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# MULTISTAGE RISK PREMIUMS IN PORTFOLIO OPTIMIZATION

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This paper deals with a multistage stochastic programming portfolio selection problem with a new type of risk premium constraints. These risk premiums are constructed on the multistage scenario tree. Two ways of the construction are introduced and compared. The risk premiums are incorporated in the multistage stochastic programming portfolio selection problem. The problem maximizes the multivariate (multiperiod) utility function under condition that the multistage risk premiums are smaller than a prescribed level. The problem does not assume any separability of the multiperiod utility function. The performance of the suggested models is demonstrated for several kinds of multiperiod utility functions and several formulations of the multistage risk premium constraints. In all cases, including the risk premium constraints avoids the riskier positions.

*Keywords:* multistage risk premium, utility function, portfolio optimization, multistage stochastic programming

*Classification:* 91B28, 91B30

## 1. INTRODUCTION

Multistage stochastic portfolio optimization is determined to find the optimal strategy (periodically rebalancing portfolios) in order to reach a desired target. The comprehensive overview on multistage stochastic optimization can be found e. g. in [5] or [15]). The objective of the portfolio optimization problem is to maximize a utility function of the investor's wealth received in the course of the investment horizon. Therefore, we have to consider multidimensional (multivariate) utility functions, namely we focus on their non-separable class. Each variate corresponds to the investor's wealth at a particular time (end of a period). Multivariate utility functions were studied and analyzed in e. g. [1]. This multistage portfolio selection problem can provide a reasonable investment strategy since the shape of a utility function express also an information about the decision maker's risk attitude. However, the solution cannot guarantee that the decision maker does not find herself in an undesirably risky position during the investment's life. To control the risk of the positions within the investment horizon we introduce multistage risk premium constraints. The constrains can be seen as an alternative to risk measures constraints, chance constraints or stochastic dominance constraints, see

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e. g. [2, 4, 9] or [10] for a recent application of these constraints. The advantage of the risk premium constraints is that risk premiums are directly related to (derived from) the utility function unlike risk measures or chance constraints. The usage of stochastic dominance constraints is computationally very demanding (mainly for higher number of stages/scenarios) so it is mainly used in static optimization problems. Therefore, we introduce new formulations of portfolio selection problems using multistage risk premiums.

One-dimensional risk premium was firstly introduced and analyzed in [12]. It was later extended to multidimensional version (see, for instance, [1]). However, as far as we know, there is no suitable risk premium construction for multistage stochastic programming problems. The main attention was paid to multivariate generalizations of the risk premium for multidimensional utility functions which are applied for a static decision making, see e. g. [3, 8] or [13]. Therefore, this paper introduces a new approach to risk premiums suitable for multistage stochastic programming problems. Two methods of a multistage risk premium construction are presented and analyzed.

Stochastic features of multistage portfolio optimization problems are often provided through scenario approach which stems from scenario tree generation. The generation can be done in several different ways. In this paper, we use moment matching method suggested in [6]. We consider monthly returns of six Fama and French US representative portfolios based on size and book to market [7]. These representative portfolios serve as the base assets.

In the computational study we aim to compare performance of the optimal investment strategy stemming from two formulations. The first one did not take into account any explicit constraints on investor's risk exposure. The second one considers the constraints in the form employing the risk premium concept. As the result of the computation, we expect that the second formulation provides investor with lower expected wealth achieved throughout the horizon which performs lower risk. Moreover, we compare the results of these models for various choices of multivariate utility functions and multistage risk premium constructions.

The paper is organized as follows. Chapter 2 is devoted to model formulation, where we departure from very basic multistage deterministic portfolio optimization problem. As the next step we introduce scenario tree generation in order to capture stochastic features and provide a definition of a risk premium as we aim to incorporate it into the portfolio optimization problem. Chapter 3 presents the computational experiment together with discussion on obtained results accompanied by numerous graphs and tables. Finally, chapter 4 provides a brief concluding statement and considerations about the future work.

## 2. MODEL FORMULATION

Multistage portfolio optimization problems assume that there are $K$ risky assets and one riskless asset, usually represented by a bond or a bank account paying a fixed interest rate, traded during a finite time horizon. In order to achieve a desired target the investor can rebalance her portfolio at fixed times of the considered investment horizon. To simplify the situation, we assume that the investor rebalances her portfolio at $T$ equidistant moments so as she maximizes the expected utility over wealth achieved throughout the investment horizon. In this study we focus on a class of non-separable utility functions which makes it impossible to adopt a standard recursive dynamic programming solution methods.

Consider an investment horizon with $T$ equidistant periods such that the rebalancing of the portfolio is held at their beginning. Let $u$ be a $T$-dimensional non-separable utility function and let $w = (w_1, \dots, w_T)$ be the $T$-dimensional wealth vector, where $w_t$, $t = 1, \dots, T$ corresponds to wealth achieved at the end of time period $t$. We consider initial wealth $w_0$ to be given. The wealth fraction invested into asset $k$ at the beginning of period $t$ is denoted as $x_{tk}$ ($k = 0$ always corresponds to the fraction invested into the riskless asset). Similarly, we denote as $r_{tk}$ the one-period return of the $k$th asset corresponding to period $t$. Our aim is to find an investment strategy represented by matrix $x = (x_{tk})$ which maximizes utility of achieved wealth. We conveniently denote as $x_{t\cdot}$ $t$-column of the investment strategy matrix $x$, i. e., $x_{t\cdot} = (x_{t0}, \dots, x_{tK})$. The analogous notation is adopted for $r_{t\cdot}$. The problem can be mathematically formulated as follows:

$$\begin{aligned}
&\underset{x}{\text{maximize}} && u(w_1, \dots, w_T) && \\
&\text{subject to} && w_t = w_{t-1} + r_{t\cdot}^T x_{t\cdot} && t = 1, \dots, T, && (1)\\
& && w_{t-1} = 1^T x_{t\cdot} && t = 1, \dots, T, && (2)\\
& && x_{t\cdot} \geq 0 && t = 1, \dots, T. && (3)
\end{aligned}$$

Constraint (2) ensures that the whole wealth achieved at the end of time period $t-1$ is reallocated between risky and riskless assets at the beginning of time period $t$. Constraint (1) describes wealth development during the investment horizon corresponding to specific investment decisions. Finally constraint (3) guarantees that short positions are not allowed.

