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ON RANDOM PROCESSES AS AN IMPLICIT SOLUTION OF EQUATIONS

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Random processes with convenient properties are often employed to model observed data, particularly, coming from economy and finance. We will focus our interest in random processes given implicitly as a solution of a functional equation. For example, random processes AR, ARMA, ARCH, GARCH are belonging in this wide class. Their common feature can be expressed by requirement that stated random process together with incoming innovations must fulfill a functional equation.

Functional dependence is linear for AR, ARMA. We consider a general functional dependence, but, existence of a forward and a backward equivalent rewritings of the given functional equation is required. We present a concept of solution construction giving uniqueness of assigned solution.

We introduce a class of implicit models where forward and backward equivalent rewritings exist. Illustrative examples are included.

Keywords: econometric models, ARMA process, implicit definition

Classification: 62M10, 91B70

1. INTRODUCTION

Linear models are based on properties of the Hilbert space L_2 and the calculus of polynomials in the backward shift operator, see e. g. [1, 2, 3]. We will consider more complex models written as a functional equation; see e. g. [4, 7, 8, 9, 10, 11]. This theoretical description and consequent properties possess relevant impact to practice. Particularly, modern analysis of financial and economic data is based on this setup.

Our realization is focused to models given as an implicit solution of a functional equation. We start to treat this subject in [5] and continue in [6]. In the present paper, we construct a solution in pointwise sense. The construction is actually a numerical algorithm based on recurrent plugging in a forward or a backward formula. Finally, the construction is adapted to derive all random solutions of a given equation with random innovations.

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2. DESCRIPTION OF THE MODEL

We consider a random real vector time-series of dimension $J \in \mathbb{N}$ explained by random real vector innovations of dimension $K \in \mathbb{N}$. Spaces of their values will be denoted $\mathcal{Y} = \mathbb{R}^J$, $\mathcal{Z} = \mathbb{R}^K$. More precisely, we consider a random real vector time-series $\mathsf{Y} = (\mathsf{Y}(t), t \in \mathbb{Z})$, where for each time $t \in \mathbb{Z}$ we have $\mathsf{Y}(t) = (\mathsf{Y}_1(t), \mathsf{Y}_2(t), \ldots, \mathsf{Y}_J(t))^\top$; i.e. $\mathsf{Y}(t) \in \mathcal{Y}$. We suppose Y is tightly related to a series of innovations $\mathsf{Z} = (\mathsf{Z}(t), t \in \mathbb{Z})$, where for each time $t \in \mathbb{Z}$ we have $\mathsf{Z}(t) = (\mathsf{Z}_1(t), \mathsf{Z}_2(t), \ldots, \mathsf{Z}_K(t))^\top$; i.e. $\mathsf{Z}(t) \in \mathcal{Z}$. The relation is implicitly described by a functional equation

$$\forall t \in \mathbb{Z} \quad \mathsf{f}(\mathsf{Y}(t), \mathsf{Y}(t-1), \ldots, \mathsf{Y}(t-p); \mathsf{Z}(t), \mathsf{Z}(t-1), \ldots, \mathsf{Z}(t-q)) = \mathbf{0} \in \mathbb{R}^r, \tag{1}$$

where $p \in \mathbb{N}_0$, $q \in \mathbb{N}_0$, $r \in \mathbb{N}$, $\mathsf{f} : \mathcal{Y}^{p+1} \times \mathcal{Z}^{q+1} \to \mathbb{R}^r$ is a function.

To abbreviate forthcoming formulas, we introduce a notation

$$\begin{aligned} \mathsf{Y}(t \circledcirc k) &= \mathsf{Y}(t), \mathsf{Y}(t-1), \ldots, \mathsf{Y}(t-k), \\ \mathsf{Z}(t \circledcirc k) &= \mathsf{Z}(t), \mathsf{Z}(t-1), \ldots, \mathsf{Z}(t-k). \end{aligned} \tag{2}$$

Hence, the relation (1) converts to an abbreviate form

$$\forall\, t \in \mathbb{Z} \qquad \mathsf{f}(\mathsf{Y}(t \circledcirc p); \mathsf{Z}(t \circledcirc q)) = \mathbf{0} \in \mathbb{R}^r. \tag{3}$$

Since $\mathsf{Z}(t \circledcirc q)$ is always appearing in one block we abbreviate our problem to

$$\forall\, t \in \mathbb{Z} \qquad \mathsf{f}(\mathsf{Y}(t \circledcirc p); \mathsf{Z}(t)) = \mathbf{0} \in \mathbb{R}^r. \tag{4}$$

Replacing $\mathsf{Z}(t)$ by $\mathsf{Z}(t \circledcirc q)$ we immediately jump from (4) to (3).

