Wei-Feng Xuan

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Mathematica Bohemica, Vol. 142 (2017), No. 4, 445–448

Persistent URL: `http://dml.cz/dmlcz/146980`

142 (2017) MATHEMATICA BOHEMICA No. 4, 445–448

A NOTE ON STAR LINDELÖF, FIRST COUNTABLE AND NORMAL SPACES

Wei-Feng Xuan, Nanjing

Received January 30, 2017. First published May 2, 2017.
Communicated by Pavel Pyrih

Abstract. A topological space X is said to be star Lindelöf if for any open cover $\mathcal{U}$ of X there is a Lindelöf subspace $A \subset X$ such that $\mathrm{St}(A, \mathcal{U}) = X$. The "extent" $e(X)$ of X is the supremum of the cardinalities of closed discrete subsets of X. We prove that under $V = L$ every star Lindelöf, first countable and normal space must have countable extent. We also obtain an example under $\mathrm{MA} + \neg\mathrm{CH}$, which shows that a star Lindelöf, first countable and normal space may not have countable extent.

Keywords: star Lindelöf space; first countable space; normal space; countable extent

MSC 2010: 54D20, 54E35

1. Introduction

Recently, the authors in [9], Theorem 2.9 proved that if X is a first countable star Lindelöf normal space and has a G_δ-diagonal, then the cardinality of X does not exceed $\mathfrak{c}$. This result suggests the following question.

Question 1.1 (see Question 2.10 of [9]). Let X be a first countable star Lindelöf normal space. Does X have to have countable extent?

It is well known that the cardinality of a space which has countable extent and a G_δ-diagonal is at most $\mathfrak{c}$ (see [4]). Therefore, a positive answer to Question 1.1 would imply a trivial proof of the above result.

In this paper, we prove that under $V = L$ every star Lindelöf, first countable and normal space must have countable extent. We also obtain an example under

The author is supported by NSFC project 11626131.

DOI: 10.21136/MB.2017.0012-17

MA + ¬CH, which shows that a star Lindelöf, first countable and normal space may not have countable extent. This gives a complete answer to Question 1.1.

2. Notation and terminology

All the spaces are assumed to be Hausdorff if not stated otherwise. We write ω for the first infinite cardinal and $\mathfrak{c}$ for the cardinality of the continuum.

If A is a subset of X and $\mathcal{U}$ is a family of subsets of X, then $\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U}\colon U \cap A \neq \emptyset\}$. If $A = \{x\}$ for some $x \in X$, then we write, for simplicity, $\mathrm{St}(x,\mathcal{U})$ instead of $\mathrm{St}(\{x\},\mathcal{U})$.

Definition 2.1 ([8]). Let $\mathcal{P}$ be a topological property. A topological space X is said to be star $\mathcal{P}$, if for any open cover $\mathcal{U}$ of X there is a subset $A \subset X$ with property $\mathcal{P}$ such that $\mathrm{St}(A,\mathcal{U}) = X$. The set A will be called a star kernel of the cover $\mathcal{U}$.

Therefore, a topological space X is said to be star Lindelöf if for any open cover $\mathcal{U}$ of X there is a Lindelöf subspace $A \subset X$ such that $\mathrm{St}(A,\mathcal{U}) = X$.

Definition 2.2 ([5]). The extent $e(X)$ of X is the supremum of the cardinalities of closed discrete subsets of X.

Definition 2.3 ([5]). The character of X is defined as:

$$\chi(X) = \sup\{\chi(p,X)\colon\ p \in X\} + \omega,$$

where $\chi(p,X) = \min\{|\mathcal{B}|\colon\ \mathcal{B}$ is a local base for $p\}$.

Definition 2.4 ([6]). An uncountable subset X of real line $\mathbb{R}$ is called a Q-set if every subset of X is a G_δ-set in X.

It should be pointed out that Q-set exists under certain set-theoretic assumption such as Martin Axiom and the negation of the Continuum Hypothesis (see [6], Theorem 4.2).

Definition 2.5 ([1]). A topological space X is collectionwise Hausdorff if any closed discrete set $S \subset X$ has a disjoint open expansion.

All notation and terminology not explained here is given in [2].

3. Results

We begin with an easy lemma, which will be useful later.

