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Isometries of Riemannian and sub-Riemannian structures on three-dimensional Lie groups

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Abstract. We investigate the isometry groups of the left-invariant Riemannian and sub-Riemannian structures on simply connected three-dimensional Lie groups. More specifically, we determine the isometry group for each normalized structure and hence characterize for exactly which structures (and groups) the isotropy subgroup of the identity is contained in the group of automorphisms of the Lie group. It turns out (in both the Riemannian and sub-Riemannian cases) that for most structures any isometry is the composition of a left translation and a Lie group automorphism.

1 Introduction

For any left-invariant Riemannian structure on a simply connected nilpotent Lie group, the isometry group decomposes as a semidirect product of the group of left translations and the isotropy subgroup of the identity; moreover, the isotropy subgroup of the identity is contained in the group of automorphisms of the Lie group ([29]). The same property has been shown to hold true for a certain class of sub-Riemannian structures on simply connected nilpotent Lie groups, namely the sub-Riemannian Carnot groups ([14], [16], see also [19]). In fact, recently it has been shown that this property generalizes to any nilpotent Lie group equipped with a left-invariant distance that induces the manifold topology ([17]). On the other hand, for left-invariant Riemannian structures on simple Lie groups, it is known that the connected component of the identity of the isometry group is contained in the group of left and right translations of the Lie group ([10]).

In this paper we investigate the isometry groups of the left-invariant Riemannian and sub-Riemannian structures on three-dimensional simply connected Lie groups. In particular, we wish to characterize those structures and groups (beyond the nilpotent ones) for which the isotropy subgroup of the identity is contained in

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the group of automorphisms of the Lie group. Towards that end, we first classify the Riemannian (resp. sub-Riemannian) structures on each Lie group up to isometric Lie group automorphism. We then determine the isotropy subgroup of the identity for each normalized structure (the isometry group is generated by the group of left translations and the isotropy subgroup). In the Riemannian case, the (linearized) isotropy subgroup is essentially determined by finding the group of linear isomorphisms of the Lie algebra preserving the metric, the curvature tensor R, and its covariant derivative ∇R. In the sub-Riemannian case, we show that any isometry of the structure is an isometry of some associated Riemannian structure. We find, in both the Riemannian case (Theorem 1) and the sub-Riemannian case (Theorem 2), that for most groups and structures the isotropy subgroup of the identity is contained in the group of automorphisms of the Lie group. The isometry groups for the Riemannian structures on *unimodular* simply connected Lie groups have previously been described in [13] (see also [25]); we correct a small mistake made in [13] with respect to the isometry groups of the Riemannian structures on $\widetilde{\mathsf{SL}}(2,\mathbb{R})$.

Two Riemannian structures on a completely solvable simply connected Lie group are isometric if and only if there exists an isometry between them that is also a Lie group isomorphism [2], [3] (see also [11]). We briefly show that this characterization essentially holds true for Riemannian (Proposition 2) and sub-Riemannian (Proposition 6) structures on any simply connected three-dimensional Lie group.

The structure of the paper is as follows. Section 2 contains the preliminaries. The Riemannian structures are treated in Section 3. We describe the procedure used in calculating the isotropy subgroup of the identity and give a classification of the structures; we then give a general characterization of when the isotropy subgroup is contained in the group of automorphisms of the Lie group. Details for a typical case follows, after which the exceptional cases are treated. The sub-Riemannian structures are likewise treated in Section 4. A classification of the three-dimensional Lie algebras is supplied in Appendix A. We present, in Appendix B, a catalogue of the Riemannian and sub-Riemannian structures on each simply connected three-dimensional Lie group; this includes a full description of the normalized structures and their isotropy subgroups. In Tables 2 and 3 an index, by isotropy subgroup, of the Riemannian and sub-Riemannian structures is provided. In Figures 1, 2, and 3 the (normalized) principle Ricci curvatures are plotted for each Riemannian structure. As a simple byproduct to the paper, we give a classification of the symmetric Riemannian structures on simply-connected three-dimensional Lie groups in Appendix C. Finally, in Appendix D, we briefly discuss (with reference to the paper [7]) the classification of the Hamilton–Poisson systems associated to the Riemannian and sub-Riemannian structures. We note that MATHEMATICA was used to facilitate most of the computations for this paper.

2 Preliminaries

2.1 Invariant sub-Riemannian structures on Lie groups

A *left-invariant sub-Riemannian structure* on a (real, finite-dimensional, connected) Lie group G with identity $\mathbf{1}$ is a triplet $(\mathsf{G}, \mathcal{D}, \mathbf{g})$, where $\mathcal{D}$ is a smooth nonintegrable left-invariant distribution on G and $\mathbf{g}$ is a left-invariant Riemannian metric on $\mathcal{D}$. In other words, $\mathcal{D}(\mathbf{1})$ is a linear subspace of the Lie algebra $\mathfrak{g}$ of G and

$\mathcal{D}(x) = x\mathcal{D}(\mathbf{1})$ for $x \in \mathsf{G}$; $\mathbf{g_1}$ is a (positive definite) inner product on $\mathcal{D}(\mathbf{1})$ and $\mathbf{g}_x(xA, xB) = \mathbf{g_1}(A, B)$ for $xA, xB \in \mathcal{D}(x)$. Here the product xA is given by $T_\mathbf{1}L_x \cdot A$, where L_x is the left translation by x and $T_\mathbf{1}L_x$ is the tangent map of L_x at the identity (indeed, $T\mathsf{G}$ has the left trivialization $T\mathsf{G} \cong \mathsf{G}\times\mathfrak{g}$, $T_\mathbf{1}L_x\cdot A \leftrightarrow (x, A)$). Whenever convenient, we identify $A \in \mathfrak{g}$ with its corresponding left-invariant vector field $X(x) = xA$. When $\mathcal{D} = T\mathsf{G}$, we have a *left-invariant Riemannian structure*, which we simply denote $(\mathsf{G}, \mathbf{g})$. Any left-invariant structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ on G is uniquely determined by the subspace $\mathcal{D}(\mathbf{1}) \subseteq \mathfrak{g}$ and the inner product $\mathbf{g_1}$ on $\mathcal{D}(\mathbf{1})$. A list of k smooth vector fields $(X_1, \ldots, X_k)$ is said to be an orthonormal frame for $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ if $\mathcal{D}(x) = \operatorname{span}(X_1(x), \ldots, X_k(x))$ and $\mathbf{g}(X_i, X_j) = \delta_{ij}$; we note that any left-invariant sub-Riemannian structure admits a global orthonormal frame of left-invariant vector fields.

An absolutely continuous curve $x(\cdot) : [0, t_1] \to \mathsf{G}$ is called a $\mathcal{D}$-curve if $\dot{x}(t) \in \mathcal{D}(x(t))$ for almost every $t \in [0, t_1]$. We shall assume that $\mathcal{D}$ satisfies the bracket generating condition, i.e., $\mathcal{D}(\mathbf{1})$ generates $\mathfrak{g}$; by the Chow–Rashevskii theorem this condition is necessary and sufficient for any two points in G to be connected by a $\mathcal{D}$-curve (see, e.g., [21]). The length of a $\mathcal{D}$-curve $x(\cdot)$ is given by $\ell(x(\cdot)) = \int_0^{t_1} \sqrt{\mathbf{g}(\dot{x}(t), \dot{x}(t))}\, dt$. Any sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is endowed with a natural metric space structure, namely the Carnot–Carathéodory distance:

$$d(x_1, x_2) = \inf\big\{\ell(x(\cdot)) : x(\cdot) \text{ is a } \mathcal{D}\text{-curve curve joining } x_1 \text{ and } x_2\big\}.$$

By left invariance $d(x_1, x_2) = d(\mathbf{1}, x_1^{-1}x_2)$. A $\mathcal{D}$-curve curve $x(\cdot)$ that realizes the Carnot–Carathéodory distance between two points is called a minimizing geodesic. For left-invariant sub-Riemannian structures on Lie groups, the Carnot–Carathéodory metric is complete (cf. [4], [5]). Hence, any two points in G can be joined by a minimizing geodesic (see, e.g., [21]).

An *isometry* between two left-invariant sub-Riemannian (or Riemannian) structures $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $(\overline{\mathsf{G}}, \overline{\mathcal{D}}, \overline{\mathbf{g}})$ is a diffeomorphism $\phi : \mathsf{G} \to \overline{\mathsf{G}}$ such that

$$\phi_*\mathcal{D} = \overline{\mathcal{D}} \qquad \text{and} \qquad \mathbf{g} = \phi^*\overline{\mathbf{g}}$$

i.e., $T_x\phi \cdot \mathcal{D}(x) = \overline{\mathcal{D}}(\phi(x))$ and $\mathbf{g}_x(xA, xB) = \overline{\mathbf{g}}_{\phi(x)}(T_x\phi \cdot xA, T_x\phi \cdot xB)$. If ϕ is additionally a Lie group isomorphism, then we say that ϕ is an *$\mathfrak{L}$-isometry*. If instead $\mathbf{g} = r\phi^*\overline{\mathbf{g}}$ for some $r > 0$, then we say the structures are isometric up to rescaling. We denote the group of isometries (resp. $\mathfrak{L}$-isometries) of a structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ by $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ (resp. $\mathfrak{L}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$). The isotropy subgroup of $x \in \mathsf{G}$ (i.e., the subgroup of isometries fixing x) will be denoted by $\mathsf{Iso}_x(\mathsf{G}, \mathcal{D}, \mathbf{g})$.

Remark 1. Isometries are distance preserving (i.e., for any isometry ϕ, we have that $d(x, y) = d(\phi(x), \phi(y))$). Conversely, every distance-preserving diffeomorphism ϕ is an isometry (see, e.g., [27]). Moreover, if all geodesics are normal, then any distance-preserving homeomorphism is smooth ([14], see also [9]).

Every left translation is an isometry. Hence, the group of isometries $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is generated by the group of left translations and the isotropy subgroup of the identity. Indeed, any isometry $\phi \in \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ can be written as $\phi = L_{\phi(\mathbf{1})} \circ \overline{\phi}$, where $\overline{\phi} \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}.\mathcal{D}, \mathbf{g})$. When the isotropy subgroup $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g})$ is a subgroup of the automorphism group $\mathsf{Aut}(\mathsf{G})$, then the group of left translations is normal in $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ decomposes as a semidirect product of the group of left translations and the isotropy subgroup $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$. Consequently, in order to describe the full isometry group $\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$, it is enough to describe the isotropy subgroup $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$.

Any isometry of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is uniquely determined by its tangent map at a point ([19, Section 2.2]). Accordingly, we shall denote by

$$d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \{T_{\mathbf{1}}\phi : \phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})\}$$

the corresponding linearized isotropy subgroup; we note that

$$\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \cong d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})\,.$$

We likewise denote by $d\,\mathfrak{L}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ the group of linearized $\mathfrak{L}$-isometries

$$d\,\mathfrak{L}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \{T_{\mathbf{1}}\phi : \phi \in \mathfrak{L}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})\}\,;$$

on a simply connected Lie group, we have that

$$d\,\mathfrak{L}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g}) = \{\psi \in \mathsf{Aut}(\mathfrak{g}) : \psi \cdot \mathcal{D}(\mathbf{1}) = \mathcal{D}(\mathbf{1}), \psi^* \mathbf{g}_{\mathbf{1}} = \mathbf{g}_{\mathbf{1}}\}\,.$$

2.2 Some elements of invariant Riemannian geometry

Let ∇ denote the Riemannian connection associated to $(\mathsf{G}, \mathbf{g})$. As $\mathbf{g}$ is left-invariant, it follows that ∇ is also left-invariant. In particular, ∇ gives rise to a bilinear map $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (given by the action of ∇ on left-invariant vector fields). Accordingly, the curvature and Ricci tensors are also left invariant and are entirely described by their restrictions to the Lie algebra $\mathfrak{g}$; we shall exclusively work with these restricted versions in all computations. We briefly recall some useful formulae for the connection, curvature tensor, and Ricci tensor in this context ([20]).

For left-invariant vector fields Y, Z, and W (or elements $Y, Z, W \in \mathfrak{g}$), we have

$$\mathbf{g}(\nabla_Y Z, W) = \frac{1}{2}\Big(\mathbf{g}([Y,Z],W) - \mathbf{g}([Z,W],Y) + \mathbf{g}([W,Y],Z)\Big).$$

Accordingly, if $(X_1, X_2, \ldots, X_n)$ is a left-invariant orthonormal frame for $(\mathsf{G}, \mathbf{g})$, then

$$\begin{aligned}\nabla_Y Z &= \sum_{i=1}^{n} \mathbf{g}(\nabla_Y Z, X_i) X_i \\ &= \sum_{i=1}^{n} \frac{1}{2}\Big(\mathbf{g}([Y,Z],X_i) - \mathbf{g}([Z,X_i],Y) + \mathbf{g}([X_i,Y],Z)\Big) X_i.\end{aligned}$$

The $(1,3)$-curvature tensor R for $(\mathsf{G}, \mathbf{g})$ is given by

$$R_{YZ} = \nabla_{[Y,Z]} - \nabla_Y \nabla_Z + \nabla_Z \nabla_Y\,;$$

its covariant derivative ∇R is given by

$$\begin{aligned}\nabla R(Y, Z_1, Z_2, Z_3) = & \nabla_Y R(Z_1, Z_2, Z_3) - R(\nabla_Y Z_1, Z_2, Z_3) \\ & - R(Z_1, \nabla_Y Z_2, Z_3) - R(Z_1, Z_2, \nabla_Y Z_3).\end{aligned}$$

Note that, as $(\mathsf{G}, \mathbf{g})$ is complete, it is symmetric if and only if $\nabla R \equiv 0$. The $(1,1)$-Ricci tensor is given by $\mathrm{Ric}(Y) = \sum_{i=1}^{3} R(X_i, Y, X_i)$; the principle Ricci curvatures are the eigenvalues of this linear endomorphism, whereas the scalar curvature is its trace.

3 Riemannian structures

We investigate the left-invariant Riemannian structures on the simply connected three-dimensional Lie groups. First, we present a classification of these structures up to $\mathfrak{L}$-isometry and rescaling. This essentially amounts to normalizing an arbitrary (positive definite) inner product on each Lie algebra by the group of automorphisms. The automorphism group for each Lie algebra, and the normalized metric for that Lie algebra, is exhibited in Appendix B.

Proposition 1 (cf. [12]; see also [23]). *Any left-invariant Riemannian structure on a simply connected three-dimensional Lie group is $\mathfrak{L}$-isometric, up to rescaling, to exactly one of the following Riemannian structures:*

$$3\mathfrak{g}_1 : (E_1, E_2, E_3) \tag{1}$$
$$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 : \left(E_1, E_2, \tfrac{1}{\sqrt{1-\beta^2}}(\beta E_1 - E_3)\right),\ 0 \leq \beta < 1 \tag{2}$$
$$\mathfrak{g}_{3.1} : (E_1, E_2, E_3) \tag{3}$$
$$\mathfrak{g}_{3.2} : (\tfrac{1}{\sqrt{\beta}} E_1, E_2, E_3),\ \beta > 0 \tag{4}$$
$$\mathfrak{g}_{3.3} : (E_1, E_2, E_3) \tag{5}$$
$$\mathfrak{g}_{3.4}^0 : (\tfrac{1}{\sqrt{\beta}} E_1, E_2, E_3),\ 0 < \beta \leq 1 \tag{6}$$
$$\mathfrak{g}_{3.4}^\alpha : (\tfrac{1}{\sqrt{\beta}} E_1, E_2, E_3),\ 0 < \beta \leq 1 \tag{7}$$
$$\mathfrak{g}_{3.5}^0 : (\tfrac{1}{\sqrt{\beta}} E_1, E_2, E_3),\ 0 < \beta \leq 1 \tag{8}$$
$$\mathfrak{g}_{3.5}^\alpha : (\tfrac{1}{\sqrt{\beta}} E_1, E_2, E_3),\ 0 < \beta \leq 1 \tag{9}$$
$$\mathfrak{g}_{3.6} : (\tfrac{1}{\sqrt{\beta_1}} E_1, \tfrac{1}{\sqrt{\beta_2}} E_2, E_3),\ \beta_1 \geq \beta_2 > 0 \tag{10}$$
$$\mathfrak{g}_{3.7} : (\tfrac{1}{\sqrt{\beta_1}} E_1, \tfrac{1}{\sqrt{\beta_2}} E_2, E_3),\ \beta_1 \geq \beta_2 \geq 1. \tag{11}$$

Here each normalized Riemannian structure $(\mathsf{G}, \mathbf{g})$ *is specified by the Lie algebra* $\mathfrak{g}$ *of* G *and an orthonormal frame for* $\mathbf{g}$ *in terms of the basis for* $\mathfrak{g}$ *as given in Appendix A.*

For each of the normal forms given above, we carry out the following program.