3. MAIN RESULTS

We will introduce a case in which we are able to construct all solutions. It is the case if we possess a forward and a backward equivalent rewriting of (1). Let us start with an explanation of the term "equivalent rewriting".

At first we consider two lists of statements.

Definition 3.1. Let $n \in \mathbb{N}_0$ and two lists of statements $R = (R(t), t \in \mathbb{Z})$, $\tilde{R} = (\tilde{R}(t), t \in \mathbb{Z})$ be given. We say R is n-implying $\tilde{R}$ whenever for all $t \in \mathbb{Z}$ we have if $R(s)$ is true for all $s \in \{t-n, t-n+1, \ldots, t+n\}$ then $\tilde{R}(t)$ is true.

Definition 3.2. Let two lists of statements $R = (R(t), t \in \mathbb{Z})$, $\tilde{R} = (\tilde{R}(t), t \in \mathbb{Z})$ be given. We say R is shift-equivalent to $\tilde{R}$ if and only if there is $n \in \mathbb{N}_0$ such that R is n-implying $\tilde{R}$ and $\tilde{R}$ is n-implying R.

Definition 3.3. Let two lists of statements $R = (R(t), t \in \mathbb{Z})$, $\tilde{R} = (\tilde{R}(t), t \in \mathbb{Z})$ be given. We say R is 0-equivalent to $\tilde{R}$ if and only if R is 0-implying $\tilde{R}$ and $\tilde{R}$ is 0-implying R.

Equivalently, 0-equivalence is characterized by a property for all $t \in \mathbb{Z}$ we have $R(t)$ is true if and only if $\tilde{R}(t)$ is true.

The paper is based on shift-equivalence, but, former paper [6] is dealing with 0-equivalence, only.

3.1. Deterministic solution

At first, we discuss existence of a deterministic solution of

$$\forall\, t \in \mathbb{Z} \qquad \mathsf{f}\left(\mathsf{y}\left(t \circledcirc p\right); \mathsf{z}\left(t\right)\right) = \mathbf{0} \in \mathbb{R}^r, \tag{5}$$

where $\mathsf{y} = \left(\mathsf{y}\left(t\right), t \in \mathbb{Z}\right) \in \mathcal{Y}^{\mathbb{Z}}$ and $\mathsf{z} = \left(\mathsf{z}\left(t\right), t \in \mathbb{Z}\right) \in \mathcal{Z}^{\mathbb{Z}}$. The task is to construct y if f and z are given.

For construction of a solution, we require shift-equivalent rewritings of (5). Forward solution needs (6) and backward solution is based on (9).

3.1.1. Forward solution

Our construction of a forward solution of (5) is based on a shift-equivalent rewriting of (5) into a form

$$\forall\, t \in \mathbb{Z} \qquad \mathsf{y}\left(t\right) = \mathsf{F}\left(\mathsf{y}\left(\left(t-1\right) \circledcirc \left(p-1\right)\right); \mathsf{z}\left(t\right)\right). \tag{6}$$

Let us call (6) a forward form of (5). Now, we will construct a forward solution of (5).

Fix $\mathsf{z} \in \mathcal{Z}^{\mathbb{Z}}$, $\tau \in \mathbb{Z}$ and $\xi \in \mathcal{Y}^p$. Initial values of a solution are

$$\mathsf{y}\left(\tau\right) = \xi_1,\ \mathsf{y}\left(\tau+1\right) = \xi_2,\ \ldots,\ \mathsf{y}\left(\tau+p-1\right) = \xi_p. \tag{7}$$

Using recurrent plugging in (6), we prolong the sequence $\mathsf{y}\left(t\right)$ for all $t \in \mathbb{Z}$, $t \geq \tau + p$ by

$$\mathsf{y}\left(t\right) = \mathsf{F}\left(\mathsf{y}\left(\left(t-1\right) \circledcirc \left(p-1\right)\right); \mathsf{z}\left(t\right)\right). \tag{8}$$

Hence for given $\mathsf{z} \in \mathcal{Z}^{\mathbb{Z}}$, $\tau \in \mathbb{Z}$ and $\xi \in \mathcal{Y}^p$, a sequence $\mathsf{y}\left(t\right)$ is uniquely determined for all $t \in \mathbb{Z}$, $t \geq \tau$ such that (6) is fulfilled for all $t \in \mathbb{Z}$, $t \geq \tau + p$. Because (6) is n-implying (5), (5) is fulfilled for all $t \in \mathbb{Z}$, $t \geq \tau + p + n$.