The above formulation relates to a deterministic problem where the returns on assets are non-random. In further text we develop a multistage stochastic portfolio optimization problem where the return on the riskless asset remains deterministic but the returns on risky assets become a random process. In order to derive a stochastic formulation we adopt a scenario approach with a specific choice of a scenario tree.

### 2.1. Scenario tree generation and scenario approach formulation

We assume that the one-period return on assets represented by random vector $R^* = (R_0^*, \dots, R_K^*)$ follows the same distribution at each time instance. The element $R_0^*$ is the return corresponding to the riskless asset and thus it is not a random variable. Although we assume the return on this asset to be deterministic and constant over the investment horizon, it is convenient to include the asset into the formulation and scenario tree generation. We assume that $R^*$ has discrete distribution with realizations $(r_{1\cdot}^*, \dots, r_{M\cdot}^*)$ where $r_{m\cdot}^* = (r_{m0}^*, \dots, r_{mK}^*)$ is taken with probability $p_m^*$, $m = 1, \dots, M$. The aim is to estimate parameters $r_{m\cdot}^*$, $p_m^*$, $m = 1, \dots, M$ so as they correspond to available data. For estimating the parameters we use moment matching method suggested in [6].

Let $\mu_i^n$ be the $n$th sample moment of $i$th asset, $i = 1, \dots, K, n = 1, \dots, N$ and $c_{ij}$ sample covariance of $i$th and $j$th asset, $i = 1, \dots, K$, $j = 1, \dots, K$. The parameters $r_{m\cdot}^*$

and $p_m^*$ are set to solve the following optimization problem $(ST)$

$$\begin{aligned}
\underset{r_{m\cdot}^*, p_m^*}{\text{minimize}} \quad & \sum_{i=1}^{K}\sum_{j=1}^{K}\left(\sum_{m=1}^{M} p_m^*(r_{mi}^* - \mu_{1i})(r_{mj}^* - \mu_{1j}) - c_{ij}\right)^2 \\
\text{subject to} \quad & \sum_{m=1}^{M} p_m^*(r_{mi}^*)^n = \mu_i^n \qquad i = 1,\dots,K,\ n = 1,\dots,N \\
& \sum_{m=1}^{M} p_m^* = 1 \\
& r_{m0}^* = r_0 \qquad m = 1,\dots,M \\
& p_m^* \geq 0 \qquad m = 1,\dots,M
\end{aligned}$$

where $r_0$ denotes risk free return. Generally, the above stated problem is non-convex. In order to obtain the parameters we employ global optimization techniques implemented in software GAMS which are dedicated to solve non-linear (possibly non-convex) optimization problems.

We consider a regular structure of the scenario tree which can be described as follows. At the beginning there is one node which branches out into $M$ followers, each of the follower is however $K+1$ dimensional. Each follower then branches out again into $M$ nodes and so on. Moreover, every ramification subjects to the same distribution determined by parameters $r_{m\cdot}^*$, $p_m^*$, $m = 1,\dots,M$. On the basis of this tree we define a scenario $s$ to be a specific path from the initial node to a particular terminal node. In further text $S$ denotes the set of all possible scenarios $s$. Obviously, the total number of scenarios equals to the total number of terminal nodes which can be computed as $M^T$. Figure 1 demonstrates the structure of the scenario tree for $M = 2$ and $T = 3$.

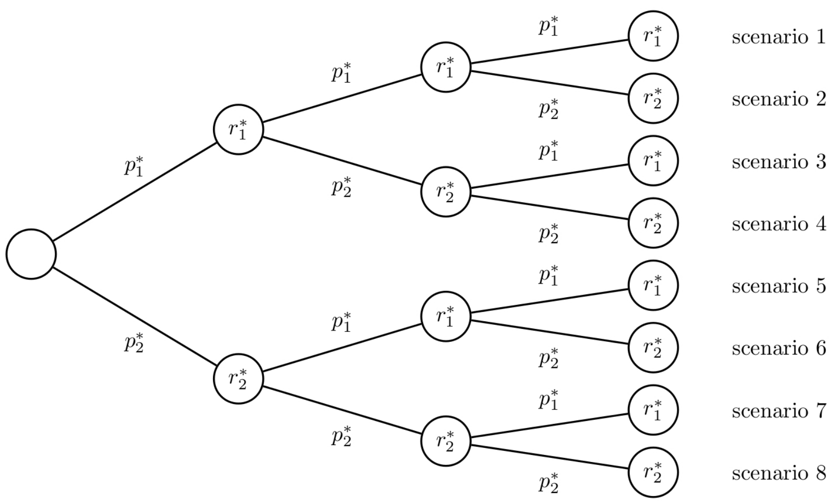


**Fig. 1.** Example of scenario tree for $M = 2$ and $T = 3$.

Using the scenario approach, $T$-dimensional wealth vector becomes scenario dependent, i. e., $w^s = (w_1^s, \ldots, w_T^s)$. Scenario dependency also enters into $T \times (K+1)$-dimensional decision matrix $x^s = \{x_{tk}^s\}_{t=1,k=0}^{T,K}$ and $T \times (K+1)$-dimensional return matrix $r^s = \{r_{tk}^s\}_{t=1,k=0}^{T,K}$. We recall that $r_{t0}^s$ corresponds to risk free return for all $t = 1, \ldots, T$ and $s \in S$.

In this paper we aim to solve multistage optimization problems. The optimal solution of these problems provide investor the optimal investment strategy which is an advise how to choose assets into her portfolio at each time instance of the investment horizon so as to achieve desired target. The main characteristic of multistage problems is the fact that decision at a particular time instance reacts the whole past, i. e. development of returns of assets in the past, and is not dependent on future since it is unknown. With respect to this characteristics, we have to consider the non-anticipativity constraints on the investment decisions and the wealth which are both scenario dependent. For more details about general construction of non-anticipativity constraints for multistage portfolio optimization problems we refer to [5] or [15]. The constraints for our specific choice of the scenario tree can be formulated in the following form:

$$\begin{aligned}
x_{t\cdot}^s &= x_{t\cdot}^q \qquad t = 1, \ldots, T,\ q = 1, \ldots, M^{t-1},\ s = (q-1)M^{T-t+1} + 1, \ldots, qM^{T-t+1}, \\
w_t^s &= w_t^q \qquad t = 1, \ldots, T,\ q = 1, \ldots, M^{t},\ s = (q-1)M^{T-t} + 1, \ldots, qM^{T-t}.
\end{aligned}$$

