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MAPS ON UPPER TRIANGULAR MATRICES PRESERVING ZERO PRODUCTS

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Abstract. Consider $\mathcal{T}_n(F)$—the ring of all $n \times n$ upper triangular matrices defined over some field F. A map φ is called a zero product preserver on $\mathcal{T}_n(F)$ in both directions if for all $x, y \in \mathcal{T}_n(F)$ the condition $xy = 0$ is satisfied if and only if $\varphi(x)\varphi(y) = 0$. In the present paper such maps are investigated. The full description of bijective zero product preservers is given. Namely, on the set of the matrices that are invertible, the map φ may act in any bijective way, whereas for the zero divisors and zero matrix one can write φ as a composition of three types of maps. The first of them is a conjugacy, the second one is an automorphism induced by some field automorphism, and the third one transforms every matrix x into a matrix x' such that $\{y \in \mathcal{T}_n(F) \colon xy = 0\} = \{y \in \mathcal{T}_n(F) \colon x'y = 0\}$, $\{y \in \mathcal{T}_n(F) \colon yx = 0\} = \{y \in \mathcal{T}_n(F) \colon yx' = 0\}$.

Keywords: zero product preserver; upper triangular matrix

MSC 2010: 15A99, 16U99

1. Stating the results

Let $\mathcal{A}$ be an algebra or simply a ring. If $\varphi\colon \mathcal{A} \to \mathcal{A}$ satisfies condition

$$xy = 0 \quad \Rightarrow \quad \varphi(x)\varphi(y) = 0 \qquad \forall x, y \in \mathcal{A}, \tag{1.1}$$

then we say that φ preserves zero products.

If $\varphi\colon \mathcal{A} \to \mathcal{A}$ fulfills the condition

$$xy = 0 \quad \Leftrightarrow \quad \varphi(x)\varphi(y) = 0 \qquad \forall x, y \in \mathcal{A},$$

then we say that φ preserves zero products in both directions.

Problem of describing maps satisfying (1.1) was considered first by Wong in [15] for the case when $\mathcal{A}$ is a finite-dimensional simple associative algebra and φ is semilinear.

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Similar but a bit different result was obtained in [3], where the maps preserving the nilpotent matrices were described. In this paper the considered algebra consists of matrices with trace 0. Such matrices were also investigated in [12], where the linear maps preserving square-zero matrices were studied. This was extended in [6] to the matrices over commutative rings. Quite a lot of authors studied zero product preservers on Banach algebras, see [1], [5], but also on some Hilbert spaces, see [9], or topological spaces, see [7].

The notion of a zero-divisor graph was introduced by Beck in [2]. If R is a ring, then we identify it with a simple graph such that its vertices are elements of R and two distinct $x, y \in R$ are adjacent if and only if $xy = 0$.

The properties of the zero-divisor graphs of the full matrix rings were studied in [4] and the zero-divisor graphs of $\mathcal{T}_n(F)$-upper triangular matrices in [10], [11], [8].

One can see that if we know the form of all automorphisms of a zero-divisor graph of some ring R, then we know the form all the bijective maps that preserve zero products on R. Therefore such automorphisms are of our interest. In [16] we can find how the automorphisms of the zero divisor graph of $\mathcal{T}_n(F)$ act on rank one triangular matrices. Using the main theorem of the latter paper and some other arguments, Wang in [14] described all the automorphisms of the zero divisor graphs of $\mathcal{T}_n(F)$. The results from [16] and [14] are valid when F is a finite field. In this article we will show how we can easily extend them to the case when F is infinite and, thanks to it, obtain the description of all the maps preserving zero products on $\mathcal{T}_n(F)$.

Before presenting the result, let us mention that by $\mathcal{ZD}(R)$ we will denote the set of zero divisors of R.

Moreover, below we introduce the maps that will appear in our theorem.

- ▷ For any invertible $t \in \mathcal{T}_n(F)$ the map $\varphi\colon\ \mathcal{T}_n(F) \to \mathcal{T}_n(F)$ such that $\varphi(x) = t^{-1}xt$ will be denoted by $\mathcal{I}nn_t$. This is simply an inner automorphism of the ring $\mathcal{T}_n(F)$.
- ▷ If σ is an automorphism of a field F, then we will write $\overline{\sigma}$ for the map on $\mathcal{T}_n(F)$ such that $(\overline{\sigma}(x))_{ij} = \sigma(x_{ij})$. We will call $\overline{\sigma}$ a field automorphism.
- ▷ Let $x \in \mathcal{T}_n(F)$. Let us introduce the following sets:

$$\overrightarrow{N}_{\mathcal{T}_n(F)}(x) = \{y \in \mathcal{T}_n(F)\colon\ xy = 0\}$$
$$\overleftarrow{N}_{\mathcal{T}_n(F)}(x) = \{y \in \mathcal{T}_n(F)\colon\ yx = 0\}.$$

If for some x_1, x_2 we have $\overrightarrow{N}_{\mathcal{T}_n(F)}(x_1) = \overrightarrow{N}_{\mathcal{T}_n(F)}(x_2)$ and $\overleftarrow{N}_{\mathcal{T}_n(F)}(x_1) = \overleftarrow{N}_{\mathcal{T}_n(F)}(x_2)$, then we will say that x_1, x_2 are twin matrices (in [14] analogous vertices of the zero divisor graphs are called twin points). If a map φ on $\mathcal{T}_n(F)$ acts in such a way that $\varphi(x) = y$ implies that x, y are twin matrices, then we call it a regular automorphism.