3.1.2. Backward solution

Our construction of a backward solution of (5) is based on a shift-equivalent rewriting of (5) into a form

$$\forall\, t \in \mathbb{Z} \qquad \mathsf{y}\left(t-p\right) = \mathsf{G}\left(\mathsf{y}\left(t \circledcirc \left(p-1\right)\right); \mathsf{z}\left(t\right)\right). \tag{9}$$

Let us call (9) a backward form of (5). Now, we will construct a backward solution of (5).

Fix $\mathsf{z} \in \mathcal{Z}^{\mathbb{Z}}$, $\tau \in \mathbb{Z}$ and $\xi \in \mathcal{Y}^p$. We set initial values of a solution

$$\mathsf{y}\left(\tau\right) = \xi_1,\ \mathsf{y}\left(\tau+1\right) = \xi_2,\ \ldots,\ \mathsf{y}\left(\tau+p-1\right) = \xi_p. \tag{10}$$

Using recurrent plugging in (9), we prolong the sequence $\mathsf{y}\left(t\right)$ for all $t \in \mathbb{Z}$, $t \leq \tau - 1$ by

$$\mathsf{y}\left(t\right) = \mathsf{G}\left(\mathsf{y}\left(\left(t+p\right) \circledcirc \left(p-1\right)\right); \mathsf{z}\left(t+p\right)\right). \tag{11}$$

Hence for given $\mathsf{z} \in \mathcal{Z}^{\mathbb{Z}}$, $\tau \in \mathbb{Z}$ and $\xi \in \mathcal{Y}^p$, a sequence $\mathsf{y}\left(t\right)$ is uniquely determined for all $t \in \mathbb{Z}$, $t \leq \tau + p - 1$ such that (9) is fulfilled for all $t \in \mathbb{Z}$, $t \leq \tau - 1$. Because (9) is n-implying (5), (5) is fulfilled for all $t \in \mathbb{Z}$, $t \leq \tau - n - 1$.

3.1.3. Both-side solution

Our construction of a solution of (5) requires existence of shift-equivalent rewritings (6), (9). Assume $n \in \mathbb{N}_0$ such that (6) is n-implying (9).

For given $\mathsf{z} \in \mathcal{Z}^{\mathbb{Z}}$, $\tau \in \mathbb{Z}$ and $\xi \in \mathcal{Y}^p$., we construct a solution of (5) in three steps:

Step 0 Set initial values

$$\eta(\tau) = \xi_1,\ \eta(\tau+1) = \xi_2,\ \ldots,\ \eta(\tau+p-1) = \xi_p.$$

Step 1 Using the forward construction (8), we are receiving a uniquely determined sequence $\eta(t)$, $t \in \mathbb{Z}$, $t \geq \tau$ such that (6) is fulfilled for all $t \in \mathbb{Z}$, $t \geq \tau + p$. Because (6) is n-implying (9), (9) is fulfilled for all $t \in \mathbb{Z}$, $t \geq \tau + p + n$.

Step 2 Set $\mathsf{y}(t) = \eta(t)$ for all $t \in \mathbb{Z}$, $t \geq \tau + p + n$. Now we prolong this sequence using backward construction (11). Now, our sequence $\mathsf{y}(t)$ is correctly determined for all $t \in \mathbb{Z}$, moreover, it fulfills (9) for all $t \in \mathbb{Z}$.

The constructed sequence fulfills (5), (6), (9) for all $t \in \mathbb{Z}$.

Theorem 3.4. Let (5) possess shift-equivalent rewritings (6) and (9). For given $\mathsf{z} \in \mathcal{Z}^{\mathbb{Z}}$, $\tau \in \mathbb{Z}$ and $\xi \in \mathcal{Y}^p$, our construction is giving a sequence $\mathsf{y}(t)$ uniquely determined for all $t \in \mathbb{Z}$ and fulfilling (5), (6), (9) for all $t \in \mathbb{Z}$.