1. The (restricted) connection $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is determined. The scalar curvature ρ and principle Ricci curvatures $\lambda_1 \geq \lambda_2 \geq \lambda_3$ are calculated; when $\rho \neq 0$ the

Riemannian metric is rescaled such that $|\rho| = 1$. We also determine whether or not the structure is symmetric. In Figures 1, 2, and 3 the (normalized) principle Ricci curvatures $\lambda_1 \geq \lambda_2$ are plotted for each structure (clearly $\lambda_3 = \rho - \lambda_1 - \lambda_2$). Table 2 lists the Riemannian structures indexed by their isotropy subgroups.

2. The subgroup of Lie algebra isomorphisms preserving the metric, namely $d\,\mathfrak{L}\mathsf{Iso}(\mathsf{G}, \mathbf{g})$, is determined.

3. The linearized isotropy subgroup $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g})$ is determined (as described below).

4. For the symmetric structures, we compute the Riemannian exponential map $\mathrm{Exp} : \mathfrak{g} \to \mathsf{G}$ and its inverse. A set of generators for $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g})$ is also computed in this case.

The first two points are a fairly straightforward computation. The fourth point is somewhat more involved, although the procedure is standard (we make use of the approach given by geometric control theory in our calculations, see, e.g., [5], [15]). For the third point, which is the main interest of this paper, we proceed as follows.

Any isometry $\phi \in \mathsf{Iso}(\mathsf{G}, \mathbf{g})$ preserves the metric, the curvature tensor R, and its covariant derivative ∇R. Consequently, for any $\psi \in d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g})$ we have that $\psi^* \mathbf{g}_{\mathbf{1}} = \mathbf{g}_{\mathbf{1}}$, $\psi^* R = R$, and $\psi^* \nabla R = \nabla R$, i.e.,

$$\begin{gathered}\mathbf{g}_{\mathbf{1}}(\psi \cdot A_1, \psi \cdot A_2) = \mathbf{g}_{\mathbf{1}}(A_1, A_2) \\ R(\psi \cdot A_1, \psi \cdot A_2, \psi \cdot A_3) = \psi \cdot R(A_1, A_2, A_3) \\ \nabla R(\psi \cdot A_1, \psi \cdot A_2, \psi \cdot A_3, \psi \cdot A_4) = \psi \cdot \nabla R(A_1, A_2, A_3, A_4)\end{gathered}$$

for $A_1, \ldots, A_4 \in \mathfrak{g}$. Accordingly, let

$$\mathsf{Sym}(\mathsf{G}, \mathbf{g}) = \{\psi \in \mathsf{GL}(\mathfrak{g}) \,:\, \psi^* \mathbf{g}_{\mathbf{1}} = \mathbf{g}_{\mathbf{1}},\, \psi^* R = R,\, \psi^* \nabla R = \nabla R\}.$$

We have that $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g})$ is a subgroup of $\mathsf{Sym}(\mathsf{G}, \mathbf{g})$. When the structure is symmetric, we make use of a well-known result.

Lemma 1. *If $\nabla R \equiv 0$, then $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) = \mathsf{Sym}(\mathsf{G}, \mathbf{g})$.*

Proof. The Riemannian structure $(\mathsf{G}, \mathbf{g})$ is complete and the group G is simply connected by assumption. Hence, for any linear map $\psi \in \mathsf{GL}(\mathfrak{g})$ preserving the curvature tensor at the identity, there exists a unique isometry $\phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g})$ such that $T_{\mathbf{1}}\phi = \psi$ (see, e.g., [24, Chapter 8, Theorem 55]). It therefore follows that $\mathsf{Sym}(\mathsf{G}, \mathbf{g}) \leq d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g})$ and thus $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) = \mathsf{Sym}(\mathsf{G}, \mathbf{g})$. □

On the other hand, if $\mathsf{Sym}(\mathsf{G}, \mathbf{g}) \leq \mathsf{Aut}(\mathfrak{g})$, then the following holds true.

Lemma 2. *If $\mathsf{Sym}(\mathsf{G}, \mathbf{g}) \leq \mathsf{Aut}(\mathfrak{g})$, then $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) = \mathsf{Sym}(\mathsf{G}, \mathbf{g})$ and $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) = \mathfrak{L}\mathsf{Iso}(\mathsf{G}, \mathbf{g})$.*

Proof. Let $\psi \in \mathsf{Sym}(\mathsf{G}, \mathbf{g}) \leq \mathsf{Aut}(\mathfrak{g})$. As G is assumed to be simply connected, there exits an automorphism $\phi \in \mathsf{Aut}(\mathsf{G})$ such that $T_{\mathbf{1}}\phi = \psi$. Hence, as $T_x\phi \cdot xA = T_{\mathbf{1}}L_{\phi(x)} \cdot \psi(A)$, we find that $\phi \in \mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g})$. Thus $\mathsf{Sym}(\mathsf{G}, \mathbf{g}) \leq d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g})$ and so $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) = \mathsf{Sym}(\mathsf{G}, \mathbf{g})$. Accordingly, as $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) \leq \mathsf{Aut}(\mathfrak{g})$, it follows that $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) \leq \mathsf{Aut}(\mathsf{G})$ and so $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) = \mathfrak{L}\mathsf{Iso}(\mathsf{G}, \mathbf{g})$. □

Remark 2. The Lie algebra $\mathfrak{sym}(\mathsf{G}, \mathbf{g})$ of $\mathsf{Sym}(\mathsf{G}, \mathbf{g})$ consists of linear maps $\psi \in \mathfrak{gl}(\mathfrak{g})$ such that

$$\begin{aligned}
0 &= \mathbf{g}_{\mathbf{1}}(\psi \cdot A_1, A_2) + \mathbf{g}_{\mathbf{1}}(A_1, \psi \cdot A_2)\\
\psi \cdot R(A_1, A_2, A_3) &= R(\psi \cdot A_1, A_2, A_3) + R(A_1, \psi \cdot A_2, A_3) + R(A_1, A_2, \psi \cdot A_3)\\
\psi \cdot \nabla R(A_1, A_2, A_3, A_4) &= \nabla R(\psi \cdot A_1, A_2, A_3, A_4) + \nabla R(A_1, \psi \cdot A_2, A_3, A_4)\\
&\quad + \nabla R(A_1, A_2, \psi \cdot A_3, A_4) + \nabla R(A_1, A_2, A_3, \psi \cdot A_4)
\end{aligned}$$

for all $A_1, \dots A_4 \in \mathfrak{g}$. In several cases $\mathfrak{sym}(\mathsf{G}, \mathbf{g})$ can quite easily be shown to be trivial; consequently, we have that $\mathsf{Sym}(\mathsf{G}, \mathbf{g})$ is a discrete subgroup of $\mathsf{O}(3)$ in those cases. By [13, Corollary 2.8], it then follows that $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) = \mathfrak{L}\mathsf{Iso}(\mathsf{G}, \mathbf{g})$.

It turns out that most structures have the property that $\mathsf{Sym}(\mathsf{G}, \mathbf{g}) \leq \mathsf{Aut}(\mathfrak{g})$ or are symmetric. Accordingly, in these cases the isotropy subgroup may be determined by means of Lemma 1 or 2; a typical case is treated in Section 3.1. In fact, only on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ do there exist structures such that $\mathsf{Sym}(\mathsf{G}, \mathbf{g}) \not\leq \mathsf{Aut}(\mathfrak{g})$ and $\nabla R \not\equiv 0$; this case is considered exceptional and treated separately in Section 3.2.1. The results for the Riemannian structures on each simply connected Lie group are catalogued in Appendix B; Table 2 lists the Riemannian structures indexed by their isotropy subgroups. With these results at hand, we are able to make the following general claims.

Theorem 1. *Let* $(\mathsf{G}, \mathbf{g})$ *be a Riemannian structure on a simply connected three-dimensional Lie group* G.

1. *If* $\nabla R \not\equiv 0$ *and* $\mathsf{G} \not\cong \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, *then* $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) \leq \mathsf{Aut}(\mathsf{G})$.
2. *If* $\nabla R \not\equiv 0$ *and* $\mathsf{G} \cong \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, *then* $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) \not\leq \mathsf{Aut}(\mathsf{G})$.
3. *If* $\nabla R \equiv 0$ *and* G *is non-Abelian, then* $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) \not\leq \mathsf{Aut}(\mathsf{G})$.
4. *If* G *is Abelian, then* $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) \leq \mathsf{Aut}(\mathsf{G})$ *trivially.*
5. *In all cases,* $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) = \mathsf{Sym}(\mathsf{G}, \mathbf{g})$.

Corollary 1. *Let* G *be non-Abelian simply connected three-dimensional Lie group. For any left-invariant Riemannian structure* $(\mathsf{G}, \mathbf{g})$, *we have that* $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g}) \leq \mathsf{Aut}(\mathsf{G})$ *if and only if* $\mathsf{G} \not\cong \mathsf{Aff}(\mathbb{R}) \times \mathbb{R}$ *and* $\nabla R \not\equiv 0$.

Proposition 2. *Let* G *be a simply connected three-dimensional Lie group. Two Riemannian structures* $\mathbf{g}^1$ *and* $\mathbf{g}^2$ *on* G *are isometric up to rescaling if and only if they are* $\mathfrak{L}$*-isometric up to rescaling.*

Proof. It suffices to show that no two of the normal forms up to $\mathfrak{L}$-isometry (and rescaling) are isometric. In Appendix B, the structures with nonzero scaler curvature ρ are rescaled so that $|\rho| = 1$; the associated principle Ricci curvatures are given in each case. It turns out that in almost all cases, if two (normalized) structures are not $\mathfrak{L}$-isometric, then their principle Ricci curvatures differ. Indeed, only on $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ does there exist a family of (normalized) structures, no two of which are $\mathfrak{L}$-isometric, with identical principle Ricci curvatures. We show that these structures are not isometric by identifying an additional scalar invariant; details are given in Section 3.2.2. $\square$

Remark 3. There exists Riemannian structures on nonisomorphic Lie groups which are nevertheless isometric. For example, we have an isometry between a Riemannian structure on $\widetilde{\mathsf{SE}}(2)$ and the Euclidean structure on $\mathbb{R}^3$; there also exists an isometry between the Riemannian structures on $\mathsf{G}_{3.5}^{\alpha}$ and $\mathsf{G}_{3.3}$ (see Appendix B.8). Additionally, there exists an isometry between some Riemannian structures on $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ and $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ (see Section 3.2.1). The only group which admits Riemannian structures which are isometric but not $\mathfrak{L}$-isometric is $\widetilde{\mathsf{SE}}(2)$; indeed, the Euclidean structures (i.e., with $R \equiv 0$) are all isometric, but not $\mathfrak{L}$-isometric (they are, however, $\mathfrak{L}$-isometric up to rescaling).

3.1 A typical case

We consider the simply connected (universal covering of the) Euclidean group

$$\widetilde{\mathsf{SE}}(2) = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_1 & \cos x_3 & -\sin x_3 & 0 \\ x_2 & \sin x_3 & \cos x_3 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{x_3} \end{bmatrix} \;:\; x_1, x_2, x_3 \in \mathbb{R} \right\}$$

as a typical case. Its Lie algebra

$$\mathfrak{g}_{3.5}^{0} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & -a_3 & 0 \\ a_2 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_3 \end{bmatrix} = a_1 E_1 + a_2 E_2 + a_3 E_3 \;:\; a_1, a_2, a_3 \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_2, E_3] = E_1$ and $[E_3, E_1] = E_2$. Any left-invariant Riemannian structure on $\widetilde{\mathsf{SE}}(2)$ is $\mathfrak{L}$-isometric to the left-invariant Riemannian structure given by $\mathbf{g_1} = r\,\mathrm{diag}(\beta, 1, 1)$ for some $r > 0$ and $0 < \beta \leq 1$ (Proposition 1); the Riemannian structure $(\widetilde{\mathsf{SE}}(2), \mathbf{g})$ admits the orthonormal frame $\frac{1}{\sqrt{r}}(\frac{1}{\sqrt{\beta}}E_1, E_2, E_3)$. The associated (restricted) connection $\nabla : \mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_{3.5}^0 \to \mathfrak{g}_{3.5}^0$ is given by

$$\begin{aligned}\nabla_A B = &\frac{(\beta-1)a_2 b_3 - (\beta+1)a_3 b_2}{2\beta} E_1 + \frac{(\beta-1)a_1 b_3 + (\beta+1)a_3 b_1}{2} E_2 \\ &- \frac{(\beta-1)(a_1 b_2 + a_2 b_1)}{2} E_3.\end{aligned}$$

Here $A = a_1E_1 + a_2E_2 + a_3E_3$ and $B = b_1E_1 + B_2E_2 + b_3E_3$. The associated $(1,3)$-curvature tensor $R : \mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_{3.5}^0 \oplus \mathfrak{g}_{3.5}^0 \to \mathfrak{g}_{3.5}^0$ is given by

$$\begin{aligned} R(A,B,C) = &\tfrac{(\beta-1)(-a_1((\beta-1)b_2c_2+(\beta+3)b_3c_3)+a_2(\beta-1)b_1c_2+a_3(\beta+3)b_1c_3)}{4\beta}E_1 \\ &+ \tfrac{(\beta-1)(a_1(\beta-1)\beta b_2c_1+a_2(b_3(3\beta c_3+c_3)-(\beta-1)\beta b_1c_1)-a_3(3\beta+1)b_2c_3)}{4\beta}E_2 \\ &+ \tfrac{(\beta-1)(b_3(a_1\beta(\beta+3)c_1-a_2(3\beta c_2+c_2))+a_3(b_2(3\beta c_2+c_2)-\beta(\beta+3)b_1c_1))}{4\beta}E_3. \end{aligned}$$

We have that $\nabla R \equiv 0$ if and only if $\beta = 1$; also, note that $R \equiv 0$ when $\beta = 1$. The principle Ricci curvatures are $\left\{\frac{1-\beta^2}{2\beta r}, -\frac{(1-\beta)^2}{2\beta r}, -\frac{1-\beta^2}{2\beta r}\right\}$ and the scalar curvature is $\rho = -\frac{(1-\beta)^2}{2\beta r}$. Assuming $0 < \beta < 1$ and taking $r = \frac{(1-\beta)^2}{2\beta}$, we obtain normalized curvatures $\rho = -1$ and $\lambda_1 = \frac{1+\beta}{1-\beta} \geq \lambda_2 = -1 \geq \lambda_3 = -\frac{1+\beta}{1-\beta}$. Note that the principle Ricci curvatures uniquely determine β.