The optimization problem $(P1)$ using scenario approach can be formulated as:

$$\begin{aligned}
\underset{x^s}{\text{maximize}} \quad & \sum_{s \in S} p_s \cdot u\left(w_1^s, \ldots, w_T^s\right) & & \\
\text{subject to} \quad & w_t^s = w_{t-1}^s + (r_{t\cdot}^s)^T x_{t\cdot}^s & & t = 1, \ldots, T,\ s \in S, \qquad (4)\\
& w_{t-1}^s = 1^T x_{t\cdot}^s & & t = 1, \ldots, T,\ s \in S, \\
& w_0^s = w_0 & & s \in S, \\
& x_{t\cdot}^s \geq 0 & & t = 1, \ldots, T,\ s \in S, \\
& x_{t\cdot}^s = x_{t\cdot}^q & & t = 1, \ldots, T,\ q = 1, \ldots, M^{t-1}, \qquad (5)\\
& & & s = (q-1)M^{T-t+1} + 1, \ldots, qM^{T-t+1},
\end{aligned}$$

where $p_s$ denotes a probability of scenario $s$. We note that the non-anticipativity constraints for the wealth are redundant, they are ensured by (4) and (5).

We refer to this formulation later in the computational part in order to compare its solution with a solution of the model incorporating constraints on the risk exposure.

### 2.2. Concept of risk premiums

Investor's risk exposure at a specific investment position can be evaluated using various tools, for instance, one can simply calculate a portfolio variance or adopt more sophisticated risk measures as VaR or CVaR, see e. g. [14]. When dealing with investor's preferences captured by means of utility functions there is another convenient method to evaluate risk exposure which utilizes risk premium framework. A risk premium associated with a random outcome of an investment is defined as the maximal amount of money that the investor would be willing to pay in order to avoid the investment and receive its expected outcome instead. Considering univariate utility function $u$, the risk premium $\Pi$ at wealth level $w$ associated with random outcome $X$ can be formally defined as:

$$\mathbb{E}_x\; u(w+X) = u\;(w+\mathbb{E}_x X - \Pi).$$

There are several ways how to extend the above stated definition for multidimensional utility functions. One possible way is to define a risk premium at time instance $t$ to be the maximum amount of money the investor would be willing to pay in order to avoid a one-period random investment occurring at this time. In our specific model we aim to measure risk exposure at each node of the scenario tree. To this purpose, we firstly define risk premium associated with time instance $t$ and scenario $s$ for all $t = 1,\dots,T$ and $s \in S$ and secondly we propose a formula to calculate the premium at each node.

Consider a particular scenario $s$, wealth vector associated with scenario $s$, $w^s$, and investment strategy associated with scenario $s$, $x_t^s\cdot$, $t = 1,\dots,T$. We define risk premium $\pi_t^s$ at time instance $t$ as the solution of the following equation:

$$\begin{aligned} u\left(w_1^s,\dots,w_{t-1}^s, w_{t-1}^s + \mathbb{E}_R R_{t\cdot}^T x_{t\cdot}^s - \pi_t^s, w_{t+1}^s,\dots,w_T^s\right) \\ = \mathbb{E}_R\; u\left(w_1^s,\dots,w_{t-1}^s, w_{t-1}^s + R_{t\cdot}^T x_{t\cdot}^s, w_{t+1}^s,\dots,w_T^s\right). \quad (6)\end{aligned}$$

Thus the risk premium at time $t$ under scenario $s$ is the amount of money the investor would pay to avoid random investment $R_{t\cdot}^T x_{t\cdot}^s$ and receive its expectation $\mathbb{E}_R R_{t\cdot}^T x_{t\cdot}^s$ instead. We emphasize that the risk premium only measures the investor's risk exposure at specific time, however in reality the investor cannot pay this amount in order to substitute the random revenue by the certain one. Contrary to the classical notion of multivariate risk premium used in [1, 3, 8] or [13], our risk premium $\pi_t^s$ is not only time-dependent but also scenario-dependent to be applicable in multistage stochastic programming problems. This scenario-dependent feature makes the classical multivariate risk premiums inappropriate for these problems.

As indicated by the previous interpretation, risk premium is always associated with a moment in which the investor makes her decision. Thus, bearing in mind the structure of our decision tree, there always has to be exactly one risk premium associated with each node. There are several ways how to move from time-scenario dependent formulation of risk premium to time-node dependent. In this paper we present two possible approaches.

First one assumes that a premium corresponding to a particular node can be formulated as a weighted sum of all risk premiums $\pi_t^s$ where the summation is taken over all scenarios which have the node lying on their path. Mathematically, for each node we define the risk premium $\Pi_t^q$, $t = 1,\dots,T$, $q = 1,\dots,M^{t-1}$ as:

$$\Pi_t^q = \sum_{s=(q-1)M^{T-t+1}+1}^{qM^{T-t+1}} p_s^n \pi_t^s, \qquad (7)$$

where $\pi_t^s$ solves equation (6) and $p_s^n$ is a conditional probability, i. e.,

$$p_s^n = \left( \sum_{s=(q-1)M^{T-t+1}+1}^{qM^{T-t+1}} p_s \right)^{-1} p_s.$$

The second approach suggests to define risk premium corresponding to a particular node as the maximum of all risk premiums $\pi_t^s$ over all scenarios $s$ which have the node lying on their path. Mathematically, we formulate the premium $\Pi_t^q$, $t = 1, \dots, T$, $q = 1, \dots, M^{t-1}$ as:

$$\Pi_t^q = \max_{s \in \{(q-1)M^{T-t+1}+1, \dots, M^{T-t+1}\}} \pi_t^s. \tag{8}$$

Similarly, one can assume arbitrary mappings $\Phi_t : (\pi_t^1, \dots, \pi_t^{M^t}) \to (\Pi_t^1, \dots, \Pi_t^{M^{t-1}})$, $t = 1, \dots, T$ in order to move from the scenario to the nodal representation. The only requirements are given by the non-anticipativity constraints. The first approach (7) is suitable for an investor which considers the risk in the "average way", that is, she just makes a weighted average of $\pi_t^s$ over the scenarios corresponding to the given node. The second approach (8) can be seen as the worst-case modification, because now the investor choose the worst scenario for the given node, i. e. the scenario which have the node lying on their path and has the highest $\pi_t^s$.