Proof. Evidently, constructed sequence is uniquely defined. It remains to check (5). Assume $n \in \mathbb{N}_0$ such that (6) is n-implying (9) and take $t \in \mathbb{Z}$ for that.

1. Let $t \geq \tau + p + n$.

   According to forward construction, we have (6) is true for all $s \in \{t-n, t-n+1, \ldots, t+n\}$.

   Because (6) is n-implying (9), (9) is fulfilled for this t.

2. Let $t < \tau + p + n$.

   According to backward construction, (9) is fulfilled for this t.

We have checked the constructed sequence y fulfills (9) for all $t \in \mathbb{Z}$.
Since (9) is shift-equivalent to (5), and (9) is shift-equivalent to (6), (5) and (6) are fulfilled for all $t \in \mathbb{Z}$. $\square$

If $n = 0$, the initial vector ξ and the vector $(\mathsf{y}(\tau), \mathsf{y}(\tau+1), \ldots, \mathsf{y}(\tau+p-1))$ must coincide; see Theorem 2 in [6]. If $n \in \mathbb{N}$, the initial vector ξ and the vector $(\mathsf{y}(\tau), \mathsf{y}(\tau+1), \ldots, \mathsf{y}(\tau+p-1))$ can differ.

Our construction is describing all solutions of (5).

Theorem 3.5. Let (5) possess shift-equivalent rewritings (6) and (9), $\mathsf{z} \in \mathcal{Z}^{\mathbb{Z}}$ be given and y be a solution of (5). Then for initials $\tau \in \mathbb{Z}$ and $\xi = (\mathsf{y}(\tau), \mathsf{y}(\tau+1), \ldots, \mathsf{y}(\tau+p-1))$, our construction is giving back the sequence y.

Proof. We know (5) is shift-equivalent to (6), (5) is shift-equivalent to (9) and y is fulfilling (5) for all $t \in \mathbb{Z}$. Since, forward and backward constructions uniquely determine values of constructed sequences, these values must coincide with original sequence y. $\square$

3.2. Random solution

Starting with random initials, our construction is giving all solutions of (1).

Theorem 3.6. Let (5) possess shift-equivalent rewritings (6) and (9). Let a random sequence $\mathsf{Z} \in \mathcal{Z}^{\mathbb{Z}}$ and a random vector $\xi \in \mathcal{Y}^p$ be given. Let $\tau \in \mathbb{Z}$ and functions F, G be measurable. We apply our construction to each random elementary event ω to receive a sequence $\mathsf{Y}(\omega) = (\mathsf{Y}(t,\omega), t \in \mathbb{Z})$. The procedure is giving to us $\mathsf{Y} = (\mathsf{Y}(t), t \in \mathbb{Z})$ a sequence of random variables which is a solution of (1). Moreover, each solution of (1) can be found by this procedure for a convenient choice of ξ.

P r o o f. Statement follows immediately Theorem 3.5 applied to random initials. Measurability of received sequence follows since F, G are measurable. $\square$

4. EXAMPLES

Here we present some examples.

Example 4.1. Consider an example with $J = K = r = 1$, $p = 2$, $q = 1$ given by a linear equation connecting to an ARMA process:

$$\mathsf{y}(t) + \frac{2}{3}\mathsf{y}(t-1) - \frac{1}{8}\mathsf{y}(t-2) - \mathsf{z}(t) + \frac{1}{3}\mathsf{z}(t-1) \quad = \quad 0.$$

Forms of (6), (9) are straightforward:

$$\begin{aligned} \mathsf{y}(t) &= -\frac{2}{3}\mathsf{y}(t-1) + \frac{1}{8}\mathsf{y}(t-2) + \mathsf{z}(t) - \frac{1}{3}\mathsf{z}(t-1), \\ \mathsf{y}(t-2) &= 8\mathsf{y}(t) + \frac{16}{3}\mathsf{y}(t-1) - 8\mathsf{z}(t) + \frac{8}{3}\mathsf{z}(t-1). \end{aligned}$$

