalgebra of $\overline{\mathcal{N}} = (\mathcal{N}, G)$, determined by φ. Then, the map $\psi : M/K^\varphi \to N$, given by $\psi(K^\varphi[x]) := \varphi(x)$ is an Ω-homomorphism from $\overline{\mathcal{M}/K^\varphi}$ onto $\overline{\varphi(\mathcal{M})}$.

Proof. The map ψ is an Ω-set-function:

$$E^{\mu^E/K^\varphi}(K^\varphi[a], K^\varphi[b]) = K^\varphi(a,b) \leqslant G(\varphi(a), \varphi(b)) = G(\psi(K^\varphi[a]), \psi(K^\varphi[b])),$$

by (17), and $(\mu^E/K^\varphi)([a]) = E^{\mu^E/K^\varphi}(K^\varphi[a], K^\varphi[a]) = K^\varphi(a,a) = G(\varphi(a), \varphi(a)) \wedge \mu^E(a)$ $= \mu^G(\varphi(a)) = \mu^G(\psi(K^\varphi[a])$, by the definitions of an Ω-kernel, and of the function ψ. It is an Ω-homomorphism:

$$\begin{aligned} & \bigwedge_{i=1}^n (\mu^E/K^\varphi)([a_i]) = \bigwedge_{i=1}^n K_\varphi(a_i, a_i) = \bigwedge_{i=1}^n \mu^E(a_i) \\ \leqslant\ & G(\varphi(f(a_1, \ldots, a_n)), f(\varphi(a_1), \ldots, \varphi(a_n))) \\ =\ & G(\psi(K^\varphi[f(a_1, \ldots, a_n)]), f(\psi(K^\varphi[a_1]), \ldots, \psi(K^\varphi[a_n]))), \end{aligned}$$

since φ is an Ω-homomorphism. □

Remark 3.15. Let us mention that Theorem 3.12 shows that the quotient Ω-algebra over an Ω-congruence turns out to be essentially different from the classical quotient algebra. As indicated in Remark 3.7, equal block (functions) need not be compatible under operations, hence they are treated as different objects, denoted by elements of the underlying algebra. Therefore, in the case of an Ω-homomorphism, the quotient Ω-structure with respect to the kernel is not isomorphic with the image subalgebra, as in the classical case.

Theorem 3.16. An Ω-set-map $\varphi : M \to N$ from an Ω-algebra $(\mathcal{M}, E)$ to an Ω-algebra $(\mathcal{N}, G)$ of the same type is an Ω-homomorphism, if and only if for every $p \in \Omega$, the mapping $\overline{\varphi} : \mu_p^E/E_p \to \mu_p^G/G_p$, such that $\overline{\varphi}([x]_{E_p}) := [\varphi(x)]_{G_p}$ is a classical homomorphism.

Proof. Assume that φ is an Ω-homomorphism. Then for $a_1 \ldots, a_n \in \mu_p^E$ we have

$$G(\varphi(f(a_1, \ldots, a_n)), f(\varphi(a_1), \ldots, \varphi(a_n))) \geqslant \mu^E(a_1) \wedge, \ldots, \wedge \mu^E(a_n) \geqslant p,$$

implying

$$(\varphi(f(a_1, \ldots, a_n)), f(\varphi(a_1), \ldots, \varphi(a_n))) \in G_p.$$

Hence, for an n-ary fundamental operation f,

$$\begin{aligned} & f(\overline{\varphi}([a_1]_{E_p}), \ldots, \overline{\varphi}([a_n]_{E_p})) = f([\varphi(a_1)]_{G_p}, \ldots, [\varphi(a_n)]_{G_p}) \\ =\ & [f(\varphi(a_1), \ldots, \varphi(a_n))]_{G_p} = [\varphi(f(a_1, \ldots, a_n))]_{G_p} \\ =\ & \overline{\varphi}([f(a_1, \ldots, a_n)]_{E_p}) = \overline{\varphi}(f([a_1]_{E_p}, \ldots, [a_n]_{E_p})). \end{aligned}$$

Therefore, the mapping $\overline{\varphi}$ is a homomorphism.

For the converse, assume that for every $p \in \Omega$, $\overline{\varphi}$ is a homomorphism and let $a_1 \dots, a_n \in M$ such that $\mu^E(a_1) \wedge \dots \wedge \mu^E(a_n) = q$, hence $a_1 \dots, a_n \in \mu^E_q$. Now, since $\overline{\varphi}$ is a homomorphism, we have

$$\overline{\varphi}(f([a_1]_{E_q}, \dots, [a_n]_{E_q})) = f(\overline{\varphi}([a_1]_{E_q}), \dots, \overline{\varphi}([a_n]_{E_q})),$$

implying

$$[\varphi(f(a_1, \dots, a_n))]_{G_q} = f([\varphi(a_1)]_{G_q}, \dots, [\varphi(a_n)]_{G_q}) = [f(\varphi(a_1), \dots, \varphi(a_n))]_{G_q}.$$

Hence,

$$G(\varphi(f(a_1, \dots, a_n)), f(\varphi(a_1), \dots, \varphi(a_n))) \geqslant q = \mu^E(a_1) \wedge \dots \wedge \mu^E(a_n).$$

Therefore, the mapping φ is an Ω-homomorphism. This completes the proof. $\square$

The following example illustrates the definition of an Ω-congruence on an Ω-groupoid, the corresponding quotient Ω-groupoid, Theorems 3.10 and 3.11.