### 2.3. Multistage model with risk premium constraints

In this section we finally arrive at the multistage portfolio optimization model considering risk exposure constraints. The model stems from formulation $(P1)$ adding constraints on risk premium at each node derived in the previous section. Note that in the sequent formulation we use (7), in this case we denote the final model as $(P2_{avg})$, however one can obtain the alternative model using (8) instead and arrive to the final model denoted by $(P2_{max})$. The model $(P2_{avg})$ has the form:

$$\begin{aligned}
\underset{x^s, \Pi_t^q}{\text{maximize}} \quad & \sum_{s \in S} p_s \cdot u\left(w_1^s, \dots, w_T^s\right) & \\
\text{subject to} \quad & w_t^s = w_{t-1}^s + (r_{t\cdot}^s)^T x_{t\cdot}^s & t = 1, \dots, T,\ s \in S, \\
& w_{t-1}^s = 1^T x_{t\cdot}^s & t = 1, \dots, T,\ s \in S, \\
& w_0^s = w_0 & s \in S, \\
& x_{t\cdot}^s \geq 0 & t = 1, \dots, T,\ s \in S, \\
& x_{t\cdot}^s = x_{t\cdot}^q & t = 1, \dots, T,\ q = 1, \dots, M^{t-1}, \\
& & s = (q-1)M^{T-t+1} + 1, \dots, qM^{T-t+1}, \\
& \Pi_t^q \leq C & t = 1, \dots, T,\ q = 1, \dots, M^{t-1}, \quad (9) \\
& (6), (7). &
\end{aligned}$$

$C$ is a deterministic constant restricting investor's risk exposure at each node. For simplicity, we assume that the maximal accepted level of the risk exposure expressed by multistage risk premiums is the same at each node. This assumption is reasonable in our model because all the conditional one-period distributions are the same. In the case of a more general structure of the tree one can modify (9) to $\Pi_t^q \leq C_t^q$, $t = 1, \dots, T$, $q = 1, \dots, M^{t-1}$.

Finally, we state model $(P2_{max})$ which has, due to (8), slightly different form than the previous model:

$$
\begin{aligned}
\underset{x^s,\Pi_t^s}{\text{maximize}} \quad & \sum_{s\in S} p_s \cdot u\left(w_1^s,\dots,w_T^s\right) \\
\text{subject to} \quad & w_t^s = w_{t-1}^s + (r_{t\cdot}^s)^T x_{t\cdot}^s && t=1,\dots,T,\ s\in S, \\
& w_{t-1}^s = 1^T x_{t\cdot}^s && t=1,\dots,T,\ s\in S, \\
& w_0^s = w_0 && s\in S, \\
& x_{t\cdot}^s \geq 0 && t=1,\dots,T,\ s\in S, \\
& x_{t\cdot}^s = x_{t\cdot}^q && t=1,\dots,T,\ q=1,\dots,M^{t-1}, \\
& && s=(q-1)M^{T-t+1}+1,\dots,qM^{T-t+1}, \\
& \Pi_t^s \leq C && t=1,\dots,T,\ s\in S, \\
& (6),(8).
\end{aligned}
\tag{10}
$$

$C$ is again a deterministic constant restricting investor's risk exposure at each node.

### 2.4. Risk premiums for selected utility functions

As an example of the theoretical framework presented above, we will analyze the following three types of utility functions which belong to the most popular ones in the decision making theory and applications:

- negative exponential utility function:

$$u(w_1,\dots,w_T) = -\exp\left(-\alpha\sum_{t=1}^{T} v^t w_t\right), \tag{11}$$

- logarithmic utility function:

$$u(w_1,\dots,w_T) = \log\left(\sum_{t=1}^{T} v^t w_t\right), \tag{12}$$

- power utility function:

$$u(w_1,\dots,w_T) = \frac{\left(\sum_{t=1}^{T} v^t w_t\right)^{1-\theta}}{1-\theta}, \tag{13}$$

where parameters $\alpha$ and $\theta$ specify the strength of investor's risk aversion and parameter $v$ is an appropriate discount factor which captures time value of investor's wealth.

Each choice of utility function enables us to calculate the exact formula for risk premium $\pi_t^s$ for arbitrary choice of $t=1,\dots,T$ and $s\in S$. Comparing both sides of (6), taking into account the assumption on distribution of asset returns, we arrive at the following expressions:

- negative exponential utility function:

$$\pi_t^s = \sum_{m=1}^{M} p_m^* (r_{m\cdot}^*)^T x_{t\cdot}^s + \frac{1}{\alpha v^t} \log \left( \sum_{m=1}^{M} p_m^* \exp\left(-\alpha v^t (r_{m\cdot}^*)^T x_{t\cdot}^s\right)\right), \tag{14}$$

- logarithmic utility function:

$$\begin{aligned}\pi_t^s = \frac{1}{v^t} \sum_{j=1, j\neq t}^{T} v^j w_j + w_{t-1} + \sum_{m=1}^{M} p_m^* (r_{m\cdot}^*)^T x_{t\cdot}^s \\ - \frac{1}{v^t} \prod_{m=1}^{M} \left( \sum_{j=1, j\neq t}^{T} v^j w_j + v^t w_{t-1} + v^t (r_{m\cdot}^*)^T x_{t\cdot}^s \right)^{p_m^*}\end{aligned}, \tag{15}$$

- power utility function:

$$\begin{aligned}\pi_t^s = \frac{1}{v^t} \sum_{j=1, j\neq t}^{T} v^j w_j + w_{t-1} + \sum_{m=1}^{M} p_m^* (r_{m\cdot}^*)^T x_{t\cdot}^s \\ - \frac{1}{v^t} \left( \sum_{m=1}^{M} p_m^* \left( \sum_{j=1, j\neq t}^{T} v^j w_j + v^t w_{t-1} + v^t (r_{m\cdot}^*)^T x_{t\cdot}^s \right)^{1-\theta} \right)^{\frac{1}{1-\theta}}\end{aligned}. \tag{16}$$

To sum up, (6) in the optimization problem ($P2_{avg}$), or in ($P2_{max}$) respectively, with specific choice of utility function can be substituted by corresponding expression for associated risk premium given by (14) or (15) or (16). For exponential utility function given, the optimization problem ($P2_{avg}$) simplifies even further in the following sense:

**Proposition 2.1.** Consider the utility function to be given by (11). Constraint (9) in problem ($P2_{avg}$) is equivalent to:

$$\pi_t^s \leq C, \qquad t = 1, \dots, T, s \in S,$$

where the explicit formula for risk premium $\pi_t^s$ is given by (14).