We now proceed to finding the isotropy subgroup. If $\beta = 1$, then $\nabla R \equiv 0$ and $R \equiv 0$; so $\mathsf{Sym}(\widetilde{\mathsf{SE}}(2), \mathbf{g}) \cong \mathsf{O}(3)$. Consequently, by Lemma 1, we have that $\mathsf{Iso}_\mathbf{1}(\widetilde{\mathsf{SE}}(2), \mathbf{g}) = \mathsf{Sym}(\widetilde{\mathsf{SE}}(2), \mathbf{g})$. Assume that $0 < \beta < 1$. Let $\psi \in \mathsf{Sym}(\mathsf{G}, \mathbf{g})$. As $\psi^*\mathbf{g_1} = \mathbf{g_1}$, we have that ϕ is an orthogonal transformation and hence can be written as

$$\begin{aligned}\psi = &\begin{bmatrix} \cos\theta_1 & -\frac{1}{\sqrt{\beta}}\sin\theta_1 & 0 \\ \sqrt{\beta}\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & 0 & -\frac{1}{\sqrt{\beta}}\sin\theta_2 \\ 0 & 1 & 0 \\ \sqrt{\beta}\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \\ &\times \begin{bmatrix} \cos\theta_3 & -\frac{1}{\sqrt{\beta}}\sin\theta_3 & 0 \\ \sqrt{\beta}\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma \end{bmatrix}\end{aligned}$$

for some $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$. We have that

$$\begin{aligned} 0 &= \psi \cdot R(E_2, E_1, E_1) - R(\psi \cdot E_2, \psi \cdot E_1, \psi \cdot E_1) \\ &= \frac{(1-\beta)\sin\theta_1 \sin^2\theta_2 \cos\theta_3}{\sqrt{\beta}}E_1 + (1-\beta)\beta\cos\theta_1 \sin^2\theta_2 \cos\theta_3 E_2 \\ &\quad - \frac{1}{2}(1-\beta)\Big((1+\beta)\sin(2\theta_1)\cos\theta_2\cos\theta_3 + \big((1+\beta)\cos(2\theta_1) + \beta - 1\big)\sin\theta_3\Big)\sin\theta_2 E_3. \end{aligned}$$

Hence $\sin\theta_1 \sin\theta_2 \cos\theta_3 = 0$ and $\cos\theta_1 \sin\theta_2 \cos\theta_3 = 0$; thus $\sin\theta_2 \cos\theta_3 = 0$. Similarly, as $\psi \cdot R(E_2, E_1, E_2) - R(\psi \cdot E_2, \psi \cdot E_1, \psi \cdot E_2) = 0$, we get $\sin\theta_2 \sin\theta_3 = 0$. It therefore follows that $\sin\theta_2 = 0$. Thus

$$\psi = \begin{bmatrix} \sigma_1\cos\theta & -\frac{1}{\sqrt{\beta}}\sin\theta & 0 \\ \sigma_1\sqrt{\beta}\sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix}$$

for some $\theta \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$. From

$$\psi \cdot R(E_3, E_1, E_1) - R(\psi \cdot E_3, \psi \cdot E_1, \psi \cdot E_1) = 0\,,$$

it then follows that $\sin\theta = 0$. Consequently, we have that $\psi = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)$ for some $\sigma_1, \sigma_2, \sigma_3 \in \{-1, 1\}$ (moreover, for any such ψ we have that $\psi^*R = R$). We note, however, that one can show that $\psi = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)$ more simply in this case by noting that ψ must preserve the eigenspaces of the $(1,1)$-Ricci curvature tensor (cf. [13]), which are $\langle E_1 \rangle$, $\langle E_2 \rangle$, and $\langle E_3 \rangle$. We have that

$$\psi \cdot \nabla R(E_1, E_1, E_3, E_2) - \nabla R(\psi \cdot E_1, \psi \cdot E_1, \psi \cdot E_3, \psi \cdot E_2) = \frac{(1-\beta)^2 \sigma_1 (\sigma_1 \sigma_2 \sigma_3 - 1)}{2\beta} E_1 \,.$$

Thus $\sigma_3 = \sigma_1 \sigma_2$ and so $\psi = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_1 \sigma_2)$. Moreover, for any such ψ we find that $\psi^* \nabla R = \nabla R$. Consequently, when $0 < \beta < 1$, we have that

$$\mathsf{Sym}(\widetilde{\mathsf{SE}}(2), \mathbf{g}) = \{\operatorname{diag}(\sigma_1, \sigma_2, \sigma_1 \sigma_2) \,:\, \sigma_1, \sigma_2 = \pm 1\} \leq \mathsf{Aut}(\mathfrak{g}_{3.5}^0).$$

It therefore follows, by Lemma 2, that

$$d\,\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}(2), \mathbf{g}) = \mathsf{Sym}(\widetilde{\mathsf{SE}}(2), \mathbf{g}) = d\,\mathfrak{L}\mathsf{Iso}(\widetilde{\mathsf{SE}}(2), \mathbf{g}).$$

3.2 The exceptional cases

3.2.1 The case of $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$

We consider the simply connected (universal covering) Lie group

$$\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & \mathrm{e}^{x_3} \end{bmatrix} \,:\, x_1, x_3 \in \mathbb{R},\ x_2 > 0 \right\}.$$

Its Lie algebra

$$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_1 & -a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} = a_1 E_1 + a_2 E_2 + a_3 E_3 \,:\, a_1, a_2, a_3 \in \mathbb{R} \right\}$$

has nonzero commutator relations $[E_1, E_2] = E_1$. Any left-invariant Riemannian structure on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ is $\mathfrak{L}$-isometric to the left-invariant Riemannian structure $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathbf{g})$ given by

$$\mathbf{g}_{\mathbf{1}} = \begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}, \qquad 0 \leq \beta < 1.$$

When $\beta = 0$, then the structure is symmetric (and the isotropy subgroup can be determined by means of Lemma 1). Accordingly, assume $0 < \beta < 1$.

We claim that $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathbf{g})$ is isometric to the structure $(\widetilde{\mathsf{SL}}(2, \mathbb{R}), \overline{\mathbf{g}})$, where

$$\overline{\mathbf{g}}_{\mathbf{1}} = \frac{\beta^2}{1 - \beta^2} \begin{bmatrix} \frac{1}{\beta^2} - 1 & 0 & 0 \\ 0 & \frac{1}{\beta^2} - 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

Here $\overline{\mathbf{g}}_{\mathbf{1}}$ is expressed in terms of a basis $(\tilde{E}_1, \tilde{E}_2, \tilde{E}_3)$ for the Lie algebra $\widetilde{\mathfrak{sl}}(2, \mathbb{R})$ of $\widetilde{\mathsf{SL}}(2, \mathbb{R})$ which has commutator relations

$$[\tilde{E}_2, \tilde{E}_3] = \tilde{E}_1, \quad [\tilde{E}_3, \tilde{E}_1] = \tilde{E}_2, \quad [\tilde{E}_1, \tilde{E}_2] = -\tilde{E}_3 \,.$$

As $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ is not linearizable, we find it convenient to work with the matrix Lie group $\mathsf{SL}(2,\mathbb{R}) = \{x \in \mathbb{R}^{2\times 2} \,:\, \det x = 1\}$. The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $\mathsf{SL}(2,\mathbb{R})$ is given by

$$\mathfrak{sl}(2,\mathbb{R}) = \left\{ \frac{1}{2}\begin{bmatrix} -a_1 & a_2 + a_3 \\ a_2 - a_3 & a_1 \end{bmatrix} = a_1\bar{E}_1 + a_2\bar{E}_2 + a_3\bar{E}_3 \,:\, a_1, a_2, a_3 \in \mathbb{R} \right\}$$

and has commutator relations

$$[\bar{E}_2, \bar{E}_3] = \bar{E}_1, \quad [\bar{E}_3, \bar{E}_1] = \bar{E}_2, \quad [\bar{E}_1, \bar{E}_2] = -\bar{E}_3\,.$$

Let $\overline{\mathbf{g}}'$ be the left-invariant Riemannian metric on $\mathsf{SL}(2,\mathbb{R})$ such that $\overline{\mathbf{g}}'_{\mathbf{1}}$ has matrix (12) with respect to $(\bar{E}_1, \bar{E}_2, \bar{E}_3)$. Let $q : \widetilde{\mathsf{SL}}(2,\mathbb{R}) \to \mathsf{SL}(2,\mathbb{R})$ be the Lie group covering homomorphism such that $T_{\mathbf{1}}q \cdot \tilde{E}_i = \bar{E}_i$. We have that q is a Riemannian covering, i.e., $q^*\overline{\mathbf{g}}' = \overline{\mathbf{g}}$. Let $\phi : \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R} \to \mathsf{SL}(2,\mathbb{R})$,

$$\begin{bmatrix} 1 & 0 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & \mathrm{e}^{x_3} \end{bmatrix}$$

$$\longmapsto \frac{1}{\sqrt{x_2}} \begin{bmatrix} \cos(\alpha x_3) & \sin(\alpha x_3) \\ -x_1\sqrt{1-\beta^2}\cos(\alpha x_3) - x_2 \sin(\alpha x_3) & -x_1\sqrt{1-\beta^2}\sin(\alpha x_3) + x_2\cos(\alpha x_3) \end{bmatrix}$$

where $\alpha = \frac{\sqrt{1-\beta^2}}{2\beta}$. It is not difficult to show that

$$\mathbf{g}(xA, xA) = \overline{\mathbf{g}}'(T_x\phi \cdot xA, T_x\phi \cdot xA)$$

for $A \in \mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ and $x \in \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, i.e., $\mathbf{g} = \phi^*\overline{\mathbf{g}}'$. There exist a unique (universal covering) diffeomorphism $\tilde{\phi} : \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R} \to \widetilde{\mathsf{SL}}(2,\mathbb{R})$ such that $\tilde{\phi}(\mathbf{1}) = \mathbf{1}$ and such that the diagram

$$\begin{tikzcd} \mathsf{Aff}(\mathbb{R}) \times \mathbb{R} \arrow[r, "\tilde{\phi}"] \arrow[dr, "\phi"'] & \widetilde{\mathsf{SL}}(2,\mathbb{R}) \arrow[d, "q"] \\ & \mathsf{SL}(2,\mathbb{R}) \end{tikzcd}$$

commutes. Consequently, $\tilde{\phi}^*\overline{\mathbf{g}} = \mathbf{g}$, and so $\tilde{\phi}$ is an isometry between $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathbf{g})$ and $(\widetilde{\mathsf{SL}}(2,\mathbb{R}), \overline{\mathbf{g}})$. We note that $\tilde{\phi}$ is essentially the sub-Riemannian isometry described in [1], adapted to the associated Riemannian structures (see Proposition 4, Remark 5, and Section 4.2).

We have that $\mathsf{Sym}(\widetilde{\mathsf{SL}}(2,\mathbb{R}), \overline{\mathbf{g}}) \leq \mathsf{Aut}(\widetilde{\mathfrak{sl}}(2,\mathbb{R}))$ and consequently find that $\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SL}}(2,\mathbb{R}), \overline{\mathbf{g}}) \cong \mathsf{O}(2)$ (this corresponds to the case $\beta_1 = \beta_2 = \frac{1}{\beta^2} - 1$ for the Riemannian structure discussed in Appendix B.9). As $\tilde{\phi}$ is an isometry, it follows that $\mathsf{Iso}_{\mathbf{1}}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathbf{g}) \cong \mathsf{O}(2)$. Furthermore, we have that $\mathsf{dIso}_{\mathbf{1}}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathbf{g})$ is a subgroup of $\mathsf{Sym}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathbf{g})$, which is given by

$$\begin{aligned} \mathsf{Sym}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathbf{g}) &= \left\{ \begin{bmatrix} \sigma\cos\theta & \frac{\sigma\sin\theta}{\sqrt{1-\beta^2}} & 0 \\ -\sqrt{1-\beta^2}\sin\theta & \cos\theta & 0 \\ \sigma(\beta - \beta\cos\theta) & -\frac{\sigma\beta\sin\theta}{\sqrt{1-\beta^2}} & \sigma \end{bmatrix} : \theta \in \mathbb{R},\ \sigma = \pm 1 \right\} \\ &\cong \mathsf{O}(2). \end{aligned}$$

Thus we conclude that $\mathsf{dIso}_{\mathbf{1}}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathbf{g}) = \mathsf{Sym}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathbf{g})$.

3.2.2 An additional scalar invariant

We consider here again the universal covering $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ of the special orthogonal group; its Lie algebra $\widetilde{\mathfrak{sl}}(2,\mathbb{R})$ has commutator relations

$$[\tilde{E}_2, \tilde{E}_3] = \tilde{E}_1, \quad [\tilde{E}_3, \tilde{E}_1] = \tilde{E}_2, \quad [\tilde{E}_1, \tilde{E}_2] = -\tilde{E}_3 \,.$$

The left-invariant Riemannian structures $(\widetilde{\mathsf{SL}}(2,\mathbb{R}), \mathbf{g}^{\beta_1})$ given by

$$\mathbf{g}_{\mathbf{1}}^{\beta_1} = r\,\mathrm{diag}(\beta_1, \beta_2, 1), \quad \beta_2 = \beta_1 - 1 > 0, \quad r = \frac{2}{\beta_1 - 1} \tag{13}$$

all have principle Ricci curvatures $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -1$. Hence, in order to show that these structures are not isometric, we identify an additional scalar invariant.

The covariant derivative $\nabla\,\mathrm{Ric}$ of the $(1,1)$-Ricci tensor is a $(1,2)$-tensor. For a fixed left-invariant vector field X, we get a $(1,1)$-tensor $\nabla\,\mathrm{Ric}(X,\cdot)$. In three-dimensions, this linear endomorphism (which depends on X) has characteristic polynomial

$$p(\mu) = \mu_1\mu_2\mu_3 - (\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3)\mu + (\mu_1 + \mu_2 + \mu_3)\mu^2 - \mu^3.$$

Here μ_1, μ_2, μ_3 are the corresponding eigenvalues. The coefficient $(\mu_1 + \mu_2 + \mu_3)$ of μ^2 is linear in X, whereas the coefficient $(\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3)$ of μ is quadratic in X. That is to say, there exists a unique symmetric $(0,2)$-tensor S such that $S(X,X) = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3$. Raising the index, we obtain a $(1,1)$-tensor $\overline{S}$. The trace of $\overline{S}$ is accordingly a scalar invariant for the Riemannian structure (indeed, the set eigenvalues of $\overline{S}$ is an isometric invariant).

For the structure $(\widetilde{\mathsf{SL}}(2,\mathbb{R}), \mathbf{g}^{\beta_1})$, the $(1,1)$-tensors Ric and $\nabla\,\mathrm{Ric}(X,\cdot)$ have matrices

$$\mathrm{Ric} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \nabla\,\mathrm{Ric}(X,\cdot) = \begin{bmatrix} 0 & a_3 & 0 \\ \frac{a_3\beta_1}{\beta_1 - 1} & 0 & \frac{a_1}{\beta_1 - 1} \\ 0 & a_1 & 0 \end{bmatrix}$$

Here $X = a_1\tilde{E}_1 + a_2\tilde{E}_2 + a_3\tilde{E}_3$. Accordingly

$$\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 = \frac{a_1^2 + \beta_1 a_3^2}{\beta_1 - 1}$$

and so $\overline{S}$ has matrix

$$\overline{S} = \mathrm{diag}\left(\frac{1}{2\beta_1}, 0, \frac{\beta_1}{2}\right).$$

The trace of $\overline{S} = \frac{\beta_1^2+1}{2\beta_1}$; we have that $\frac{\beta_1^2+1}{2\beta_1} = \frac{\beta_1'^2+1}{2\beta_1'}$ for some $\beta_1, \beta_1' > 1$ only if $\beta_1 = \beta_1'$. Hence we can conclude that no two of the Riemannian metrics (13) are isometric.

Remark 4. The trace of $\overline{S}$ (and more generally its eigenvalues) can be used to distinguish between some structures on distinct nonisomorphic Lie groups whose principle Ricci curvatures coincide. For instance, the structure (6), $\beta = 1$ on $\mathsf{SE}(1,1)$ also has principle Ricci curvatures $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -1$. However, for this structure the trace of $\overline{S}$ is equal to 1; thus this structure is not isometric to (12) for any $\beta_1 > 1$ (as $1 < \frac{\beta_1^2+1}{2\beta_1}$ for $\beta_1 > 1$).

4 Sub-Riemannian structures

We now proceed to investigate the isometry groups of the sub-Riemannian structures on simply connected three-dimensional Lie groups. Following the work of Agrachev and Barilari [1], one can associate a contact structure to each such sub-Riemannian structure; by promoting the corresponding Reeb vector field to a orthonormal compliment of the distribution we (canonically) associate a Riemannian structure to a given sub-Riemannian structure. Accordingly, we can then use the results of the foregoing section in the investigation of the isometry groups.