Now we can present our main result.

Theorem 1.1. *Let F be any field and $n \in \mathbb{N}$. The bijective map $\varphi\colon \mathcal{T}_n(F) \to \mathcal{T}_n(F)$ preserves zero products in both directions if and only if the two conditions below are fulfilled.*

(1) *There exist an upper triangular invertible matrix $t \in \mathcal{T}_n(F)$, an automorphism σ of F, and a regular automorphism ϱ of $\mathcal{T}_n(F)$ such that for every $x \in \mathcal{ZD}(\mathcal{T}_n(F)) \cup \{0\}$ we have*

$$\varphi(x) = \mathcal{I}nn_t \cdot \overline{\sigma} \cdot \varrho(x). \tag{1.2}$$

(2) *The map φ cut to $\mathcal{T}_n(F) \setminus (\mathcal{ZD}(\mathcal{T}_n(F)) \cup \{0\})$ is a bijection on this set.*

2. Proof

Let us start with a few more remarks about notation. The set F^* is meant to be $F \setminus \{0\}$. By x^{T} we understand the transpose of a matrix x. We will write e_{ij} for the matrix whose (i,j) entry is equal to 1 and all the other entries are zeroes. The symbol $0_{n\times m}$ will be used for $n \times m$ matrix whose all entries are zeroes. We will denote by $\mathcal{M}_{n\times m}(F)$ the set of all $n \times m$ matrices over F. We will also write x^a for the element $a^{-1}xa$.

As we have already mentioned, the proof of our theorem will be based on the results of Wang in [14] and Wong, Ma and Zhou in [16]. First we cite the theorem from the former.

Proposition 2.1 ([14], Theorem 1.1)**.** *A mapping φ on $V(\mathcal{T})$ is an automorphism of $\mathcal{T}$ if and only if it can be uniquely decomposed into the product of an inner automorphism, a field automorphism and a regular automorphism.*

Let us now explain the notation and assumptions of paper [14].

$\mathcal{T}$ is the set of all zero divisors of $\mathcal{T}_n(F)$, i.e. it consists of all noninvertible triagular matrices. The graph $\Gamma(\mathcal{T}_n(F))$ is defined as follows: there is an edge between x and y if and only if $xy = 0$ and no loops are deleted. (Note that in the definition of the zero divisor graph it is assumed that it is simple. However, in our case it is even more convenient to assume that $\Gamma(\mathcal{T}_n(F))$ contains loops.) The symbol $V(\mathcal{T})$ stands for the set of vertices of $\Gamma(\mathcal{T}_n(F))$. The field F is finite. The proof is based on the result from [16] in which this assumption appears and on some results of the author where this assumption is not necessary. Hence, we can make use of Proposition 2.1 on the condition that the result from [16] will hold for any field. Let us cite it.

Proposition 2.2 ([16], Theorem 3.3). *When $n \geqslant 3$, σ is an automorphism of $\mathcal{R}_n(q)$ if and only if $\sigma = \sigma_P \cdot \sigma_\pi$, where σ_P is an inner automorphism of $\mathcal{R}_n(q)$ and σ_π is a field automorphism of $\mathcal{R}_n(q)$.*

When $n = 2$, σ is an automorphism of $\mathcal{R}_n(q)$ if and only if $\sigma = \sigma_U \cdot \sigma_\omega$, where ω is a permutation on F_q fixing 0, U is a 2×2 unit upper triangular matrix (all diagonal elements are 1) over F_q.

In [16] $\mathcal{R}_n(q)$ is the ring of all $n \times n$ upper triangular rank one matrices over the field of q elements. Clearly, we wish this theorem would hold for any field. Let us look closer at the proof. The sufficiency follows immediately. The proof of the necessity is given in 9 steps (claims). When proving claims 3–9, the authors make use of claims 1, 2 and lemmas in which the finiteness of the field is not demanded. In the proofs of claims 1, 2 the assumption that the field is finite is used. Therefore it suffices to prove that these two claims are true when the field is infinite. Let us then cite and discuss these claims.

Claim 1. Each $\Sigma(s,t)$ is stabilized by σ for $1 \leqslant s \leqslant t \leqslant n$.