Proof. Baring in mind (14) it suffices to show that $\Pi_t^q = \pi_t^s$ for all choices of $q = 1, \dots, M^{t-1}$ and $s = (q-1)M^{T-t+1} + 1, \dots, qM^{T-t+1}$. In this case $\Pi_t^q$ in equation (9) can be replaced by (14). Equality between risk premiums is however straightforward as far as one realizes that according to (14) the premium is a function of single variable $x_{t\cdot}^s$ and thus subjects to the same non-anticipativity constraints. □

## 3. EMPIRICAL STUDY

In the computational part we used data from the Kenneth French library [7], namely "six portfolios formed on size and book to market." These six portfolios are considered as the base risky assets (instead of stocks) in our computational experiment. We consider a series of 1071 monthly returns of the assets observed between July 1926 and September 2015.

Based on the optimization problem $(ST)$ we generate scenario tree which splits up into 5 branches at each node ($M = 5$) and is design so as first four moments fit the data ($N = 4$). Other parameters of the models are set as follows:

- riskless rate of return: $r_f = 0$,
- discount factor: $v = 0.99$,
- investor's initial wealth: $w_0 = 1000$,
- length of investment horizon: $T = 3$,
- risk aversion parameter of exponential utility function: $\alpha = 1.5 \cdot 10^{-4}$,
- risk aversion parameter of power utility function: $\gamma = 0.4$.

The computations were done in software GAMS 24.4.

### 3.1. Computational results for exponential utility function

In this part we discuss in detail the results obtained for the exponential utility function. All results are presented for the average formulation of the risk premium. In the case of exponential utility function, one can easily see from Proposition 1 that the average formulation is equivalent to the maximum formulation, hence both lead to the same optimization problem.

Table 1 summarizes percentage of the initial wealth invested in the risky assets for different choices of the risk constraint $C$ and for the model excluding the constraint limiting risk exposure (in the table we refer to this model by symbol '/'). In the table we present results obtained when applying the average formulation of the risk premium. We observe that the amount of the initial wealth invested into the riskless asset in the first stage (denoted as $x_{10}$) decreases as the limitations on the risk exposure becomes more lose. On the other hand, the amount of the initial wealth invested into the risky assets increases. We observe that the investor prefers the third and fourth risky assets since the amount of the wealth invested in the others is negligible. We thus conclude that these two assets provide the best balance between the revenue and the risk. As the limiting constraint $C$ increases we also observe that the investor prefers to invest the wealth into the third asset. Table 1 shows another important results. The amount of the wealth invested into a particular asset does not have to be monotone as the constant $C$ increases as indicates the proportion invested into the second of the fifth assets. We recall that this behaviour is acceptable since the objective of the optimization problem is not the maximization of utility of the expected wealth achieved at a particular time instance but the maximization of utility of discounted wealth achieved over the whole investment horizon.

Table 2 shows the proportion of the wealth invested into the riskless asset in all considered stages and different choices of the limitation on the risk exposure. In the table we present results obtained when applying the average formulation of the risk premium. Since the investment strategy is dependent on the particular development of the scenario tree, we present here only results for one selected scenario. Based on the results we conclude that for the second and the third stage there is the same trend as for the first stage decision, i. e. the proportion invested into the riskless asset decreases as the constraint $C$ increases. As we consider only one riskless asset in the optimization problem, this trend should be present for arbitrary choice of the scenario. We recall that the same conclusion is however not valid for the risky asset as we have already emphasized in the previous paragraph.

| C | 0.01 | 0.1 | 0.5 | 1 | / |
|---|---|---|---|---|---|
| $x_{10}$ | 0.9246 | 0.7608 | 0.4672 | 0.2492 | 0.0867 |
| $x_{11}$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $x_{12}$ | 0.0113 | 0.0000 | 0.0130 | 0.0000 | 0.0000 |
| $x_{13}$ | 0.0370 | 0.1463 | 0.3176 | 0.4697 | 0.5744 |
| $x_{14}$ | 0.0271 | 0.0921 | 0.1945 | 0.2811 | 0.3389 |
| $x_{15}$ | 0.0000 | 0.0008 | 0.0076 | 0.0000 | 0.0000 |
| $x_{16}$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

**Tab. 1.** First stage decision (fraction of the initial wealth invested in the assets) for different choices of risk constraint (exponential utility function, average risk premium).

| C | 0.01 | 0.1 | 0.5 | 1 | / |
|---|---|---|---|---|---|
| stage 1 | 0.9246 | 0.7608 | 0.4672 | 0.2492 | 0.0867 |
| stage 2 | 0.9233 | 0.7571 | 0.4931 | 0.2518 | 0.0000 |
| stage 3 | 0.9219 | 0.7411 | 0.4692 | 0.3637 | 0.0000 |

**Tab. 2.** Fraction of the wealth invested in the riskless asset under scenario 1 depending on the choice of risk constraint (exponential utility function, average risk premium).

Table 3 displays the mean wealth achieved at each stage for different choice of the risk constraint. We observe that for our specific choice of model parameters, the amount of wealth achieved at a particular stage raises comparing to the previous stage. This conclusion is however dependent on the model parameters and for different choice we could observe the opposite. One could provide the same explanation as in the previous paragraph, i. e. the objective of the optimization problem is not the maximization of utility of the expected wealth achieved at a particular time instance but the maximization of utility of discounted wealth achieved over the whole investment horizon. With raising parameter $C$ we observe that the investor achieve higher wealth at each stage. This is totally consistent with the expectation. For completeness, in Table 4 we also provide results for the wealth variance corresponding to different stages and different choice of the risk constraint. Conclusions relating to the variance are the same as for the wealth mean.

| C | 0.01 | 0.1 | 0.5 | 1 | / |
|---|---|---|---|---|---|
| stage 1 | 1000.76 | 1002.42 | 1005.40 | 1007.63 | 1009.29 |
| stage 2 | 1001.51 | 1004.82 | 1010.63 | 1015.21 | 1020.14 |
| stage 3 | 1002.26 | 1007.04 | 1015.94 | 1022.65 | 1031.27 |

**Tab. 3.** Comparison of the wealth mean for different choices of risk constraint (exponential utility function, average risk premium).

| C | 0.01 | 0.1 | 0.5 | 1 | / |
|---|---|---|---|---|---|
| stage 1 | 3.67 | 11.59 | 25.87 | 36.54 | 44.49 |
| stage 2 | 5.18 | 16.43 | 36.78 | 51.80 | 69.07 |
| stage 3 | 6.35 | 20.17 | 45.12 | 63.56 | 90.72 |

**Tab. 4.** Comparison of the wealth variance for different choices of risk constraint (exponential utility function, average risk premium).