Let (Y_1, Y_2) be an orthonormal frame for a left-invariant sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$. There exists a unique contact one-form ω on G (i.e., $d\omega \wedge \omega$ is a nonvanishing volume form) such that

$$\ker \omega = \mathcal{D} = \operatorname{span}(Y_1, Y_2) \qquad \text{and} \qquad d\omega(Y_1, Y_2) = 1.$$

Any other orthonormal frame (Y_1, Y_2) yields the same one-form, up to a change of sign.

Lemma 3. *If $\phi \in \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$, then $\phi^*\omega = \pm\omega$.*

The Reeb vector field associated to the contact one-form ω is the unique vector field Y_0 such that $\omega(Y_0) = 1$ and $i_{Y_0} d\omega = 0$; the Reeb vector field is uniquely determined up to a change of sign.

Lemma 4. *If $\phi \in \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$, then $\phi_* Y_0 = \pm Y_0$.*

As left translations are (orientation preserving) isometries for any invariant sub-Riemannian structure, it follows that the Reeb vector field is left invariant. Accordingly, we associate to $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ the Riemannian structure $(\mathsf{G}, \tilde{\mathbf{g}})$ admitting orthonormal frame (Y_0, Y_1, Y_2). Note that $(\mathsf{G}, \tilde{\mathbf{g}})$ does not depend on the choice of orthonormal frame (Y_1, Y_2). We show that the isometries of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ are exactly the isometries of $(\mathsf{G}, \tilde{\mathbf{g}})$ preserving $\mathcal{D}$.

Proposition 3. *$\phi \in \mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ if and only if $\phi \in \mathsf{Iso}(\mathsf{G}, \tilde{\mathbf{g}})$ and $\phi_*\mathcal{D} = \mathcal{D}$.*

Proof. Suppose ϕ is an isometry of a sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ with orthonormal frame (Y_1, Y_2). Then $(\phi_* Y_1, \phi_* Y_2)$ is an orthonormal frame for $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and so $(\pm Y_0, \phi_* Y_1, \phi_* Y_2)$ is an orthonormal frame for $(\mathsf{G}, \tilde{\mathbf{g}})$. That is, ϕ pushes forward the orthonormal frame (Y_0, Y_1, Y_2) of $(\mathsf{G}, \tilde{\mathbf{g}})$ to an orthonormal frame $(\pm Y_0, \phi_* Y_1, \phi_* Y_2)$ of $(\mathsf{G}, \tilde{\mathbf{g}})$. Thus ϕ is an isometry of $(\mathsf{G}, \tilde{\mathbf{g}})$ such that $\phi_*\mathcal{D} = \mathcal{D}$.

Conversely, suppose ϕ is an isometry of $(\mathsf{G}, \tilde{\mathbf{g}})$ such that $\phi_*\mathcal{D} = \mathcal{D}$. Let (Y_1, Y_2) be an orthonormal frame for $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and (Y_0, Y_1, Y_2) be the corresponding orthonormal frame for $(\mathsf{G}, \tilde{\mathbf{g}})$. We have that $(\phi_* Y_0, \phi_* Y_1, \phi_* Y_2)$ is an orthonormal frame for $(\mathsf{G}, \tilde{\mathbf{g}})$. Moreover, as $\phi_* \mathcal{D} = \mathcal{D}$, we have that $(\phi_* Y_1, \phi_* Y_2)$ is an orthonormal frame for $\mathcal{D}$ with respect to $\mathbf{g}$ (as $\tilde{\mathbf{g}}\big|_{\mathcal{D}} = \mathbf{g}$). Hence ϕ is an isometry of $(\mathsf{G}, \mathcal{D}, \mathbf{g})$. □

The above argument can easily be adapted to isometries between two distinct structures.

Proposition 4. *Let* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ *be two left-invariant sub-Riemannian structures with associated Riemannian structures* $(\mathsf{G}, \tilde{\mathbf{g}})$ *and* $(\mathsf{G}', \tilde{\mathbf{g}}')$*, respectively. A diffeomorphism* ϕ *is an isometry between* $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *and* $(\mathsf{G}', \mathcal{D}', \mathbf{g}')$ *if and only if* ϕ *is an isometry between* $(\mathsf{G}, \tilde{\mathbf{g}})$ *and* $(\mathsf{G}', \tilde{\mathbf{g}}')$ *such that* $\phi_* \mathcal{D} = \mathcal{D}'$*.*

Remark 5. Rescaling the metric $\mathbf{g}$ of a sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ by a constant $r > 0$ rescales the associated Reeb vector field by $\frac{1}{r}$ (and any orthonormal frame by $\frac{1}{\sqrt{r}}$). Accordingly, the Riemannian metric associated to $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ is generally not related by rescaling to the Riemannian metric associated to $(\mathsf{G}, \mathcal{D}, r\mathbf{g})$.

Next, we give a classification of the sub-Riemannian structures up to $\mathfrak{L}$-isometry; this essentially amounts to normalizing the bracket generating subspaces and the (positive definite) inner products on these subspaces. Again, we note that the automorphism group for each Lie algebra, and the normalized sub-Riemannian structure for that Lie algebra, is exhibited in Appendix B.

Proposition 5 (cf. [28, p. 52]; see also [1]). *Any left-invariant sub-Riemannian structure on a simply connected three-dimensional Lie group is* $\mathfrak{L}$*-isometric, up to rescaling, to exactly one of the following sub-Riemannian structures:*

$$
\begin{aligned}
3\mathfrak{g}_1 &: \varnothing \\
\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 &: (E_1 + E_3, E_2) && (14)\\
\mathfrak{g}_{3.1} &: (E_2, E_3) && (15)\\
\mathfrak{g}_{3.2} &: (E_2, E_3) && (16)\\
\mathfrak{g}_{3.3} &: \varnothing \\
\mathfrak{g}_{3.4}^0 &: (E_2, E_3) && (17)\\
\mathfrak{g}_{3.4}^\alpha &: (E_2, E_3) && (18)\\
\mathfrak{g}_{3.5}^0 &: (E_2, E_3) && (19)\\
\mathfrak{g}_{3.5}^\alpha &: (E_2, E_3) && (20)\\
\mathfrak{g}_{3.6} &: (\tfrac{1}{\sqrt{\beta}} E_1, E_2),\ 0 < \beta \leq 1 && (21)\\
& \ (\tfrac{1}{\sqrt{\beta}} E_2, E_3),\ 0 < \beta && (22)\\
\mathfrak{g}_{3.7} &: (\tfrac{1}{\sqrt{\beta}} E_2, E_3),\ 0 < \beta \leq 1. && (23)
\end{aligned}
$$

Here each normalized sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$ *is specified by the Lie algebra* $\mathfrak{g}$ *of* G *and an orthonormal frame in terms of the basis for* $\mathfrak{g}$ *as given in Appendix A.*

For each of the normal forms given above, we determine the linearized isotropy subgroup $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g})$. This is accomplished by determining the associated Riemannian structure and making use of the results of the previous section. A typical case in treated in Section 4.1; the sub-Riemannian structure on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ presents an exception and is treated in Section 4.2. Full results are catalogued (along with the Riemannian results) in Appendix B; Table 3 lists the sub-Riemannian structures indexed by their isotropy subgroups. With these results at hand, we are able to make the following general claims.

Theorem 2. *Let G be a simply connected three-dimensional Lie group. For any left-invariant sub-Riemannian structure $(\mathsf{G}, \mathcal{D}, \mathbf{g})$, we have that $\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathcal{D}, \mathbf{g}) \leq \mathsf{Aut}(\mathsf{G})$ if and only if $\mathsf{G} \not\cong \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$.*

Proposition 6 (cf. [1]). *Two sub-Riemannian structures on the same simply connected three-dimensional Lie group are isometric if and only if they are $\mathfrak{L}$-isometric.*

Proof. Again it suffices to show that no two of the normal forms up to $\mathfrak{L}$-isometry are isometric. This can be accomplished by studying the associated Riemannian structures; alternatively, one can make use of the classification given in [1]. □

4.1 A typical case

We again consider the simply connected Euclidean group $\widetilde{\mathsf{SE}}(2)$ (see Section 3.1). Any left-invariant sub-Riemannian structure on $\widetilde{\mathsf{SE}}(2)$ is $\mathfrak{L}$-isometric to exactly one of the left-invariant structures $(\widetilde{\mathsf{SE}}(2), \mathcal{D}, r\mathbf{g})$, $r > 0$ given by $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3 \rangle$ and $r\mathbf{g}_{\mathbf{1}} = r\,\mathrm{diag}(1, 1)$. The structure $(\widetilde{\mathsf{SE}}(2), \mathcal{D}, r\mathbf{g})$ admits the orthonormal frame $\frac{1}{\sqrt{r}}(E_2, E_3)$. Let (ν_1, ν_2, ν_3) be the (Maurer–Cartan) coframe dual to the given frame (E_1, E_2, E_3) of left-invariant vector fields. The contact one-form ω associated to $(\widetilde{\mathsf{SE}}(2), \mathcal{D}, r\mathbf{g})$ is given by $\omega = -r\nu_1$ and has exterior derivative $d\omega = r\nu_2 \wedge \nu_3$. Accordingly, the corresponding Reeb vector field is $-\frac{1}{r}E_1$. It therefore follows that the Riemannian structure $(\widetilde{\mathsf{SE}}(2), \tilde{\mathbf{g}}^r)$ associated to $(\widetilde{\mathsf{SE}}(2), \mathcal{D}, r\mathbf{g})$ is given by $\tilde{\mathbf{g}}^r_{\mathbf{1}} = r\,\mathrm{diag}(r, 1, 1)$. We have, by Proposition 3, that

$$\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}(2), \mathcal{D}, r\mathbf{g}) \leq \mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}(2), \tilde{\mathbf{g}}^r)\,.$$

Also, as shown in Section 3.1, we have that

$$d\,\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}(2), \tilde{\mathbf{g}}^r) = \{\mathrm{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2) \,:\, \sigma_1, \sigma_2 = \pm 1\} \leq \mathsf{Aut}(\mathfrak{g}^0_{3.5})$$

whenever $r < 1$. As the isotropy subgroups $\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}(2), \mathcal{D}, r\mathbf{g})$ are identical for all $r > 0$, it follows that

$$d\,\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}(2), \mathcal{D}, r\mathbf{g}) \leq \{\mathrm{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2) \,:\, \sigma_1, \sigma_2 = \pm 1\} \leq \mathsf{Aut}(\mathfrak{g}^0_{3.5}).$$

Consequently, we find that

$$\begin{aligned}\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}(2), \mathcal{D}, r\mathbf{g}) &\cong d\,\mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SE}}(2), \mathcal{D}, r\mathbf{g}) \\ &= d\,\mathfrak{L}\mathsf{Iso}(\widetilde{\mathsf{SE}}(2), \mathcal{D}, r\mathbf{g}) \\ &= \{\mathrm{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2) \,:\, \sigma_1, \sigma_2 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2.\end{aligned}$$

4.2 The exceptional case

We again consider the group $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ (see Section 3.2.1). Any left-invariant sub-Riemannian structure on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ is $\mathfrak{L}$-isometric up to rescaling to the structure $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ given by $\mathcal{D}(\mathbf{1}) = \langle E_1 + E_3, E_2 \rangle$ and $\mathbf{g_1} = \mathrm{diag}(1,1)$; this structure has orthonormal frame $(E_1 + E_3, E_2)$. Proceeding as in a typical case (i.e., by considering the associated Riemannian structure) we find that

$$\begin{aligned} d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g}) &\leq \left\{ \begin{bmatrix} \sigma\cos\theta & -\sigma\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ -\sigma + \sigma\cos\theta & -\sigma\sin\theta & \sigma \end{bmatrix} : \theta \in \mathbb{R}, \sigma = \pm 1 \right\} \\ &\cong \mathsf{O}(2). \end{aligned}$$

However, as these maps are not all Lie algebra automorphisms, we shall make use of the fact that $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ is isometric to a sub-Riemannian structure on $\widetilde{\mathsf{SL}}(2,\mathbb{R})$ ([1]) to establish equality. Indeed the mapping $\phi : \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R} \to \mathsf{SL}(2,\mathbb{R})$,

$$\begin{bmatrix} 1 & 0 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & \mathrm{e}^{x_3} \end{bmatrix} \longmapsto \frac{1}{\sqrt{x_2}} \begin{bmatrix} \cos\frac{x_3}{2} & \sin\frac{x_3}{2} \\ x_1\cos\frac{x_3}{2} - x_2\sin\frac{x_3}{2} & x_2\cos\frac{x_3}{2} + x_1\sin\frac{x_3}{2} \end{bmatrix}$$

is a sub-Riemannian covering from $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ to the sub-Riemannian structure $(\mathsf{SL}(2,\mathbb{R}), \overline{\mathcal{D}}', \overline{\mathbf{g}}')$ admitting orthonormal frame $(\bar{E}_1, \bar{E}_2)$. Proceeding as in Section 3.2.1, we therefore have a sub-Riemannian isometry $\tilde{\phi} : \mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R} \to \widetilde{\mathsf{SL}}(2,\mathbb{R})$ between $(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g})$ and the structure $(\widetilde{\mathsf{SL}}(2,\mathbb{R}), \overline{\mathcal{D}}, \overline{\mathbf{g}})$ admitting orthonormal frame $(\tilde{E}_1, \tilde{E}_2)$ such that $\tilde{\phi}(\mathbf{1}) = \mathbf{1}$. Accordingly,

$$\mathsf{Iso}_{\mathbf{1}}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g}) \cong \mathsf{Iso}_{\mathbf{1}}(\widetilde{\mathsf{SL}}(2,\mathbb{R}), \overline{\mathcal{D}}, \overline{\mathbf{g}}) \cong \mathsf{O}(2)$$

and so

$$d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}, \mathcal{D}, \mathbf{g}) = \left\{ \begin{bmatrix} \sigma\cos\theta & -\sigma\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ -\sigma + \sigma\cos\theta & -\sigma\sin\theta & \sigma \end{bmatrix} : \theta \in \mathbb{R}, \sigma = \pm 1 \right\}.$$

A Three-dimensional Lie algebras

There are eleven types of three-dimensional real Lie algebras; in fact, nine algebras and two parametrized infinite families of algebras (see, e.g., [18], [22], [26]). In terms of an (appropriate) ordered basis (E_1, E_2, E_3), the commutation operation is given by

$$\begin{aligned} [E_2, E_3] &= n_1 E_1 - \alpha E_2 \\ [E_3, E_1] &= \alpha E_1 + n_2 E_2 \\ [E_1, E_2] &= n_3 E_3. \end{aligned}$$

The structure parameters α, n_1, n_2, n_3 for each type are given in Table 1. For each Lie algebra $\mathfrak{g}$ there exists a unique connected simply connected (universal covering) Lie group with Lie algebra $\mathfrak{g}$; with the exception of $\widetilde{\mathsf{SL}}(2,\mathbb{R})$, these groups are all linearizable. Matrix representations for each of the linearizable groups can, for instance, be found in [6].

Algebra	α	n_1	n_2	n_3	Unimodular	Nilpotent	Compl. Solvable	Exponential	Solvable	Simple	Simply connected Lie group
$3\mathfrak{g}_1$	0	0	0	0	•	•	•	•	•		$\mathbb{R}^3$
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	1	1	-1	0			•	•	•		$\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$
$\mathfrak{g}_{3.1}$	0	1	0	0	•	•	•	•	•		H_3
$\mathfrak{g}_{3.2}$	1	1	0	0			•	•	•		$\mathsf{G}_{3.2}$
$\mathfrak{g}_{3.3}$	1	0	0	0			•	•	•		$\mathsf{G}_{3.3}$
$\mathfrak{g}_{3.4}^0$	0	1	-1	0	•		•	•	•		$\mathsf{SE}(1,1)$
$\mathfrak{g}_{3.4}^\alpha$	$\alpha>0$, $\alpha\neq1$	1	-1	0			•	•	•		$\mathsf{G}_{3.4}^\alpha$
$\mathfrak{g}_{3.5}^0$	0	1	1	0	•				•		$\widetilde{\mathsf{SE}}(2)$
$\mathfrak{g}_{3.5}^\alpha$	$\alpha>0$	1	1	0				•	•		$\mathsf{G}_{3.5}^\alpha$
$\mathfrak{g}_{3.6}$	0	1	1	-1	•					•	$\widetilde{\mathsf{SL}}(2,\mathbb{R})$
$\mathfrak{g}_{3.7}$	0	1	1	1	•					•	$\mathsf{SU}(2)$

Table 1: Classification of three-dimensional Lie algebras

Note 1. We use a basis for $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$ different from the one given in Table 1. Specifically, we use the basis

$$E_1' = \tfrac{1}{2}(E_1 - E_2), \quad E_2' = -\tfrac{1}{2}E_3, \quad E_3' = \tfrac{1}{2}(E_1 + E_2)\,;$$

the only nonzero commutator is $[E_1', E_2'] = E_1'$.