Example 3.17. Let $\mathcal{G} = (\{a, b, c, d, e\}, *)$ be a groupoid given by Table 1 and let Ω be a four element Boolean lattice $\Omega = \{0, p, q, 1\}$, Figure 1. An Ω-groupoid is $\overline{\mathcal{G}} = (\mathcal{G}, E)$, where the Ω-valued equality E is presented in Table 2. In Table 3, an Ω-valued congruence Θ on $\overline{\mathcal{G}}$ is given.

$*$	a	b	c	d	e
a	a	c	b	a	d
b	b	b	c	b	c
c	b	c	b	c	b
d	d	c	b	c	d
e	e	c	b	d	d

Tab. 1.

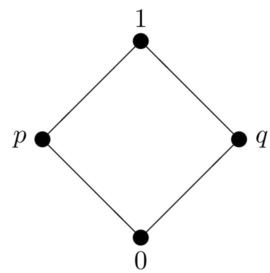

Fig. 1.

E	a	b	c	d	e
a	p	0	0	0	0
b	0	1	p	0	0
c	0	p	1	0	0
d	0	0	0	q	q
e	0	0	0	q	q

Tab. 2.

Θ	a	b	c	d	e
a	p	0	0	0	0
b	0	1	1	q	q
c	0	1	1	q	q
d	0	q	q	q	q
e	0	q	q	q	q

Tab. 3.

The quotient Ω-groupoid is $(\mathcal{G}/\Theta, E^\Theta)$, where

$$\mathcal{G}/\Theta = \{\Theta[a], \Theta[b], \Theta[c], \Theta[d], \Theta[e]\}, \quad \text{and}$$

$$\Theta[a] = \begin{pmatrix} a & b & c & d & e \\ p & 0 & 0 & 0 & 0 \end{pmatrix};\ \Theta[b] = \begin{pmatrix} a & b & c & d & e \\ 0 & 1 & 1 & q & q \end{pmatrix};\ \Theta[c] = \begin{pmatrix} a & b & c & d & e \\ 0 & 1 & 1 & q & q \end{pmatrix};$$
$$\Theta[d] = \begin{pmatrix} a & b & c & d & e \\ 0 & q & q & q & q \end{pmatrix};\ \Theta[e] = \begin{pmatrix} a & b & c & d & e \\ 0 & q & q & q & q \end{pmatrix}.$$

The operation in this groupoid is presented in Table 4, and the Ω-valued equality in Table 5.

$*$	$\Theta[a]$	$\Theta[b]$	$\Theta[c]$	$\Theta[d]$	$\Theta[e]$
$\Theta[a]$	$\Theta[a]$	$\Theta[c]$	$\Theta[b]$	$\Theta[a]$	$\Theta[d]$
$\Theta[b]$	$\Theta[b]$	$\Theta[b]$	$\Theta[c]$	$\Theta[b]$	$\Theta[c]$
$\Theta[c]$	$\Theta[b]$	$\Theta[c]$	$\Theta[b]$	$\Theta[c]$	$\Theta[b]$
$\Theta[d]$	$\Theta[d]$	$\Theta[c]$	$\Theta[b]$	$\Theta[c]$	$\Theta[d]$
$\Theta[e]$	$\Theta[e]$	$\Theta[c]$	$\Theta[b]$	$\Theta[d]$	$\Theta[d]$

Tab. 4.

E^Θ	$\Theta[a]$	$\Theta[b]$	$\Theta[c]$	$\Theta[d]$	$\Theta[e]$
$\Theta[a]$	p	0	0	0	0
$\Theta[b]$	0	1	1	q	q
$\Theta[c]$	0	1	1	q	q
$\Theta[d]$	0	q	q	q	q
$\Theta[e]$	0	q	q	q	q

.

Tab. 5.

It is easy to check that the mapping $x \mapsto \Theta[x]$ is an Ω-homomorphism from $(\mathcal{G}, E)$ to $(\mathcal{G}/\Theta, E^\Theta)$ (Theorem 3.10). To illustrate Theorem 3.11, we take a subgroupoid $\mu_p^E = \{a, b, c\}$; observe that $\mu^E(x) = E(x, x)$, $\mu_p^E = \{x \in G \mid \mu^E(x) \geqslant p\}$) and the cut Θ_p of Θ is given in Table 6:

Θ_p	a	b	c
a	1	0	0
b	0	1	1
c	0	1	1

.