Table 5 shows the achieved utility with its 95% confidence interval for different choices of risk constraint. This utility is directly calculated from the sum of discounted wealth achieved during the investment horizon. Since the difference in utility is rather small and perhaps not so informative, we present results for the sum of discounted wealth, in the rest of the paper.

| C | 0.01 | 0.1 | 0.5 | 1 | / |
|---|---|---|---|---|---|
| mean | -0.6429 | -0.6420 | -0.6403 | -0.6391 | -0.6377 |
| lower CI | -0.6455 | -0.6501 | -0.6585 | -0.6648 | -0.6717 |
| upper CI | -0.6404 | -0.6340 | -0.6227 | -0.6144 | -0.6053 |
| length of CI | 0.0051 | 0.0160 | 0.0358 | 0.0504 | 0.0664 |

**Tab. 5.** Achieved utility with the 95% confidence interval for different choices of risk constraint (exponential utility function, average risk premium).

Last table (Table 6) shows results for the expected sum of discounted wealth and its 95% confidence interval for different choices of risk constraint. In this case, the expected mean of discounted wealth, as the objective of the optimization problem, has to be an increasing function of the risk constraint $C$. We also observe that the length of the confidence interval raises as the risk constraint raises.

| C | 0.01 | 0.1 | 0.5 | 1 | / |
|---|---|---|---|---|---|
| mean | 2944.83 | 2954.34 | 2971.63 | 2984.83 | 2999.67 |
| lower CI | 2918.56 | 2871.04 | 2785.23 | 2722.26 | 2652.65 |
| upper CI | 2971.11 | 3037.64 | 3158.03 | 3247.41 | 3346.70 |
| length of CI | 52.54 | 166.59 | 372.80 | 525.15 | 694.05 |

**Tab. 6.** Expected sum of discounted wealth with the 95% confidence interval for different choices of risk constraint (exponential utility function, average risk premium).

Figure 2 depicts the expected wealth achieved at different stages with its 95% confidence interval for the formulations which use no limitation on the risk exposure or the limitation of 0.1. Figure 3 illustrates the difference between the expected wealth achieved at different stages for the average and maximum formulation of the risk premium. As mentioned at the beginning of Section 3.1, both formulations have to provide exactly the same results.

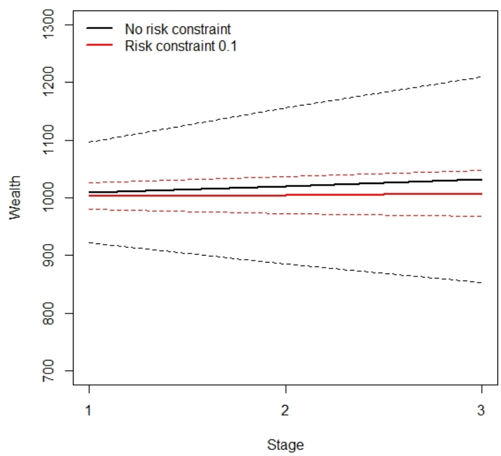


**Fig. 2.** Expected wealth development with the 95% confidence interval (exponential utility function, average risk premium).

Figure 4 depicts the dependence of the expected sum of discounted wealth on the risk constraint for two formulations of the risk premium, which for the case of exponential utility functions coincide. Figure 5 depicts the dependence of the expected wealth on the standard deviation, which could be considered as a efficient frontier of the optimization problem. Both pictures stem from the results which are partly displayed in Table 6.

### 3.2. Computational results for other utility functions

In the second part of this section we focus on comparison of the results of optimization problems which assume different class of utility functions, exponential, logarithmic and power. The results distinguish between two formulations of the risk premium, average and maximum. We recall that for the exponential class the formulations coincide, however this is not true for the logarithmic and power class of utility functions.

Figure 6 depicts the expected wealth achieved at different stages with its 95% confidence interval for the formulations of the risk premium which use no limitation on the risk exposure or the limitation of 0.1. Upper pictures provide results for the average formulations of the risk premium, whereas the lower pictures provide results for the

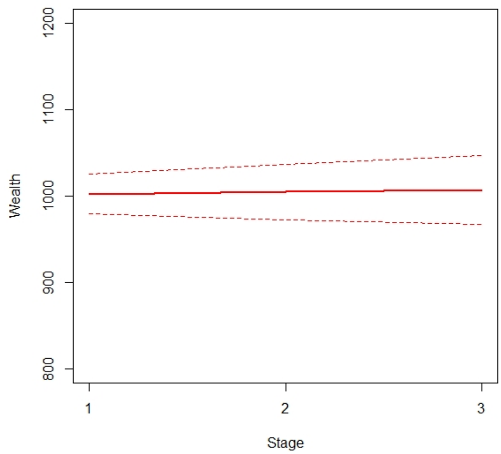


**Fig. 3.** Expected wealth development with the 95% confidence interval (exponential utility function, risk constraint 0.1).

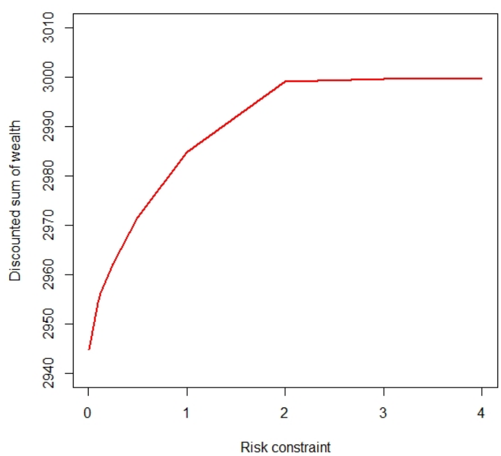


**Fig. 4.** Expected sum of discounted wealth as a function of risk constraint (exponential utility function).

maximum formulation of the risk premium. In all cases we observe that limitation on the risk exposure causes lower wealth achievement with lower deviation throughout the stages. The difference is the most significant for the exponential utility function. For the other utility functions the difference is slighter. We emphasise that in all cases we used the risk limitation $C = 0.1$.