B Catalogue of Riemannian and sub-Riemannian structures

We catalogue the results for the Riemannian and sub-Riemannian structures by Lie algebra. The following information is exhibited for each Lie algebra:

1. A matrix representation of the Lie algebra along with the commutator relations for a given basis; the group of Lie algebra automorphisms (represented with respect to the given basis).

2. The normalized (up to $\mathfrak{L}$-isometry) Riemannian structure along with an orthonormal frame and the associated restricted connection.

3. The normalized scalar curvature ρ and principle Ricci curvatures $\lambda_1 \geq \lambda_2 \geq \lambda_3$ (though we omit λ_3, as $\lambda_3 = \rho - \lambda_1 - \lambda_2$); whether or not the structure is symmetric.

4. The groups $d\,\mathfrak{L}\mathsf{Iso}(\mathsf{G}, \mathbf{g})$, $\mathsf{Sym}(\mathsf{G}, \mathbf{g})$, and $d\,\mathsf{Iso}_{\mathbf{1}}(\mathsf{G}, \mathbf{g})$; however, we shall simply write $d\,\mathfrak{L}\mathsf{Iso}$, Sym, and $d\,\mathsf{Iso}_{\mathbf{1}}$, respectively.

$\mathsf{Iso_1}$	Lie algebra	Riemannian structure	$\mathsf{Iso_1} \leq \mathsf{Aut}(\mathsf{G})$	Symmetric
$\mathsf{O}(3)$	$3\mathfrak{g}_1$	$\mathbb{E}^3$	•	•
	$\mathfrak{g}_{3.3}$	(5)		•
	$\mathfrak{g}_{3.5}^0$	(8), $\beta = 1$		•
	$\mathfrak{g}_{3.5}^\alpha$	(9), $\beta = 1$		•
	$\mathfrak{g}_{3.7}$	(11), $\beta_1 = \beta_2 = 1$		•
$\mathsf{O}(2) \times \mathbb{Z}_2$	$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	(2), $\beta = 0$		•
$\mathsf{O}(2)$	$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	(2), $0 < \beta < 1$		
	$\mathfrak{g}_{3.1}$	(3)	•	
	$\mathfrak{g}_{3.6}$	(10), $\beta_1 = \beta_2 > 0$	•	
	$\mathfrak{g}_{3.7}$	(11), $\beta_1 = \beta_2 > 1$	•	
		(11), $\beta_1 > \beta_2 = 1$	•	
D_4	$\mathfrak{g}_{3.4}^0$	(6), $\beta = 1$	•	
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathfrak{g}_{3.4}^0$	(6), $0 < \beta < 1$	•	
	$\mathfrak{g}_{3.4}^\alpha$	(7), $\beta = 1$	•	
	$\mathfrak{g}_{3.5}^0$	(8), $0 < \beta < 1$	•	
	$\mathfrak{g}_{3.6}$	(10), $\beta_1 > \beta_2 > 0$	•	
$\mathbb{Z}_2$	$\mathfrak{g}_{3.2}$	(4)	•	
	$\mathfrak{g}_{3.4}^\alpha$	(7), $0 < \beta < 1$	•	
	$\mathfrak{g}_{3.5}^\alpha$	(9), $0 < \beta < 1$	•	
	$\mathfrak{g}_{3.7}$	(11), $\beta_1 > \beta_2 > 1$	•	

Table 2: Riemannian structures indexed by isotropy subgroup

5. The normalized (up to $\mathfrak{L}$-isometry and rescaling) sub-Riemannian structure along with an orthonormal frame.

6. The Riemannian structure associated to the given sub-Riemannian structure (as defined in Section 4).

7. The groups $d\,\mathfrak{L}\mathsf{Iso}(\mathsf{G}, \mathcal{D}, \mathbf{g})$ and $d\,\mathsf{Iso_1}(\mathsf{G}, \mathcal{D}, \mathbf{g})$; however, we again simply write $d\,\mathfrak{L}\mathsf{Iso}$ and $d\,\mathsf{Iso_1}$, respectively.

When a Riemannian structure is symmetric, we additionally present the Riemannian exponential map $\mathrm{Exp} : \mathfrak{g} \to \mathsf{G}$ and its inverse for that structure (except in the case of $\mathfrak{g}_{3.5}^\alpha$ where instead an isometry to a structure on $\mathfrak{g}_{3.3}$ is provided); we also identify a set of isometries which generate $\mathsf{Iso_1}(\mathsf{G}, \mathbf{g})$ (again with the exception of $\mathfrak{g}_{3.5}^\alpha$).

A concrete matrix representation for G is supplied only when G admits a symmetric structure (in which case the parametrization for the group is used to express the Riemannian exponential map and some generators of the isotropy subgroup).

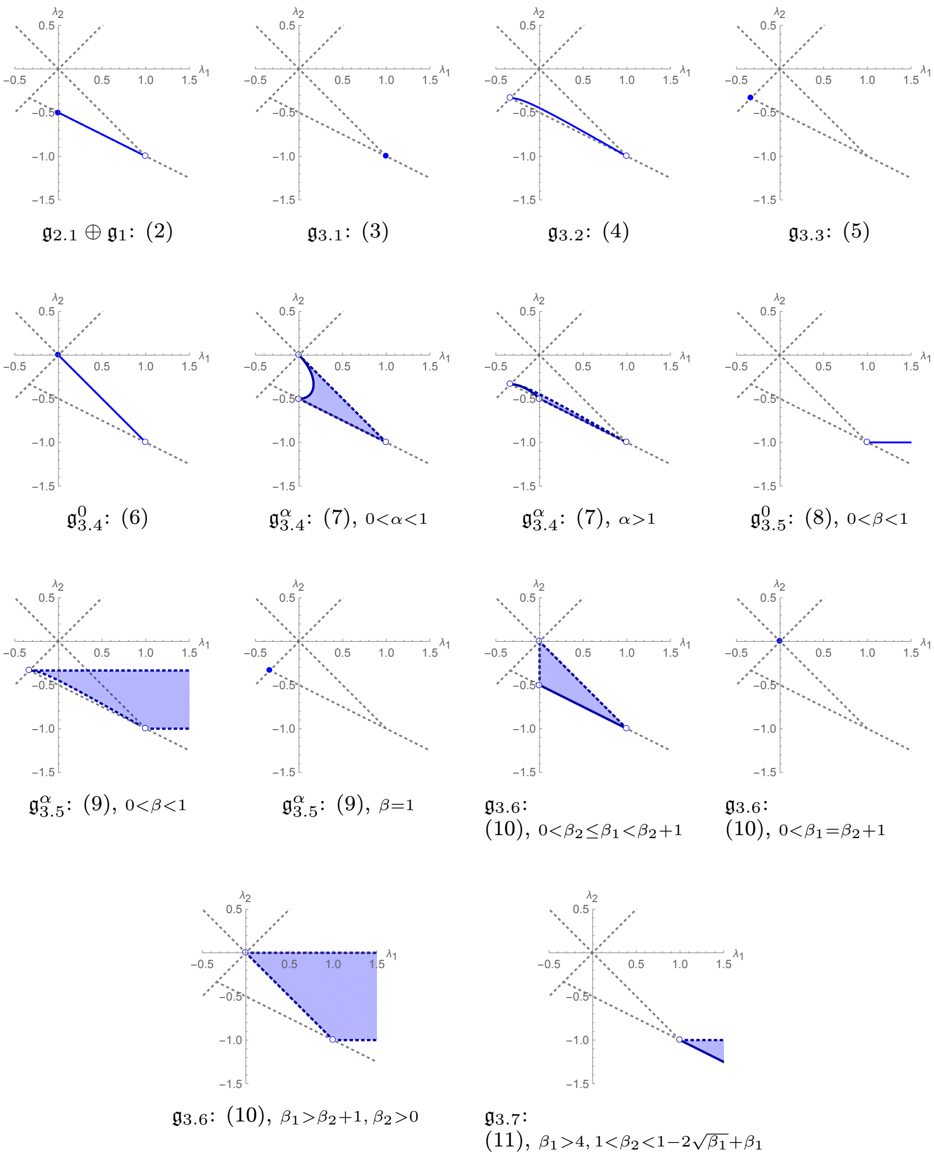

Figure 1: Normalized principle Ricci curvatures λ_1, λ_2 for Riemannian structures with scalar curvature $\rho = -1$

$\mathfrak{g}_{3.7}$: (11), $\beta_2 \geq 1, \beta_1 = 1 + 2\sqrt{\beta_2} + \beta_2, r > 0$

$\mathfrak{g}^0_{3.5}$: (8), $\beta = 1$

Figure 2: Principle Ricci curvatures λ_1, λ_2 for Riemannian structures with scalar curvature $\rho = 0$

$\mathfrak{g}_{3.7}$: (11), $1 \leq \beta_1, \beta_2, 1 - 2\sqrt{\beta_1} + \beta_1 < \beta_2 \leq \beta_1, \beta_1 < 1 + \beta_2$

$\mathfrak{g}_{3.7}$: (11), $1 \leq \beta_1, \beta_2, 1 - 2\sqrt{\beta_1} + \beta_1 < \beta_2 \leq \beta_1, \beta_1 \geq 1 + \beta_2$

Figure 3: Normalized principle Ricci curvatures λ_1, λ_2 for Riemannian structures with scalar curvature $\rho = 1$

Iso_1	Lie algebra	Sub-Riemannian structure	$\mathsf{Iso}_1 \leq \mathsf{Aut}(\mathsf{G})$	Symmetric ([27])
$\mathsf{O}(2)$	$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	(14)		•
	$\mathfrak{g}_{3.1}$	(15)	•	•
	$\mathfrak{g}_{3.6}$	(21), $\beta = 1$	•	•
	$\mathfrak{g}_{3.7}$	(23), $\beta = 1$	•	•
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathfrak{g}^0_{3.4}$	(17)	•	•
	$\mathfrak{g}^0_{3.5}$	(19)	•	•
	$\mathfrak{g}_{3.6}$	(21), $0 < \beta < 1$	•	•
		(22)	•	•
	$\mathfrak{g}_{3.7}$	(11), $0 < \beta < 1$	•	•
$\mathbb{Z}_2$	$\mathfrak{g}_{3.2}$	(16)	•	
	$\mathfrak{g}^\alpha_{3.4}$	(18)	•	
	$\mathfrak{g}^\alpha_{3.5}$	(20)	•	

Table 3: Sub-Riemannian structures indexed by isotropy subgroup

Nonetheless, for all but $\mathfrak{g}_{3.6}$, the matrix Lie algebra given exponentiates to the corresponding simply connected matrix Lie group. The universal covering Lie group for $\mathfrak{g}_{3.6}$ is not linearizable; we find it convenient to represent the Lie algebra $\mathfrak{g}_{3.6}$ as the matrix Lie algebra $\mathfrak{so}(2,1)$ of the pseudo-orthogonal group $\mathsf{SO}(2,1)$.

In Figures 1, 2, and 3 the (normalized) principle Ricci curvatures $\lambda_1 \geq \lambda_2$ are plotted for each structure. In Tables 2 and 3 an index of the Riemannian and sub-Riemannian structures by isotropy subgroup is provided. We note that the Abelian case $3\mathfrak{g}_1$ is omitted in what follows, as it is trivial (there is no bracket generating sub-Riemannian structure and only one Riemannian structure, namely the Euclidean one).

B.1 Algebra $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$

- $\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_1 & -a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} = a_1E_1 + a_2E_2 + a_3E_3 \;:\; a_1, a_2, a_3 \in \mathbb{R} \right\}$

  $[E_1, E_2] = E_1$

- $\mathsf{Aut}(\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1) = \left\{ \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & 1 & 0 \\ 0 & a_3 & a_4 \end{bmatrix} \;:\; a_1, \ldots, a_4 \in \mathbb{R},\ a_1a_4 \neq 0 \right\}$

- $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x_1 & x_2 & 0 \\ 0 & 0 & \mathrm{e}^{x_3} \end{bmatrix} \leftrightarrow (x_1, x_2, x_3) \;:\; x_1, x_3 \in \mathbb{R},\ x_2 > 0 \right\}$

Riemannian structure

- $\mathbf{g_1} = r \begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}, \qquad r > 0,\ 0 \leq \beta < 1$

  orthonormal frame: $\frac{1}{\sqrt{r}}(E_1, E_2, \frac{1}{\sqrt{1-\beta^2}}(\beta E_1 - E_3))$

- $\nabla_A B = \frac{a_1\left(\beta^2-2\right)b_2 - \beta(a_2(\beta b_1+b_3)+a_3b_2)}{2(\beta^2-1)} E_1 - (\frac{1}{2}\beta(a_1b_3 + a_3b_1) + a_1b_1)E_2$
  $+ \frac{\beta(b_2(a_1+\beta a_3)+a_2(b_1+\beta b_3))}{2(\beta^2-1)} E_3$

Normalized invariants

- $\rho = -1$ (with $r = \frac{4-3\beta^2}{2-2\beta^2}$), $\qquad \lambda_1 = \frac{\beta^2}{4-3\beta^2}, \qquad \lambda_2 = \frac{\beta^2-2}{4-3\beta^2},$
  $\nabla R \equiv 0 \iff \beta = 0$

Isometries

- $d\,\mathfrak{L}\mathsf{Iso} = \begin{cases} \{\mathrm{diag}(\sigma_1, 1, \sigma_2) \;:\; \sigma_1, \sigma_2 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } \beta = 0 \\ \{\mathrm{diag}(\sigma, 1, \sigma) \;:\; \sigma = \pm 1\} \cong \mathbb{Z}_2 & \text{if } 0 < \beta < 1 \end{cases}$

- $\mathsf{Sym} = d\,\mathsf{Iso}_{\mathbf{1}} =$
$$\begin{cases} \left\{ \begin{bmatrix} \sigma_1\cos\theta & -\sin\theta & 0 \\ \sigma_1\sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} \,:\, \theta\in\mathbb{R},\ \sigma_1,\sigma_2=\pm1 \right\} \cong \mathsf{O}(2)\times\mathbb{Z}_2 & \text{if } \beta = 0 \\ \left\{ \begin{bmatrix} \sigma\cos\theta & \frac{\sigma\sin\theta}{\sqrt{1-\beta^2}} & 0 \\ -\sqrt{1-\beta^2}\sin\theta & \cos\theta & 0 \\ \sigma(\beta-\beta\cos\theta) & -\frac{\beta\sigma\sin\theta}{\sqrt{1-\beta^2}} & \sigma \end{bmatrix} \,:\, \theta\in\mathbb{R},\ \sigma=\pm1 \right\} \cong \mathsf{O}(2) & \text{if } 0<\beta<1 \end{cases}$$
- When $\beta = 0$, $\mathsf{Iso}_{\mathbf{1}}$ is generated by the isometries
$$\phi_1(x) = \left(\tfrac{\left(x_1^2+x_2^2-1\right)\sin\theta + 2x_1\cos\theta}{x_1^2+2x_1\sin\theta-\left(x_1^2+x_2^2-1\right)\cos\theta+x_2^2+1}, \tfrac{2x_2}{x_1^2+2x_1\sin\theta-\left(x_1^2+x_2^2-1\right)\cos\theta+x_2^2+1}, x_3 \right)$$
$$\phi_2(x) = (-x_1, x_2, x_3)$$
$$\phi_3(x) = (x_1, x_2, -x_3)$$
which have linearizations
$$T_{\mathbf{1}}\phi_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_{\mathbf{1}}\phi_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_{\mathbf{1}}\phi_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Riemannian exponential map (when $\beta = 0$ and $r = 1$)

- $\mathrm{Exp}(tA) =$
$$\left(\tfrac{a_1\sinh\left(t\sqrt{a_1^2+a_2^2}\right)}{a_2\sinh\left(t\sqrt{a_1^2+a_2^2}\right)+\sqrt{a_1^2+a_2^2}\cosh\left(t\sqrt{a_1^2+a_2^2}\right)}, \tfrac{2\sqrt{a_1^2+a_2^2}\,e^{t\sqrt{a_1^2+a_2^2}}}{\left(\sqrt{a_1^2+a_2^2}+a_2\right)e^{2t\sqrt{a_1^2+a_2^2}}+\sqrt{a_1^2+a_2^2}-a_2}, a_3 t \right)$$
- $\mathrm{Exp}^{-1}(x) = \Big(2x_1\zeta(x_1,x_2), \left(1-x_1^2-x_2^2\right)\zeta(x_1,x_2), x_3\Big)$,
$$\zeta(x_1,x_2) = \frac{\operatorname{sech}^{-1}\left(\frac{2x_2}{x_1^2+x_2^2+1}\right)}{\left(x_1^2+(x_2+1)^2\right)\sqrt{1-\frac{4x_2}{x_1^2+(x_2+1)^2}}}$$

Sub-Riemannian structure

- $\mathcal{D}(\mathbf{1}) = \langle E_1+E_3, E_2\rangle, \qquad \mathbf{g}_{\mathbf{1}} = \mathrm{diag}(1,1)$
orthonormal frame: (E_1+E_3, E_2)

Associated Riemannian structure

- $\tilde{\mathbf{g}}_{\mathbf{1}} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, which is $\mathfrak{L}$-isometric to $\bar{\mathbf{g}}_{\mathbf{1}} = \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 1 \end{bmatrix}$.