Tab. 6.

Θ_p is a congruence on μ_p^E, and the map $x \mapsto [x]_{\Theta_p}$ is a classical natural homomorphism from μ_p^E onto the quotient groupoid $\mu_p^E/\Theta_p = \{\{a\}, \{b, c\}\}$. The same holds for all other corresponding cuts of μ^E and Θ.

Next, we present an example of an Ω-homomorphism which is not a classical homomorphism on the underlying groupoids. It also shows the ways identities are preserved in Ω-structures and under Ω-homomorphism (Theorems 3.13 and 3.16).

Example 3.18. Here we have two Ω-groupoids, $(\{a, b, c, d\}, E_1)$ and $(\{e, f, g\}, E_2)$, for which the corresponding operations and Ω-equalities are given by Tables 7, 8, 9 and 10. The lattice Ω is the one in Figure 1 (Example 1).

$\cdot$	a	b	c	d
a	a	b	c	d
b	a	b	d	b
c	c	d	c	c
d	d	b	d	d

Tab. 7.

$\cdot$	e	f	g
e	e	e	g
f	f	f	f
g	g	f	g

Tab. 8.

E_1	a	b	c	d
a	p	0	p	0
b	0	q	0	q
c	p	0	p	0
d	0	q	0	q

Tab. 9.

E_2	e	f	g
e	p	0	p
f	0	q	q
g	p	q	1

Tab. 10.

The underlying groupoids are evidently not commutative, while the omega ones, $(\{a,b,c,d\},E_1)$ and $(\{e,f,g\},E_2)$ are, i. e., they fulfill the lattice formula (11) which in this case has the following form:

$$\mu_i(x) \wedge \mu_i(y) \leqslant E_i(x \cdot y, y \cdot x), \quad i \in \{1,2\}.$$

This condition can be easily checked from the above tables, where

$$\mu_i(x) = E_i(x,x), \quad \text{and} \quad \mu_1 = \begin{pmatrix} a & b & c & d \\ p & q & p & q \end{pmatrix}; \quad \mu_2 = \begin{pmatrix} e & f & g \\ p & q & 1 \end{pmatrix}.$$

E.g., $\mu_1(a) \wedge \mu_1(b) = p \wedge q = 0 \leqslant E_1(a \cdot b, b \cdot a) = E_1(b,a) = 0$, and so on.

Let φ be the function $\{a,b,c,d\} \to \{e,f,g\}$, given by

$$\varphi = \begin{pmatrix} a & b & c & d \\ e & f & e & f \end{pmatrix}.$$

It is easy to check that φ is not a homomorphism among these crisp groupoids, e. g., $\varphi(a \cdot b) = \varphi(b) = f$, but $\varphi(a) \cdot \varphi(b) = e \cdot f = e$. However, it is an Ω-homomorphism from $(\{a,b,c,d\},E_1)$ to $(\{e,f,g\},E_2)$, i. e., conditions (13), (14), (21) and (22) are satisfied. In addition, the homomorphic image under this Ω-homomorphism, Ω-groupoid $(\{e,f,g\},E_2)$ is also commutative, it satisfies the same lattice-valued formula as the first Ω-groupoid (Theorem 3.13). Finally Theorem 3.16 is also illustrated, i. e., there are classical homomorphisms from quotient cut-structures of the first groupoid into the other e. g., from $(\mu_1)_p/(E_1)_p$ onto $(\mu_2)_p/(E_2)_p$.

4. CONCLUSION

The paper deals with homomorphisms and congruences in the field of Ω-algebras. We have presented the basic notions in this field from the universal algebraic aspect. It turned out that several properties are different from the classical theory, concerning e. g., quotient structures, homomorphic images etc. Still, the link between the classical homomorphisms, kernels and natural maps exists in the field of cut structures, namely their quotients over the cuts of the Ω-equalities.

Our next task is to investigate further topics in this framework, namely homomorphism and isomorphism theorems, in particular for known classes of algebras like groups, rings etc. Further, it is known that in classical algebra collections of subalgebras or congruences form algebraic lattices under inclusion. Is it the case also for collections of Ω-congruences and Ω-subalgebras under the componentwise order of functions? Finally, principal congruences, generated by pairs (a,b), a,b belonging to the underlying algebra, determine important substructures of these algebras. Their role in the framework of Ω-algebras should be investigated.

ACKNOWLEDGMENT

The authors express their gratitude to the anonymous referees whose comments improved the general quality of this paper.

The research of the 2^{nd} and the 3^{rd} author is supported by Ministry of Education, Science and Technological Development, Republic of Serbia, Grant No. 174013.

(Received October 30, 2016)

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