Lemma 3.1. *If S is a closed discrete set in a normal space X and $\mathcal{U} = \{U(x)\colon x \in S\}$ is a disjoint open expansion of S, then there is a discrete open expansion $\mathcal{V} = \{V(x)\colon x \in S\}$ of S with $\overline{\bigcup \mathcal{V}} \subset \bigcup \mathcal{U}$.*

P r o o f. By normality there exists an open set W in X such that $S \subset W \subset \overline{W} \subset \bigcup \mathcal{U}$. For all $x \in S$ let $V(x) = U(x) \cap W$. It is easily verified that $\mathcal{V} = \{V(x)\colon x \in S\}$ is a discrete open collection of cardinality $|S|$ and satisfies $\overline{\bigcup \mathcal{V}} \subset \bigcup \mathcal{U}$. $\square$

Theorem 3.2. *Assuming $V = L$, if X is a star Lindelöf and normal space with $\chi(X) \leqslant \mathfrak{c}$, then X has countable extent.*

P r o o f. Assume the contrary. It follows that there exists an uncountable closed and discrete subset S of X. Fleissner in [3] proved that under $V = L$, all normal spaces with character $\leqslant \mathfrak{c}$ are collectionwise Hausdorff, so S has a disjoint open expansion $\mathcal{U} = \{U(x)\colon x \in S\}$. We apply Lemma 3.1 to conclude that there is a discrete open expansion $\mathcal{V} = \{V(x)\colon x \in S\}$ of S satisfying $\overline{\bigcup \mathcal{V}} \subset \bigcup \mathcal{U}$.

Let $\mathcal{W} = \mathcal{V} \cup \{X \setminus S\}$. Obviously, $\mathcal{W}$ is an open cover of X. Since X is star Lindelöf, it follows that there exists a Lindelöf subset Y of X such that $\mathrm{St}(Y, \mathcal{W}) = X$. For each $x \in S$, clearly $Y \cap V(x) \neq \emptyset$; pick $\xi(x) \in Y \cap V(x)$ and let $A = \{\xi(x)\colon x \in S\}$. Since Y is Lindelöf, there exists a limit point ξ for A. Therefore

$$\xi \in \overline{A} \subset \overline{\bigcup \mathcal{V}} \subset \bigcup \mathcal{U}.$$

It follows that there is $U(x)$ for some $x \in S$ which contains ξ. This implies that infinitely many points of A are in $U(x)$, which is a contradiction. This completes the proof. $\square$

Clearly, every first countable space X satisfies $\chi(X) = \omega < \mathfrak{c}$. So the following result is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Assuming $V = L$, if X is a star Lindelöf, first countable and normal space, then X has countable extent.*

We present an example below, in which the Q-set is used.

E x a m p l e 3.4 ([7], Example F). Assume $\mathrm{MA} + \neg\mathrm{CH}$. There exists a star Lindelöf, first countable and normal space, which has an uncountable closed and discrete subset.

P r o o f. Take an uncountable Q-set A in $\mathbb{R}$. Let L be the closed upper half-plane, $L_1 = \{(x,0)\colon\ x \in R\}$. Let $X = \{(x,0)\colon\ x \in A\} \cup (L \setminus L_1)$. Define a basis for a topology on X as follows. For every $p \in X$ and $\varepsilon > 0$, let $B(p,\varepsilon)$ be the set of all points of X inside the circle of radius ε and the center at p, and define

$$U(p,\varepsilon) = B((x,\varepsilon),\varepsilon) \cup \{p\} \quad \text{for } p \in A$$

and

$$U(p,\varepsilon) = B(p,\varepsilon) \quad \text{for } p \in X \setminus A.$$

By a quick observation, we conclude that X is first countable. Moreover, $\{(x,y)\colon x,y \in \mathbb{Q}^+\} \subset X$ (where $\mathbb{Q}$ is the set of all rational numbers) witnesses that X is separable, and it follows that X is star countable, and hence star Lindelöf. It has been proved in [7], Example F, that X is normal. Finally, it is not difficult to see that $\{\{(x,0)\}\colon\ x \in A\}$ is an uncountable closed and discrete subset of X. This completes the proof. □

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Author's address: *Wei-Feng Xuan*, School of Science, Nanjing Audit University, 86 YuShan Road, Nanjing, China, 210093, e-mail: `wfxuan@nau.edu.cn`.