Isometries

- $d\,\mathfrak{L}\mathsf{Iso} = \{\mathrm{diag}(\sigma,1,\sigma) \,:\, \sigma=\pm1\}$
- $d\,\mathsf{Iso}_{\mathbf{1}} = \left\{ \begin{bmatrix} \sigma\cos\theta & -\sigma\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ -\sigma+\sigma\cos\theta & -\sigma\sin\theta & \sigma \end{bmatrix} \,:\, \theta\in\mathbb{R}, \sigma=\pm1 \right\} \cong \mathsf{O}(2)$

B.2 Algebra $\mathfrak{g}_{3.1}$

- $\mathfrak{g}_{3.1} = \left\{ \begin{bmatrix} 0 & a_2 & a_1 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{bmatrix} = a_1 E_2 + a_2 E_2 + a_3 E_3 \ : \ a_1, a_2, a_3 \in \mathbb{R} \right\}$

  $[E_2, E_3] = E_1$

- $\mathsf{Aut}(\mathfrak{g}_{3.1}) = \left\{ \begin{bmatrix} a_2 a_6 - a_5 a_3 & a_1 & a_4 \\ 0 & a_2 & a_5 \\ 0 & a_3 & a_6 \end{bmatrix} \ : \ a_1, \ldots, a_6 \in \mathbb{R},\ a_2 a_6 - a_5 a_3 \neq 0 \right\}$

Riemannian structure

- $\mathbf{g_1} = r\,\mathrm{diag}(1,1,1), \quad r > 0, \qquad \text{orthonormal frame: } \frac{1}{\sqrt{r}}(E_1, E_2, E_3)$
- $\nabla_A B = \frac{1}{2}(a_2 b_3 - a_3 b_2) E_1 + \frac{1}{2}(a_1 b_3 + a_3 b_1) E_2 + \frac{1}{2}(-a_1 b_2 - a_2 b_1) E_3$

Normalized invariants

- $\rho = -1 \text{ (with } r = \frac{1}{2}), \quad \lambda_1 = 1, \quad \lambda_2 = -1, \quad \nabla R \not\equiv 0$

Isometries

- $d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} = \left\{ \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma\cos\theta & -\sin\theta \\ 0 & \sigma\sin\theta & \cos\theta \end{bmatrix} \ : \ \theta \in \mathbb{R},\ \sigma = \pm 1 \right\} \cong \mathsf{O}(2)$

Sub-Riemannian structure

- $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3 \rangle, \qquad \mathbf{g_1} = \mathrm{diag}(1,1), \qquad \text{orthonormal frame: } (E_2, E_3)$

Associated Riemannian structure

- $\tilde{\mathbf{g}}_{\mathbf{1}} = \mathrm{diag}(1,1,1)$

Isometries

- $d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \left\{ \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma\cos\theta & -\sin\theta \\ 0 & \sigma\sin\theta & \cos\theta \end{bmatrix} \ : \ \theta \in \mathbb{R},\ \sigma = \pm 1 \right\} \cong \mathsf{O}(2)$

B.3 Algebra $\mathfrak{g}_{3.2}$

- $\mathfrak{g}_{3.2} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_2 & a_3 & 0 \\ a_1 & -a_3 & a_3 \end{bmatrix} = a_1 E_1 + a_2 E_2 + a_3 E_3 \ : \ a_1, a_2, a_3 \in \mathbb{R} \right\}$

  $[E_2, E_3] = E_1 - E_2,\ [E_3, E_1] = E_1$

- $\mathsf{Aut}(\mathfrak{g}_{3.2}) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & a_4 \\ 0 & 0 & 1 \end{bmatrix} \ : \ a_1, \ldots, a_4 \in \mathbb{R},\ a_1 \neq 0 \right\}$

Riemannian structure

- $\mathbf{g_1} = r\,\mathrm{diag}(\beta, 1, 1), \quad r, \beta > 0,$ orthonormal frame: $\frac{1}{\sqrt{r}}(\frac{1}{\sqrt{\beta}}E_1, E_2, E_3)$
- $\nabla_A B = \frac{1}{2}(b_3(a_2 - 2a_1) - a_3 b_2)E_1 + \frac{1}{2}(\beta a_1 b_3 - 2a_2 b_3 + \beta a_3 b_1)E_2$
  $- (\frac{1}{2}\beta(a_1 b_2 + a_2 b_1) + \beta a_1 b_1 + a_2 b_2)E_3$

Normalized invariants

- $\rho = -1$ (with $r = \frac{\beta}{2} + 6$), $\lambda_1 = \frac{\sqrt{\beta(\beta+4)}-4}{\beta+12}$, $\lambda_2 = \frac{8}{\beta+12} - 1$,
  $\nabla R \not\equiv 0$

Isometries

- $d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} = \{\mathrm{diag}(\sigma, \sigma, 1) \,:\, \sigma = \pm 1\} \cong \mathbb{Z}_2$

Sub-Riemannian structure

- $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3\rangle,$ $\mathbf{g_1} = \mathrm{diag}(1, 1),$ orthonormal frame: (E_2, E_3)

Associated Riemannian structure

- $\tilde{\mathbf{g}}_{\mathbf{1}} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is $\mathfrak{L}$-isometric to $\overline{\mathbf{g}}_{\mathbf{1}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Isometries

- $d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \{\mathrm{diag}(\sigma, \sigma, 1) \,:\, \sigma = \pm 1\} \cong \mathbb{Z}_2$

B.4 Algebra $\mathfrak{g}_{3.3}$

- $\mathfrak{g}_{3.3} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_2 & a_3 & 0 \\ a_1 & 0 & a_3 \end{bmatrix} = a_1 E_1 + a_2 E_2 + a_3 E_3 \,:\, a_1, a_2, a_3 \in \mathbb{R} \right\}$
  $[E_2, E_3] = -E_2$, $[E_3, E_1] = E_1$
- $\mathsf{Aut}(\mathfrak{g}_{3.3}) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix} \,:\, a_1, \ldots, a_6 \in \mathbb{R}, a_1 a_5 - a_2 a_4 \neq 0 \right\}$
- $\mathsf{G}_{3.3} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x_2 & \mathrm{e}^{x_3} & 0 \\ x_1 & 0 & \mathrm{e}^{x_3} \end{bmatrix} \leftrightarrow (x_1, x_2, x_3) \,:\, x_1, x_2, x_3 \in \mathbb{R} \right\}$

Riemannian structure

- $\mathbf{g_1} = r\,\mathrm{diag}(1, 1, 1), \quad r > 0,$ orthonormal frame: $\frac{1}{\sqrt{r}}(E_1, E_2, E_3)$
- $\nabla_A B = -a_1 b_3 E_1 - a_2 b_3 E_2 + (a_1 b_1 + a_2 b_2)E_3$

Normalized invariants

- $\rho = -1$ (with $r = 6$), $\lambda_1 = -\frac{1}{3}$, $\lambda_2 = -\frac{1}{3}$, $\nabla R \equiv 0$

Isometries

- $d\,\mathfrak{L}\mathsf{Iso}_{\mathbf{1}} = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sigma\sin\theta & \sigma\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} : \theta\in\mathbb{R},\, \sigma - \pm 1 \right\} \cong \mathsf{O}(2)$

- $d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} \cong \mathsf{O}(3)$; $d\,\mathsf{Iso}_{\mathbf{1}}$ is generated by the isometries

$$\begin{aligned}
\phi_1(x) &= (x_1\cos\theta - x_2\sin\theta,\, x_1\sin\theta + x_2\cos\theta,\, x_3) \\
\phi_2(x) &= \left(\frac{(1-\zeta_1(x))\sin\theta + 2x_1\cos\theta}{2\zeta_2(x)}, \frac{x_2}{\zeta_2(x)}, x_3 - \log\zeta_2(x) \right) \\
\phi_3(x) &= \left(\frac{x_1}{\zeta_3(x)}, \frac{(1-\zeta_1(x))\sin\theta + 2x_2\cos\theta}{2\zeta_3(x)}, x_3 - \log\zeta_3(x) \right) \\
\phi_4(x) &= \left(\frac{x_1}{\zeta_1(x)}, \frac{x_2}{\zeta_1(x)}, x_3 - \log\zeta_1(x) \right)
\end{aligned}$$

where

$$\begin{aligned}
\zeta_1(x) &= x_1^2 + x_2^2 + \mathrm{e}^{2x_3} \\
2\zeta_2(x) &= 1 + \zeta_1(x) + (1-\zeta_1(x))\cos\theta - 2x_1\sin\theta \\
2\zeta_3(x) &= 1 + \zeta_1(x) + (1-\zeta_1(x))\cos\theta - 2x_2\sin\theta
\end{aligned}$$

and

$$T_{\mathbf{1}}\phi_1 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_{\mathbf{1}}\phi_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$
$$T_{\mathbf{1}}\phi_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad T_{\mathbf{1}}\phi_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Riemannian exponential map (when $r = 1$)

- $\mathrm{Exp}(tA) =$
$\left(\frac{a_1\sinh(\delta t)}{\delta\cosh(\delta t) - a_3\sinh(\delta t)}, \frac{a_2\sinh(\delta t)}{\delta\cosh(\delta t) - a_3\sinh(\delta t)}, -\log\left(\frac{\delta\cosh(\delta t) - a_3\sinh(\delta t)}{\delta} \right) \right)$,
$\delta = \sqrt{a_1^2 + a_2^2 + a_3^2}$

- $\mathrm{Exp}^{-1}(x) = 2\zeta(x)x_1E_1 + 2\zeta(x)x_2E_2 + \zeta(x)\left(x_1^2 + x_2^2 + \mathrm{e}^{2x_3} - 1\right)E_3$
where $\zeta(x) = \dfrac{\operatorname{arcsech}\left(\frac{2\,\mathrm{e}^{x_3}}{x_1^2+x_2^2+\mathrm{e}^{2x_3}+1}\right)}{\left(x_1^2+x_2^2+\mathrm{e}^{x_3}(\mathrm{e}^{x_3}+2)+1\right)\sqrt{1-\frac{4\,\mathrm{e}^{x_3}}{x_1^2+x_2^2+\mathrm{e}^{x_3}(\mathrm{e}^{x_3}+2)+1}}}$.

Sub-Riemannian structure

- Every subspace of $\mathfrak{g}_{3.3}$ is a subalgebra; accordingly, there are no left-invariant bracket generating distributions on $\mathsf{G}_{3.3}$.

B.5 Algebra $\mathfrak{g}_{3.4}^{0}$

- $\mathfrak{g}_{3.4}^{0} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_1 & 0 & -a_3 \\ a_2 & -a_3 & 0 \end{bmatrix} = a_1 E_1 + a_2 E_2 + a_3 E_3 \,:\, a_1, a_2, a_3 \in \mathbb{R} \right\}$
  $[E_2, E_3] = E_1$, $[E_3, E_1] = -E_2$

- $\mathsf{Aut}(\mathfrak{g}_{3.4}^{0}) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ \sigma a_2 & \sigma a_1 & a_4 \\ 0 & 0 & \sigma \end{bmatrix} \,:\, a_1, \ldots, a_4 \in \mathbb{R},\, a_1^2 - a_2^2 \neq 0,\, \sigma = \pm 1 \right\}$

Riemannian structure

- $\mathbf{g_1} = r\,\mathrm{diag}(\beta, 1, 1), \quad r > 0,\, 0 < \beta \leq 1$
  orthonormal frame: $\frac{1}{\sqrt{r}}(\frac{1}{\sqrt{\beta}} E_1, E_2, E_3)$

- $\nabla_A B = \frac{a_2(\beta+1)b_3 + a_3(b_2 - \beta b_2)}{2\beta} E_1 + \frac{1}{2}\big(a_1(\beta+1)b_3 + a_3(\beta-1)b_1\big) E_2$
  $\quad - \frac{1}{2}(\beta+1)(a_1 b_2 + a_2 b_1) E_3$

Normalized invariants

- $\rho = -1$ (with $r = \dfrac{(\beta+1)^2}{2\beta}$), $\quad \lambda_1 = \dfrac{1-\beta}{\beta+1}, \quad \lambda_2 = -\dfrac{1-\beta}{\beta+1}, \quad \nabla R \not\equiv 0$

Isometries

- $d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} =$

$$\begin{cases} \{\mathrm{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2) \,:\, \sigma_1, \sigma_2 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } \beta \neq 1 \\ \left\{ \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix}, \begin{bmatrix} 0 & \sigma_2 & 0 \\ \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} \,:\, \sigma_1, \sigma_2 = \pm 1 \right\} \cong D_4 & \text{if } \beta = 1 \end{cases}$$

(The dihedral group D_4 is generated by $a = \left[\begin{smallmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right]$ and $b = \left[\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right]$; it has elements $\mathbf{1}, a, a^2, a^3, ab, a^2 b, a^3 b$.)