Example 4.2. Consider an example with $J = K = r = 2$, $p = 3$, $q = 2$ given by two equations:

$$\begin{aligned} &\mathsf{y}_1(t) + 2\mathsf{y}_1(t-1)\,\mathsf{y}_2(t-1)\,\mathsf{z}_1(t) - \mathsf{y}_1(t-2) + \mathsf{z}_1(t-1) = 0, \\ &\mathsf{y}_2(t) - 2\mathsf{y}_2(t-1)\,\mathsf{y}_2(t-2)\,(\mathsf{z}_2(t))^2 - 5\mathsf{y}_2(t-2)\,\mathsf{z}_1(t-1)\,\mathsf{z}_2(t-2) \\ &\qquad + \mathsf{y}_1(t-3) - \mathsf{y}_2(t-3) = 0. \end{aligned}$$

Form of (6), is straightforward:

$$\begin{aligned} \mathsf{y}_1(t) &= -2\mathsf{y}_1(t-1)\,\mathsf{y}_2(t-1)\,\mathsf{z}_1(t) + \mathsf{y}_1(t-2) - \mathsf{z}_1(t-1), \\ \mathsf{y}_2(t) &= 2\mathsf{y}_2(t-1)\,\mathsf{y}_2(t-2)\,(\mathsf{z}_2(t))^2 + 5\mathsf{y}_2(t-2)\,\mathsf{z}_1(t-1)\,\mathsf{z}_2(t-2) \\ &\quad - \mathsf{y}_1(t-3) + \mathsf{y}_2(t-3). \end{aligned}$$

Deriving a form of (9) needs a few steps:

$$\begin{aligned} \mathsf{y}_1(t-2) &= \mathsf{y}_1(t) + 2\mathsf{y}_1(t-1)\,\mathsf{y}_2(t-1)\,\mathsf{z}_1(t) + \mathsf{z}_1(t-1), \\ \mathsf{y}_2(t-3) &= \mathsf{y}_2(t) - 2\mathsf{y}_2(t-1)\,\mathsf{y}_2(t-2)\,(\mathsf{z}_2(t))^2 \\ &\quad - 5\mathsf{y}_2(t-2)\,\mathsf{z}_1(t-1)\,\mathsf{z}_2(t-2) + \mathsf{y}_1(t-3). \end{aligned}$$

$$\begin{aligned}
\mathsf{y}_1(t-3) &= \mathsf{y}_1(t-1) + 2\mathsf{y}_1(t-2)\,\mathsf{y}_2(t-2)\,\mathsf{z}_1(t-1) + \mathsf{z}_1(t-2)\,,\\
\mathsf{y}_2(t-3) &= \mathsf{y}_2(t) - 2\mathsf{y}_2(t-1)\,\mathsf{y}_2(t-2)\,(\mathsf{z}_2(t))^2\\
&\quad -5\mathsf{y}_2(t-2)\,\mathsf{z}_1(t-1)\,\mathsf{z}_2(t-2) + \mathsf{y}_1(t-3)\,.
\end{aligned}$$

Finally, we are receiving a form of (9)

$$\begin{aligned}
\mathsf{y}_1(t-3) &= \mathsf{y}_1(t-1) + 2\mathsf{y}_1(t-2)\,\mathsf{y}_2(t-2)\,\mathsf{z}_1(t-1) + \mathsf{z}_1(t-2)\,,\\
\mathsf{y}_2(t-3) &= \mathsf{y}_2(t) - 2\mathsf{y}_2(t-1)\,\mathsf{y}_2(t-2)\,(\mathsf{z}_2(t))^2\\
&\quad -5\mathsf{y}_2(t-2)\,\mathsf{z}_1(t-1)\,\mathsf{z}_2(t-2)\\
&\quad +\mathsf{y}_1(t-1) + 2\mathsf{y}_1(t-2)\,\mathsf{y}_2(t-2)\,\mathsf{z}_1(t-1) + \mathsf{z}_1(t-2)\,.
\end{aligned}$$

Example 4.3. Consider an example with $J = 3$, $K = 2$, $r = 1$, $p = 3$, $q = 2$ given by

$$\left\| \begin{pmatrix} \mathsf{y}_1(t)^3 \\ \mathsf{y}_2(t)^3 \\ \mathsf{y}_3(t)^5 \end{pmatrix} - \begin{pmatrix} \mathsf{y}_1(t-1)^2\,\mathsf{z}_1(t) \\ \mathsf{y}_2(t-1)^{-3}\,\mathsf{y}_2(t-2)^2\,\mathsf{z}_2(t) \\ \mathsf{y}_3(t-1)^4\,\mathsf{z}_1(t)^2\,\mathsf{z}_2(t-1)^{-2} \end{pmatrix} - \frac{1}{6}\begin{pmatrix} \mathsf{y}_3(t-3)^3 \\ \mathsf{y}_1(t-3)^{-3} \\ \mathsf{y}_2(t-3)^{-1} \end{pmatrix} \right\| = 0,$$