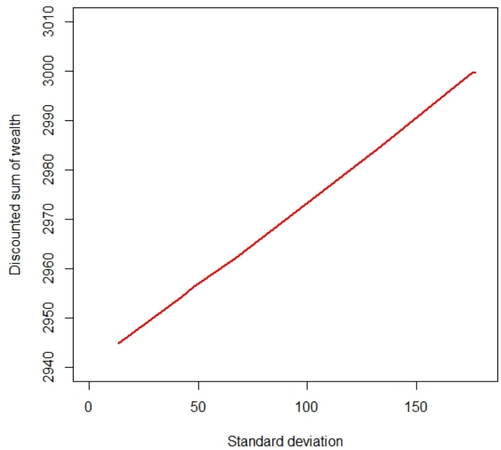


**Fig. 5.** Expected sum of discounted wealth as a function of standard deviation (exponential utility function).

Figure 7 compares the expected wealth development and its 95% confidence interval for two formulations of the risk premium, the average and maximum. We consider different types of utility functions and the risk limitation again equals to 0.1. We observe that for the exponential and logarithmic class we obtain the same results. Slight difference in the expected wealth can be seen in the third picture corresponding to the power utility function.

Figures 8 and 9 compare expected sum of discounted wealth as a function of the risk constraint for different classes of utility functions and different formulations of the risk premium. The first figure illustrates that the average formulation of the risk premium can provide investor higher expected sum of discounted wealth that the maximum formulation, the risk constraint is kept on level 0.1. This fact follows immediately from the mathematical formulation of both approaches. The second figure confirms that the logarithmic utility function corresponds to less risk averse investor than the above specified exponential one and our power utility function corresponds to even less risk averse investor.

Figures 10 and 11 compare expected sum of discounted wealth as a function of the standard deviation for different classes of utility functions and different formulations of the risk premium. We observe that in all cases the dependence is almost linear.

## 4. CONCLUSION

In this paper we presented a multistage portfolio optimization problem with a new type of constraints. The problem maximizes the investor's utility of wealth achieved throughout the horizon under condition on risk exposure expressed by newly introduced

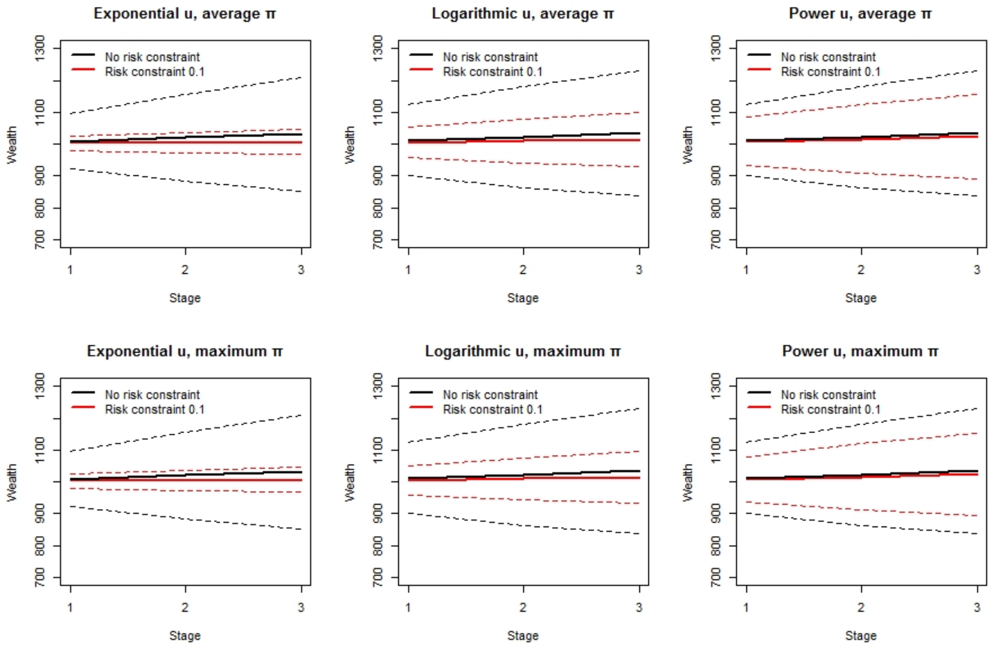


**Fig. 6.** Expected wealth development with 95% confidence interval.

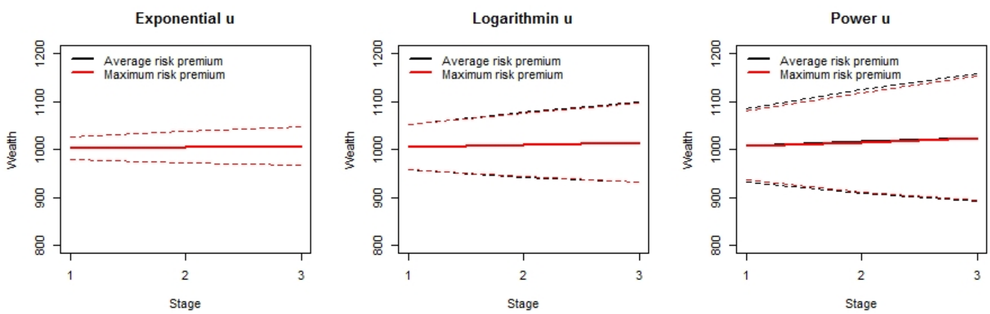


**Fig. 7.** Expected wealth development with 95% confidence interval (risk constraint 0.1).

multistage risk premiums. The premiums were constructed in two ways, using "average" or "orst-case" approach. The premiums were explicitly formulated for the exponential, logarithmic and power utility functions which are the most popular ones. To capture the stochasticity of the problem, we generated multidimensional scenario tree, which simulates development of returns on all risky assets together with their interactions