Sub-Riemannian structure

- $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3 \rangle, \qquad \mathbf{g_1} = \mathrm{diag}(1, 1), \qquad$ orthonormal frame: (E_2, E_3)

Associated Riemannian structure

- $\tilde{\mathbf{g}}_{\mathbf{1}} = \mathrm{diag}(1, 1, 1)$

Isometries

- $d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \{\mathrm{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2) \,:\, \sigma_1, \sigma_2 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

B.6 Algebra $\mathfrak{g}^{\alpha}_{3.4}$

- $\mathfrak{g}^{\alpha}_{3.4} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_1 & \alpha a_3 & -a_3 \\ a_2 & -a_3 & \alpha a_3 \end{bmatrix} \,:\, a_1E_1 + a_2E_1 + a_3E_3 \,:\, a_1, a_2, a_3 \in \mathbb{R} \right\},$
  $\alpha > 0,\ \alpha \neq 1$
  $[E_2, E_3] = E_1 - \alpha E_2,\ [E_3, E_1] = \alpha E_1 - E_2$
- $\mathsf{Aut}(\mathfrak{g}^{\alpha}_{3.4}) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_4 \\ 0 & 0 & 1 \end{bmatrix} \,:\, a_1, \ldots, a_4 \in \mathbb{R},\ a_1^2 - a_2^2 \neq 0 \right\}$

Riemannian structure

- $\mathbf{g_1} = r\,\mathrm{diag}(\beta, 1, 1), \quad r > 0,\ 0 < \beta \leq 1$
  orthonormal frame: $\frac{1}{\sqrt{r}}(\frac{1}{\sqrt{\beta}}E_1, E_2, E_3)$
- $$\nabla_A B = \tfrac{b_3(-2\alpha\beta a_1 + \beta a_2 + a_2) + a_3(b_2 - \beta b_2)}{2\beta} E_1 + \tfrac{1}{2}\big((\beta a_1 + a_1 - 2\alpha a_2)b_3 + (\beta - 1)a_3 b_1\big)E_2 + \tfrac{1}{2}\big(-a_1(-2\alpha\beta b_1 + \beta b_2 + b_2) - a_2(\beta b_1 + b_1 - 2\alpha b_2)\big)E_3$$

Normalized invariants

- $\rho = -1 \qquad (\text{with } r = \frac{1}{2}\Big(12\alpha^2 + \beta + \frac{1}{\beta} + 2\Big)), \qquad \nabla R \not\equiv 0,$
  $$\begin{cases} \begin{cases} \lambda_1 = \dfrac{(\beta+1)\sqrt{\beta\,(4\alpha^2 + \beta - 2) + 1} - 4\alpha^2\beta}{\beta\,(12\alpha^2 + \beta + 2) + 1} \\ \lambda_2 = -\dfrac{4\alpha^2\beta + (\beta+1)\sqrt{\beta\,(4\alpha^2 + \beta - 2) + 1}}{\beta\,(12\alpha^2 + \beta + 2) + 1} \end{cases} & \text{if } 0 < \alpha < 1 \\ \begin{cases} \lambda_1 = \dfrac{(\beta+1)\sqrt{\beta\,(4\alpha^2 + \beta - 2) + 1} - 4\alpha^2\beta}{\beta\,(12\alpha^2 + \beta + 2) + 1} \\ \lambda_2 = \dfrac{8\alpha^2\beta}{\beta\,(12\alpha^2 + \beta + 2) + 1} - 1 \end{cases} & \text{if } \alpha > 1 \end{cases}$$

Isometries

- $d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} =$
  $$\begin{cases} \{\mathrm{diag}(\sigma, \sigma, 1) \,:\, \sigma = \pm 1\} \cong \mathbb{Z}_2 & \text{if } \beta \neq 1 \\ \left\{ \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \,:\, \sigma = \pm 1 \right\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } \beta = 1 \end{cases}$$

Sub-Riemannian structure

- $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3 \rangle, \qquad \mathbf{g_1} = \mathrm{diag}(1, 1), \qquad$ orthonormal frame: (E_2, E_3)

Associated Riemannian structure

- $\tilde{\mathbf{g}}_{\mathbf{1}} = \begin{bmatrix} \alpha^2+1 & -\alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

  which is $\mathfrak{L}$-isometric to $\overline{\mathbf{g}}_{\mathbf{1}} = \begin{bmatrix} \frac{1}{2}\left(\alpha^4+2-\alpha^2\sqrt{\alpha^4+4}\right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Isometries

- $d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \{\mathrm{diag}(\sigma, \sigma, 1) \,:\, \sigma = \pm 1\} \cong \mathbb{Z}_2$

B.7 Algebra $\mathfrak{g}^{\mathbf{0}}_{\mathbf{3.5}}$

- $\mathfrak{g}^0_{3.5} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & -a_3 & 0 \\ a_2 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_3 \end{bmatrix} = a_1E_1 + a_2E_2 + a_3E_3 \,:\, a_1, a_2, a_3 \in \mathbb{R} \right\}$

  $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$

- $\mathsf{Aut}(\mathfrak{g}^0_{3.5}) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ -\sigma a_2 & \sigma a_1 & a_4 \\ 0 & 0 & \sigma \end{bmatrix} \,:\, a_1, \ldots, a_4 \in \mathbb{R},\, a_1^2 + a_2^2 \neq 0,\, \sigma = \pm 1 \right\}$

- $\widetilde{\mathsf{SE}}(2) = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_1 & \cos x_3 & -\sin x_3 & 0 \\ x_2 & \sin x_3 & \cos x_3 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{x_3} \end{bmatrix} \leftrightarrow (x_1, x_2, x_3) \,:\, x_1, x_2, x_3 \in \mathbb{R} \right\}$

Riemannian structure

- $\mathbf{g}_{\mathbf{1}} = r\,\mathrm{diag}(\beta, 1, 1), \qquad r > 0,\ 0 < \beta \leq 1$

  orthonormal frame: $\frac{1}{\sqrt{r}}(\frac{1}{\sqrt{\beta}}E_1, E_2, E_3)$

- $\nabla_A B = \frac{a_2(\beta-1)b_3 - a_3(\beta+1)b_2}{2\beta}E_1 + \frac{1}{2}\big(a_1(\beta-1)b_3 + a_3(\beta+1)b_1\big)E_2$
  $\qquad - \frac{1}{2}(\beta-1)(a_1b_2 + a_2b_1)E_3$

Normalized invariants

- If $0 < \beta < 1$:

  $\rho = -1$ (with $r = \dfrac{(1-\beta)^2}{2\beta}$), $\quad \lambda_1 = \dfrac{1+\beta}{1-\beta}, \quad \lambda_2 = -1, \quad \nabla R \not\equiv 0.$

- If $\beta = 1$: $\rho = 0, \quad \lambda_1 = 0, \quad \lambda_2 = 0, \quad \nabla R \equiv 0.$

Isometries

- If $0 < \beta < 1$:

  $d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} = \{\mathrm{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2) \,:\, \sigma_1, \sigma_2 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

- If $\beta = 1$:
$$d\,\mathfrak{L}\mathsf{Iso} = \left\{ \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sigma\sin\theta & \sigma\cos\theta & 0 \\ 0 & 0 & \sigma \end{bmatrix} \,:\, \theta \in \mathbb{R},\, \sigma = \pm 1 \right\} \cong \mathsf{O}(2),$$
$$d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} \cong \mathsf{O}(3), \quad \mathsf{Iso}_{\mathbf{1}} = \left\{ \operatorname{Exp} \circ \psi \circ \operatorname{Exp}^{-1} \,:\, \psi \in d\,\mathsf{Iso}_{\mathbf{1}} \right\}$$

Riemannian exponential map (when $\beta = 1$ and $r = 1$)

- $\operatorname{Exp}(tA) = (ta_1, ta_2, ta_3), \qquad \operatorname{Exp}^{-1}(x) = x_1E_1 + x_2E_2 + x_3E_3$

Sub-Riemannian structure

- $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3 \rangle, \qquad \mathbf{g}_{\mathbf{1}} = \operatorname{diag}(1,1), \qquad$ orthonormal frame: (E_2, E_3)

Associated Riemannian structure

- $\tilde{\mathbf{g}}_{\mathbf{1}} = \operatorname{diag}(1,1,1)$; however, the rescaled structure $(\widetilde{\mathsf{SE}}(2), \mathcal{D}, r\mathbf{g})$ has associated Riemannian structure $\tilde{\mathbf{g}}' = r\operatorname{diag}(r,1,1)$, see Remark 5.

Isometries

- $d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \{\operatorname{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2) \,:\, \sigma_1, \sigma_2 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

B.8 Algebra $\mathfrak{g}_{3.5}^{\alpha}$

- $$\mathfrak{g}_{3.5}^{\alpha} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a_1 & \alpha a_3 & -a_3 \\ a_2 & a_3 & \alpha a_3 \end{bmatrix} = a_1E_1 + a_2E_2 + a_3E_3 \,:\, a_1, a_2, a_3 \in \mathbb{R} \right\},$$
$\alpha > 0$
$[E_2, E_3] = E_1 - \alpha E_2$, $[E_3, E_1] = \alpha E_1 + E_3$

- $$\mathsf{Aut}(\mathfrak{g}_{3.5}^{\alpha}) = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ -a_2 & a_1 & a_4 \\ 0 & 0 & 1 \end{bmatrix} \,:\, a_1, \ldots, a_4 \in \mathbb{R},\, a_1^2 + a_2^2 \neq 0 \right\}$$

- $$\mathsf{G}_{3.5}^{\alpha} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x_1 & \mathrm{e}^{\alpha x_3}\cos x_3 & -\mathrm{e}^{\alpha x_3}\sin x_3 \\ x_2 & \mathrm{e}^{\alpha x_3}\sin x_3 & \mathrm{e}^{\alpha x_3}\cos x_3 \end{bmatrix} \,:\, x_1, x_2, x_3 \in \mathbb{R} \right\}$$

Riemannian structure

- $\mathbf{g}_{\mathbf{1}} = r\operatorname{diag}(\beta, 1, 1), \qquad r > 0,\ 0 < \beta \leq 1$
orthonormal frame: $\frac{1}{\sqrt{r}}\left(\frac{1}{\sqrt{\beta}}E_1, E_2, E_3\right)$

- $$\begin{aligned} \nabla_A B = {} & -\tfrac{(2\alpha\beta a_1 - \beta a_2 + a_2)b_3 + (\beta+1)a_3b_2}{2\beta}E_1 \\ & + \tfrac{1}{2}\big(b_3(a_1(\beta-1) - 2\alpha a_2) + (\beta+1)a_3b_1\big)E_2 \\ & + \tfrac{1}{2}\big(a_1(2\alpha\beta b_1 - \beta b_2 + b_2) + a_2(b_1 - \beta b_1 + 2\alpha b_2)\big)E_3 \end{aligned}$$

Normalized invariants

- If $0 < \beta < 1$:
  $\rho = -1$ (with $r = \frac{1}{2}\left(12\alpha^2 + \beta + \frac{1}{\beta} - 2\right)$),
  $\lambda_1 = -\frac{4\alpha^2\beta + (\beta-1)\sqrt{\beta(4\alpha^2+\beta+2)+1}}{\beta(12\alpha^2+\beta-2)+1}, \qquad \lambda_2 = \frac{8\alpha^2\beta}{\beta(12\alpha^2+\beta-2)+1} - 1, \quad \nabla R \not\equiv 0$
- If $\beta = 1$: $\rho = -1$ (with $r = 6\alpha^2$), $\lambda_1 = -\frac{1}{3}$, $\lambda_2 = -\frac{1}{3}$, $\nabla R \equiv 0$

Isometries

- If $0 < \beta < 1$: $d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} = \{\mathrm{diag}(\sigma, \sigma, 1) \,:\, \sigma = \pm 1\} \cong \mathbb{Z}_2$
- If $\beta = 1$:
  $$d\,\mathfrak{L}\mathsf{Iso} = \left\{ \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \,:\, \theta \in \mathbb{R} \right\} \cong \mathsf{SO}(2), \qquad d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} \cong \mathsf{O}(3)$$

Isometry to structure on $\mathsf{G}_{3.3}$ (when $\beta = 1$)

- The diffeomorphism $\phi : \mathsf{G}_{3.5}^{\alpha} \to \mathsf{G}_{3.3}$,
  $$\begin{bmatrix} 1 & 0 & 0 \\ x_1 & \mathrm{e}^{\alpha x_3}\cos x_3 & -\mathrm{e}^{\alpha x_3}\sin x_3 \\ x_2 & \mathrm{e}^{\alpha x_3}\sin x_3 & \mathrm{e}^{\alpha x_3}\cos x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ \alpha x_2 & \mathrm{e}^{\alpha x_3} & 0 \\ \alpha x_1 & 0 & \mathrm{e}^{\alpha x_3} \end{bmatrix}$$
  defines an isometry between $(\mathsf{G}_{3.5}^{\alpha}, \mathbf{g}), \mathbf{g}_{\mathbf{1}} = \alpha^2 r\,\mathrm{diag}(1,1,1)$ and $(\mathsf{G}_{3.3}, \overline{\mathbf{g}})$, $\overline{\mathbf{g}}_{\mathbf{1}} = r\,\mathrm{diag}(1,1,1)$.

Sub-Riemannian structure

- $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3 \rangle$, $\qquad \mathbf{g}_{\mathbf{1}} = \mathrm{diag}(1,1)$, $\qquad$ orthonormal frame: (E_2, E_3)

Associated Riemannian structure

- $\tilde{\mathbf{g}}_{\mathbf{1}} = \begin{bmatrix} \alpha^2+1 & -\alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which is $\mathfrak{L}$-isometric to
  $$\overline{\mathbf{g}}_{\mathbf{1}} = \begin{bmatrix} \frac{1}{2}\left(2 + 4\alpha^2 + \alpha^4 - \left(2\alpha + \alpha^3\right)\sqrt{4+\alpha^2}\right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Isometries

- $d\,\mathsf{Iso}_{\mathbf{1}} = \{\mathrm{diag}(\sigma, \sigma, 1) \,:\, \sigma = \pm 1\} \cong \mathbb{Z}_2$

B.9 Algebra $\mathfrak{g}_{3.6}$

- $$\mathfrak{g}_{3.6} = \left\{ \begin{bmatrix} 0 & a_3 & a_2 \\ -a_3 & 0 & a_1 \\ a_2 & a_1 & 0 \end{bmatrix} = a_1 E_1 + a_2 E_2 + a_3 E_3 \,:\, a_1, a_2, a_3 \in \mathbb{R} \right\}$$
  $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = -E_3$

- $\mathsf{Aut}(\mathfrak{g}_{3.6}) = \mathsf{SO}(2,1) = \{x \in \mathbb{R}^{3\times 3} \,:\, x^\top J x = J,\ \det x = 1\}$,
  where $J = \mathrm{diag}(1,1,-1)$

Riemannian structure

- $\mathbf{g_1} = r\,\mathrm{diag}(\beta_1, \beta_2, 1), \quad \beta_1 \geq \beta_2 > 0,\ r > 0$
  orthonormal frame: $\frac{1}{\sqrt{r}}(\frac{1}{\sqrt{\beta_1}}E_1, \frac{1}{\sqrt{\beta_2}}E_2, E_3)$
- $\nabla_A B = \frac{a_2b_3(\beta_1-\beta_2-1)-a_3b_2(\beta_1+\beta_2+1)}{2\beta_1}E_1 + \frac{a_1b_3(\beta_1-\beta_2+1)+a_3b_1(\beta_1+\beta_2+1)}{2\beta_2}E_2$
  $+ \frac{1}{2}\big(a_1b_2(-\beta_1+\beta_2-1) + a_2b_1(-\beta_1+\beta_2+1)\big)E_3$

Normalized invariants:

- If $1+\beta_2 > \beta_1 \geq \beta_2 > 0$:
  $\rho = -1$ (with $r = \frac{\beta_1^2-2\beta_1(\beta_2-1)+(\beta_2+1)^2}{2\beta_1\beta_2}$), $\quad \lambda_1 = \frac{2(\beta_1+\beta_2+1)}{\beta_1^2-2\beta_1(\beta_2-1)+(\beta_2+1)^2} - 1$,
  $\lambda_2 = \frac{\beta_1^2-(\beta_2+1)^2}{\beta_1^2-2\beta_1(\beta_2-1)+(\beta_2+1)^2}, \quad \nabla R \not\equiv 0$
- If $1+\beta_2 = \beta_1$, $\beta_2 > 0$:
  $\rho = -1$ (with $r = \frac{2}{\beta_2}$), $\quad \lambda_1 = 0, \quad \lambda_2 = 0, \quad \nabla R \not\equiv 0$ (see also Section 3.2.2)
- If $\beta_1 > 1+\beta_2$, $\beta_2 > 0$:
  $\rho = -1$ (with $r = \frac{\beta_1^2-2\beta_1(\beta_2-1)+(\beta_2+1)^2}{2\beta_1\beta_2}$), $\quad \lambda_1 = \frac{\beta_1^2-(\beta_2+1)^2}{\beta_1^2-2\beta_1(\beta_2-1)+(\beta_2+1)^2}$,
  $\lambda_2 = \frac{2(\beta_1+\beta_2+1)}{\beta_1^2-2\beta_1(\beta_2-1)+(\beta_2+1)^2} - 1, \quad \nabla R \not\equiv 0$

Isometries

- $d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} =$
  $$\begin{cases} \{\mathrm{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2) \,:\, \sigma_1, \sigma_2 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } \beta_1 > \beta_2 > 0 \\ \left\{ \begin{bmatrix} \sigma\cos\theta & -\sin\theta & 0 \\ \sigma\sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma \end{bmatrix} \,:\, \theta \in \mathbb{R},\ \sigma = \pm 1 \right\} \cong \mathsf{O}(2) & \text{if } \beta_1 = \beta_2 > 0 \end{cases}$$
  We note that in [13] it is incorrectly claimed that the isometry groups are $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathsf{O}(2) \times \mathbb{Z}_2$, respectively; the mistake lies in the claim that certain linear isomorphisms ψ are Lie algebra automorphisms, but the condition $\det \psi = 1$ is not satisfied.