where $\|.\|$ denotes Euclidean norm. Here a form of (6) is:

$$\begin{aligned}
\mathsf{y}_1(t) &= \left(\mathsf{y}_1(t-1)^2\,\mathsf{z}_1(t) + \frac{1}{6}\mathsf{y}_3(t-3)^3\right)^{\frac{1}{3}},\\
\mathsf{y}_2(t) &= \left(\mathsf{y}_2(t-1)^{-3}\,\mathsf{y}_2(t-2)^2\,\mathsf{z}_2(t) + \frac{1}{6}\mathsf{y}_1(t-3)^{-3}\right)^{\frac{1}{3}},\\
\mathsf{y}_3(t) &= \left(\mathsf{y}_3(t-1)^4\,\mathsf{z}_1(t)^2\,\mathsf{z}_2(t-1)^{-2} + \frac{1}{6}\mathsf{y}_2(t-3)^{-1}\right)^{\frac{1}{5}}.
\end{aligned}$$

Here a form of (9) is:

$$\begin{aligned}
\mathsf{y}_1(t-3) &= \left(6\mathsf{y}_2(t)^3 - 6\mathsf{y}_2(t-1)^{-3}\,\mathsf{y}_2(t-2)^2\,\mathsf{z}_2(t)\right)^{-\frac{1}{3}},\\
\mathsf{y}_2(t-3) &= \left(6\mathsf{y}_3(t)^5 - 6\mathsf{y}_3(t-1)^4\,\mathsf{z}_1(t)^2\,\mathsf{z}_2(t-1)^{-2}\right)^{-1},\\
\mathsf{y}_3(t-3) &= \left(6\mathsf{y}_1(t)^3 - 6\mathsf{y}_1(t-1)^2\,\mathsf{z}_1(t)\right)^{\frac{1}{3}}.
\end{aligned}$$

5. CONCLUSION

Implicit models with forward and backward shift-equivalent rewritings are convenient for modeling. Random processes following such models always exist and are uniquely determined by initial values. Moreover, we have found a full description of solutions of these implicit formulas.

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R E F E R E N C E S

[1] G. E. P. Box and G. M. Jenkins: Time-series Analysis Forecasting and Control. Holden-Day, San Francisco 1976.

[2] P. J. Brockwell and R. A. Davis: Time-series: Theory and Methods. Springer-Verlag, New York 1987. DOI:10.1007/978-1-4899-0004-3

[3] J. Durbin and S. J. Koopman: Time Series Analysis by State Space Models. Oxford University Press, Oxford 2001.

[4] C. Hommes: Financial markets as nonlinear adaptive evolutionary systems. Quantitative Finance *1* (2001), 1, 149–167. DOI:10.1088/1469-7688/1/1/311

[5] P. Lachout: On functional definition of time-series models. In: Proc. 32th International Conference on Mathematical Methods in Economics, Olomouc (J. Talašová, J. Stoklasa and T. Talášek, eds.), Palacký University, Olomouc 2014, pp. 560-565.

[6] P. Lachout: Discussion on implicit econometric models. In: Proc. 34th International Conference on Mathematical Methods in Economics, (A. Kocourek and M. Vavroušek, eds.), Technical University of Liberec 2016, pp. 501–505.

[7] H.-E. Liao and W. A. Sethares: Suboptimal identification of nonlinear ARMA models using an orthogonality approach. IEEE Trans. Circuits and Systems *42* (1995), 1, 14–22. DOI:10.1109/81.350792

[8] J. Liu and K. Susko: On strict stationarity and ergodicity of a non-linear ARMA model. J. Appl. Probab. *29* (1992), 2, 363–373. DOI:10.1017/s0021900200043114

[9] E. Masry and D. Tjøstheim: Nonparametric estimation and identification of nonlinear ARCH time-series. Econometr. Theory *11* (1995), 1, 258–289. DOI:10.1017/s0266466600009166

[10] M. B. Priestley: Non-Linear and Non-Stationary time-series. Academic Press, London 1988.

[11] R. H. Shumway and D. S. Stoffer: Time Series Analysis and Its Applications – With R Examples. EZ Green Edition, 2015.

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