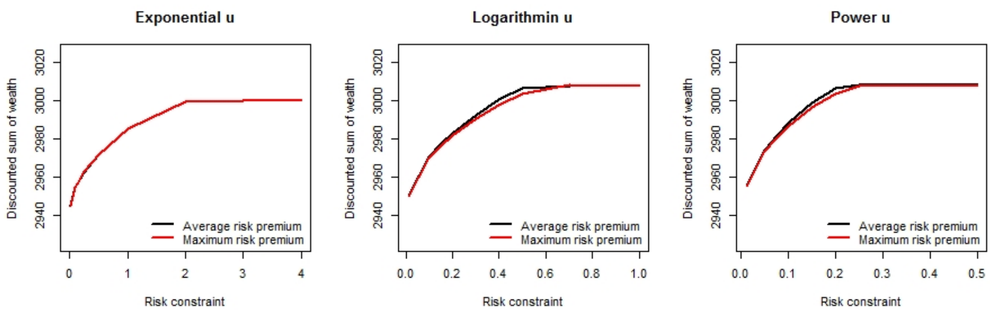


**Fig. 8.** Expected sum of discounted wealth as a function of risk constraint.

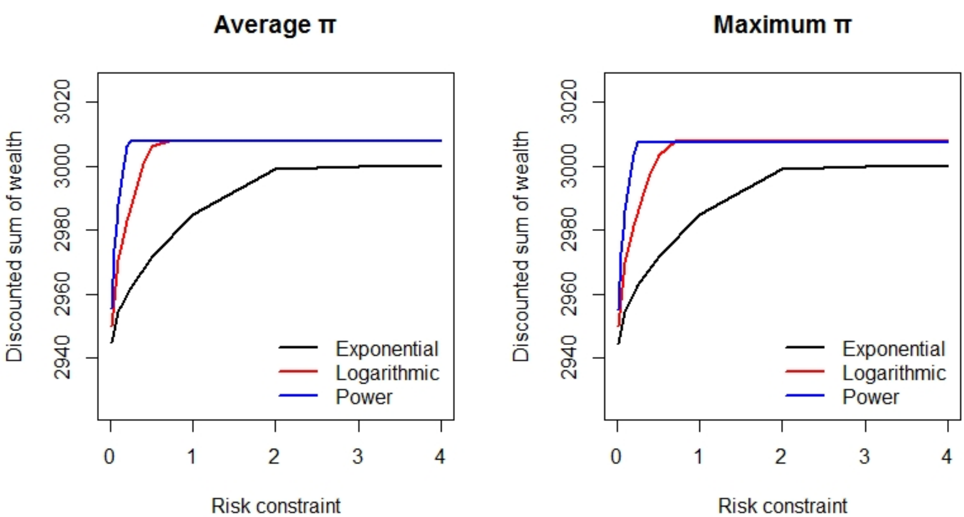


**Fig. 9.** Expected sum of discounted wealth as a function of risk constraint.

expressed via correlations. We adopted procedure suggested in [6] for a scenario tree generation.

We compared performance of 30 different formulations of the portfolio optimization problem. We considered three types of utility functions, two ways of multistage risk premium constructions and five different levels of risk limits (including the case when the limit is not active). We analyzed the here-and-now solutions, the fractions invested in riskless asset within the investment horizon, evolution of the wealth as well as the

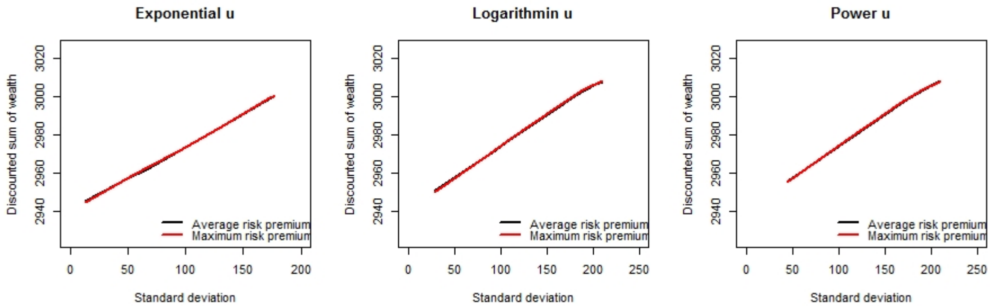


**Fig. 10.** Expected sum of discounted wealth as a function of standard deviation.

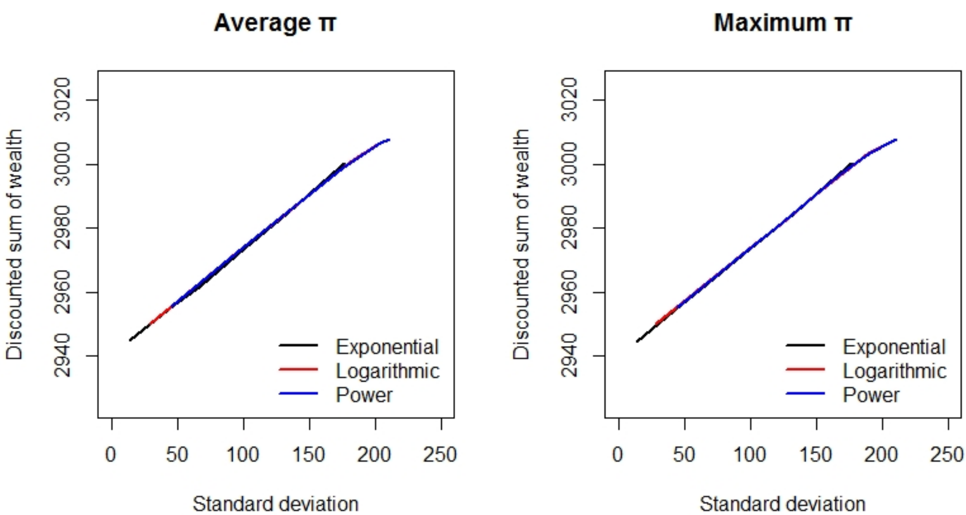


**Fig. 11.** Expected sum of discounted wealth as a function of standard deviation.

discounted sum of wealth. The obtained results are in line with the expectation. When more limiting the risk exposure, the investor chooses more conservative strategy what leads to lower level of expected wealth throughout the horizon and also lower variance. The slight differences in results for different utility functions and for the two ways of the multistage risk premium construction were observed, however, the main risk-return conclusions remain the same. Therefore, the multistage risk premium constraints may successfully serve as an alternative to well-known risk measures, especially if objective function maximizes the expected utility of the final portfolio wealth.

The problem was solved for a regular interstage independent scenario tree, however, the same approaches can be used for any kind of scenario tree. For example, more stages (periods) can be considered. Moreover, the limits on multistage risk premiums can be considered in more complicated way. However, in these cases, the final multistage portfolio optimization problem may become much more computationally demanding or even intractable.

For the future research, the newly introduced risk premiums may serve also for the further theoretical analysis of multivariate utility functions in terms of multivariate stochastic dominance relations, for example, following recent ideas in [11] or [10]. Moreover, it can be easily used in some other applications such as consumption model or pension fund management, see e. g. [16], [17] and [9].

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