Sub-Riemannian structures

- $\mathcal{D}_1(\mathbf{1}) = \langle E_1, E_2 \rangle, \qquad \mathbf{g}^1_{\mathbf{1}} = \mathrm{diag}(\beta, 1), \quad 0 < \beta \leq 1$
  orthonormal frame: $(\frac{1}{\sqrt{\beta}}E_1, E_2)$
- $\mathcal{D}_2(\mathbf{1}) = \langle E_2, E_3 \rangle, \qquad \mathbf{g}^2_{\mathbf{1}} = \mathrm{diag}(\beta, 1), \quad 0 < \beta$
  orthonormal frame: $(\frac{1}{\sqrt{\beta}}E_2, E_3)$

Associated Riemannian structures

- $\tilde{\mathbf{g}}_{\mathbf{1}}^1 = \operatorname{diag}(\beta, 1, \beta)$, which is $\mathfrak{L}$-isometric to $\overline{\mathbf{g}}_{\mathbf{1}} = \beta \operatorname{diag}(\frac{1}{\beta}, 1, 1)$.
- $\tilde{\mathbf{g}}_{\mathbf{1}}^2 = \operatorname{diag}(\beta, \beta, 1)$

Isometries

- $(\mathcal{D}_1, \mathbf{g}^1) \,:\, d\,\mathfrak{L}\mathsf{Iso}_{\mathbf{1}} = d\,\mathsf{Iso}_{\mathbf{1}} =$
$$\begin{cases} \{\operatorname{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2) \,:\, \sigma_1, \sigma_2 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } 0 < \beta < 1 \\ \left\{ \begin{bmatrix} \sigma\cos\theta & -\sin\theta & 0 \\ \sigma\sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma \end{bmatrix} \,:\, \theta \in \mathbb{R},\, \sigma = \pm 1 \right\} \cong \mathsf{O}(2) & \text{if } \beta = 1 \end{cases}$$
- $(\mathcal{D}_2, \mathbf{g}^2) \,:\, d\,\mathfrak{L}\mathsf{Iso}_{\mathbf{1}} = d\,\mathsf{Iso}_{\mathbf{1}} = \{\operatorname{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2) \,:\, \sigma_1, \sigma_2 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

B.10 Algebra $\mathfrak{g}_{3.7}$

- $\mathfrak{g}_{3.7} = \left\{ \frac{1}{2} \begin{bmatrix} ia_1 & ia_3 + a_2 \\ ia_3 - a_2 & -ia_1 \end{bmatrix} = a_1E_1 + a_2E_2 + a_3E_3 \,:\, a_1, a_2, a_3 \in \mathbb{R} \right\}$
$[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, $[E_1, E_2] = E_3$
- $\mathsf{Aut}(\mathfrak{g}_{3.7}) = \mathsf{SO}(3) = \{x \in \mathbb{R}^{3\times 3} \,:\, x^\top x = \mathbf{1},\, \det x = 1\}$
- $\mathsf{SU}(2) = \left\{x \in \mathbb{C}^{2\times 2} \,:\, x^\dagger x = \mathbf{1},\, \det x = 1\right\}$

Riemannian structure

- $\mathbf{g}_{\mathbf{1}} = r \operatorname{diag}(\beta_1, \beta_2, 1), \qquad \beta_1 \geq \beta_2 \geq 1,\, r > 0$
orthonormal frame: $\frac{1}{\sqrt{r}}(\frac{1}{\sqrt{\beta_1}}E_1, \frac{1}{\sqrt{\beta_2}}E_2, E_3)$
- $\nabla_A B = \frac{a_2b_3(\beta_1-\beta_2+1)-a_3b_2(\beta_1+\beta_2-1)}{2\beta_1}E_1 + \frac{a_1b_3(\beta_1-\beta_2-1)+a_3b_1(\beta_1+\beta_2-1)}{2\beta_2}E_2$
$+ \frac{1}{2}\big(a_1b_2(-\beta_1+\beta_2+1) + a_2b_1(-\beta_1+\beta_2-1)\big)E_3$

Normalized invariants

- If $\beta_1 > 4$ and $1 \leq \beta_2 < \beta_1 - 2\sqrt{\beta_1} + 1$:
$\rho = -1$ (with $r = \frac{\beta_1^2 - 2\beta_1(\beta_2+1)+(\beta_2-1)^2}{2\beta_1\beta_2}$),
$\lambda_1 = \frac{(\beta_1-\beta_2+1)(\beta_1+\beta_2-1)}{\beta_1^2-2\beta_1(\beta_2+1)+(\beta_2-1)^2}, \quad \lambda_2 = -\frac{2(\beta_1+\beta_2-1)}{\beta_1^2-2\beta_1(\beta_2+1)+(\beta_2-1)^2} - 1, \quad \nabla R \not\equiv 0$
- If $\beta_2 \geq 1$ and $\beta_1 = \beta_2 + 2\sqrt{\beta_2} + 1$:
$\rho = 0, \quad \lambda_1 = \frac{2}{r\sqrt{\beta_2}}, \quad \lambda_2 = -\frac{2}{r\left(\sqrt{\beta_2}+1\right)\sqrt{\beta_2}}, \quad \nabla R \not\equiv 0$
- If $\beta_2 \geq 1$ and $\beta_1 - 2\sqrt{\beta_1} + 1 < \beta_2 \leq \beta_1$:
$\rho = 1$ (with $r = \frac{2\beta_1(\beta_2+1)-\beta_1^2-(\beta_2-1)^2}{2\beta_1\beta_2}$)

$$\begin{cases} \begin{cases} \lambda_1 = -\dfrac{(\beta_1 - \beta_2 + 1)(\beta_1 + \beta_2 - 1)}{\beta_1^2 - 2\beta_1(\beta_2+1) + (\beta_2-1)^2} \\ \lambda_2 = \dfrac{(\beta_1-1)^2 - \beta_2^2}{\beta_1^2 - 2\beta_1(\beta_2+1) + (\beta_2-1)^2} \end{cases} & \text{if } \beta_1 < 1 + \beta_2 \\ \begin{cases} \lambda_1 = -\dfrac{(\beta_1 - \beta_2 + 1)(\beta_1 + \beta_2 - 1)}{\beta_1^2 - 2\beta_1(\beta_2+1) + (\beta_2-1)^2} \\ \lambda_2 = \dfrac{(\beta_1-\beta_2)^2 - 1}{\beta_1^2 - 2\beta_1(\beta_2+1) + (\beta_2-1)^2} \end{cases} & \text{if } \beta_1 \geq 1 + \beta_2 \end{cases}$$

$$\nabla R \equiv 0 \iff \beta_1 = \beta_2 = 1$$

Isometries:

- If $\beta_1 > \beta_2 > 1$:
  $d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} = \{\mathrm{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2) \,:\, \sigma_1, \sigma_2 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
- If $\beta_1 = \beta_2 > 1$:
  $$d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} = \left\{ \begin{bmatrix} \sigma\cos\theta & -\sin\theta & 0 \\ \sigma\sin\theta & \cos\theta & 0 \\ 0 & 0 & \sigma \end{bmatrix} \,:\, \theta \in \mathbb{R}, \sigma = \pm 1 \right\} \cong \mathsf{O}(2)$$
- If $\beta_1 > \beta_2 = 1$:
  $$d\,\mathfrak{L}\mathsf{Iso} = d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} = \left\{ \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \cos\theta & -\sigma\sin\theta \\ 0 & \sin\theta & \sigma\cos\theta \end{bmatrix} \,:\, \theta \in \mathbb{R}, \sigma = \pm 1 \right\} \cong \mathsf{O}(2)$$
- If $\beta_1 = \beta_2 = 1$:
  $d\,\mathsf{Iso}_{\mathbf{1}} = \mathsf{Sym} \cong \mathsf{O}(3)$, $\mathsf{Iso}_{\mathbf{1}}$ is generated by $\mathfrak{L}\mathsf{Iso}$ and the isometry $\iota : x \mapsto x^{-1}$.

Riemannian exponential map (when $\beta_1 = \beta_2 = 1$ and $r = 1$)

- $\mathrm{Exp}(tA) = \exp(tA)$ (i.e., the Riemannian exponential map is simply the Lie group exponential map).

Sub-Riemannian structure

- $\mathcal{D}(\mathbf{1}) = \langle E_2, E_3 \rangle, \qquad \mathbf{g_1} = \mathrm{diag}(\beta, 1), \quad 0 < \beta \leq 1$
  orthonormal frame: $(\frac{1}{\sqrt{\beta}} E_2, E_3)$

Associated Riemannian structure

- $\tilde{\mathbf{g}}_{\mathbf{1}} = \mathrm{diag}(\beta, \beta, 1)$, which is $\mathfrak{L}$-isometric to $\overline{\mathbf{g}}_{\mathbf{1}} = \beta\,\mathrm{diag}(\frac{1}{\beta}, 1, 1)$.
  We note however that when $\beta = 1$, then the rescaled sub-Riemannian structure $(\mathsf{SU}(2), \mathcal{D}, r\mathbf{g})$ has associated Riemannian structure $\overline{\mathbf{g}} = r\,\mathrm{diag}(r, 1, 1)$, see Remark 5.

Isometries

- $d\,\mathfrak{L}\mathsf{Iso}_{\mathbf{1}} = d\,\mathsf{Iso}_{\mathbf{1}} =$
$$\begin{cases} \{\mathrm{diag}(\sigma_1, \sigma_2, \sigma_1\sigma_2) \,:\, \sigma_1, \sigma_2 = \pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } 0 < \beta < 1 \\ \left\{ \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma\cos\theta & -\sin\theta \\ 0 & \sigma\sin\theta & \cos\theta \end{bmatrix} \,:\, \theta \in \mathbb{R},\ \sigma = \pm 1 \right\} \cong \mathsf{O}(2) & \text{if } \beta = 1 \end{cases}$$

C Classification of symmetric Riemannian structures

The only simply connected three-dimensional Lie groups which admit symmetric Riemannian structures are $\mathbb{R}^3$, $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$, $\mathsf{G}_{3.3}$, $\widetilde{\mathsf{SE}}(2)$, $\mathsf{G}_{3.5}^{\alpha}$, and $\mathsf{SU}(2)$. On each respective group, all the symmetric structures are $\mathfrak{L}$-isometric, up to rescaling. Moreover, we have that the symmetric structure on $\widetilde{\mathsf{SE}}(2)$ is isometric to the one on $\mathbb{R}^3$ (see Appendix B.7); also, the symmetric structure on $\mathsf{G}_{3.5}^{\alpha}$, for each $\alpha > 0$, is isometric to the symmetric structure on $\mathsf{G}_{3.3}$ (see Appendix B.8). Consequently, we have the following result.

Proposition 7. *Any left-invariant symmetric Riemannian structure on a simply connected three-dimensional Lie group is isometric, up to rescaling, to exactly one of the following four Riemannian structures:*

1. *The Euclidean space $\mathbb{E}^3$.*
2. *The structure on $\mathsf{Aff}(\mathbb{R})_0 \times \mathbb{R}$ admitting orthonormal frame (E_1, E_2, E_3).*
3. *The structure on $\mathsf{G}_{3.3}$ admitting orthonormal frame (E_1, E_2, E_3).*
4. *The structure on $\mathsf{SU}(2)$ admitting orthonormal frame (E_1, E_2, E_3).*

The respective Riemannian exponential maps for the above symmetric Riemannian structures are given in Appendix B. (The Euclidean case is omitted as it is trivial.)

D Associated Hamilton–Poisson systems

To each left-invariant sub-Riemannian (resp. Riemannian) structure one associates a quadratic Hamilton–Poisson system on the dual of the corresponding Lie algebra; the normal geodesics of the structure are related to the integral curves of the associated Hamiltonian vector field (see, e.g., [5], [8], [15]). Two such Hamilton–Poisson systems are considered linearly equivalent if their associated Hamiltonian vector fields are compatible with a linear isomorphism. We note that if two structures are $\mathfrak{L}$-isometric up to rescaling, then their associated Hamilton–Poisson systems are linearly equivalent (cf. [8]). The positive semidefinite quadratic Hamilton–Poisson systems in three dimensions are classified in [7], up to linear equivalence. In Tables 4 and 5 we identify the normal form (as given in [7]) of the Hamilton–Poisson system associated to each Riemannian (resp. sub-Riemannian) structure. We note that, quite remarkably, among the unimodular Lie groups the Hamilton–Poisson system associated to any Riemannian (or sub-Riemannian) structure is linearly equivalent to one of three systems, namely $\mathsf{Np}(7)$, $\mathsf{P}(8)$, or the trivial system (whose Hamiltonian vector field is constant zero).

Algebra	Riemannian structure	Hamiltonian normal form ([7])
$3\mathfrak{g}_1$	$\mathbb{E}^3$	trivial
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	(2), $0 < \beta < 1$	$\mathsf{P}(2)$
	(2), $\beta = 0$	$\mathsf{P}(1)$
$\mathfrak{g}_{3.1}$	(3)	$\mathsf{P}(8)$
$\mathfrak{g}_{3.2}$	(4)	$\mathsf{Np}(2)_{\delta=\frac{1}{\beta}}$
$\mathfrak{g}_{3.3}$	(5)	$\mathsf{P}(5)$
$\mathfrak{g}_{3.4}^0$	(6)	$\mathsf{Np}(7)$
$\mathfrak{g}_{3.4}^\alpha$	(7)	$\mathsf{Np}(6)$
$\mathfrak{g}_{3.5}^0$	(8), $0 < \beta < 1$	$\mathsf{Np}(7)$
	(8), $\beta = 1$	$\mathsf{P}(8)$
$\mathfrak{g}_{3.5}^\alpha$	(9)	$\mathsf{Np}(8)$
$\mathfrak{g}_{3.6}$	(10), $\beta_1 > \beta_2 > 0$	$\mathsf{Np}(7)$
	(10), $\beta_1 = \beta_2 > 0$	$\mathsf{P}(8)$
$\mathfrak{g}_{3.7}$	(11), $\beta_1 > \beta_2 > 1$	$\mathsf{Np}(7)$
	(11), $\beta_1 = \beta_2 > 1$ or $\beta_1 > \beta_2 = 1$	$\mathsf{P}(8)$
	(11), $\beta_1 = \beta_2 = 1$	trivial

Table 4: Normal forms for the Hamilton–Poisson systems associated to Riemannian structures

Algebra	Sub-Riemannian structure	Hamiltonian normal form ([7])
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$	(14)	$\mathsf{P}(2)$
$\mathfrak{g}_{3.1}$	(15)	$\mathsf{P}(8)$
$\mathfrak{g}_{3.2}$	(16)	$\mathsf{Np}(2)_{\delta=0}$
$\mathfrak{g}_{3.4}^0$	(17)	$\mathsf{Np}(7)$
$\mathfrak{g}_{3.4}^\alpha$	(18)	$\mathsf{Np}(6)_{\beta=0}$
$\mathfrak{g}_{3.5}^0$	(19)	$\mathsf{Np}(7)$
$\mathfrak{g}_{3.5}^\alpha$	(20)	$\mathsf{Np}(8)_{\beta=0}$
$\mathfrak{g}_{3.6}$	(21), $0 < \beta < 1$	$\mathsf{Np}(7)$
	(21), $\beta = 1$	$\mathsf{P}(8)$
	(22)	$\mathsf{Np}(7)$
$\mathfrak{g}_{3.7}$	(23), $0 < \beta < 1$	$\mathsf{Np}(7)$
	(23), $\beta = 1$	$\mathsf{P}(8)$

Table 5: Normal forms for the Hamilton–Poisson systems associated to sub-Riemannian structures

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