Claim 2. There is a unit upper triangular matrix U such that $\sigma_U \cdot \sigma$ fixes each $[E_{st}]$.

The symbol $\Sigma(s,t)$ denotes here the set of all rank one upper triangular matrices a satisfying condition

$$a_{st} \neq 0, \quad \forall\, l,\ 1 \leqslant l < t\colon\ a_{sl} = 0, \quad \forall\, k,\ s < k \leqslant n\colon\ a_{kl} = 0. \tag{2.1}$$

By $[E_{st}]$ we understand the subspace of $\mathcal{T}_n(F)$ spanned by e_{st}.

Thus, it suffices to prove the following.

Proposition 2.3. *Let F be any field and $n \in \mathbb{N}$, $n \geqslant 2$. If $\varphi\colon\ \mathcal{T}_n(F) \to \mathcal{T}_n(F)$ is bijective and it preserves zero products, then there exists an invertible $t \in \mathcal{T}_n(F)$ such that*

1. *for every $1 \leqslant i \leqslant j \leqslant n$ and $\alpha \in F^*$ there exists $\alpha' \in F^*$ such that $(\varphi(\alpha e_{ij}))^t = \alpha' e_{ij}$,*
2. *if x is a rank one matrix satisfying (2.1), then $(\varphi(x))^t$ is also a rank one matrix satisfying (2.1).*

Before proving the above proposition we present some lemmas.

Lemma 2.1. *Let F be a field, $n \in \mathbb{N}$, $n \geqslant 2$. The noninvertible matrix $x \in \mathcal{T}_n(F)$ satisfies the conditions $\overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x) = \overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x^2)$ and $\overleftarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x) = \overleftarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x^2)$ if and only if x is a multiple of some idempotent.*

P r o o f. The inclusions $\overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x) \subseteq \overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x^2)$ and $\overleftarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x) \subseteq \overleftarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x^2)$ are obvious. Suppose that $\overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x) \subsetneq \overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x^2)$. Then there exists $y \in \mathcal{T}_n(F)$ such that for every $\alpha \in F^*$ we have $x^2y = 0$ and $\alpha xy \neq 0$. This yields $(x^2 - \alpha x)y \neq 0$. Hence $y \neq 0$ and $x^2 - \alpha x \neq 0$. From the latter we get $x^2 \neq \alpha x$, so the equality can hold only in the case when $x^2 = \alpha x$. The discussion for $\overleftarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x)$ is exactly the same. This completes our proof. □

From what was proved above we get:

Corollary 2.1. *Suppose that F is a field, $n \in \mathbb{N}$, $n \geqslant 2$ and that the map $\varphi\colon \mathcal{T}_n(F) \to \mathcal{T}_n(F)$ is a bijection that preserves zero products. If x is a multiple of a noninvertible idempotent, then $\varphi(x)$ is also a multiple of a noninvertible idempotent.*

We will focus on idempotents for a moment.

Let us note that we have:

Lemma 2.2 ([13], Lemma 2.3). *Let F be any field. If $x \in \mathcal{T}_\infty(F)$ is an idempotent, then there exists an invertible matrix $t \in \mathcal{T}_\infty(F)$ such that $t^{-1}xt$ is a diagonal matrix.*

From the above lemma we get immediately:

Corollary 2.1. *Let F be any field and $n \in \mathbb{N}$. If $x \in \mathcal{T}_n(F)$ is an idempotent, then there exists an invertible matrix $t \in \mathcal{T}_n(F)$ such that $t^{-1}xt$ is a diagonal matrix.*

Clearly, the same is satisfied for multiples of idempotents.

Now we can prove the following.

Lemma 2.2. *Let F be a field, $n \in \mathbb{N}$, $n \geqslant 2$. If $x \in \mathcal{T}_n(F)$ is a noninvertible multiple of some idempotent, then there exists another multiple $y \in \mathcal{T}_n(F)$ of some idempotent such that*

(1) $\operatorname{rank}(x) < \operatorname{rank}(y)$,

(2) $\overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(y) \subsetneq \overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x)$,

(3) $\overleftarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(y) \subsetneq \overleftarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x)$.

P r o o f. By Corollary 2.2, x can be diagonalized by some $t \in \mathcal{T}_n(F)$. Consider then x^t that is equal to $\alpha \sum\limits_{i\in I} e_{ii}$ for some $\alpha \in F^*$ and $\emptyset \subsetneq I \subsetneq \{1, 2, \ldots, n\}$. Define y^t as $\sum\limits_{i\in I} e_{ii} + e_{jj}$, where j is not in I (since $I \neq \{1, 2, \ldots, n\}$, such j exists). We can see that for x^t, y^t all the claims (1), (2), (3) hold, so y is the matrix from the claim of the lemma. □

Let us note that we also have:

Lemma 2.4 ([14], Lemma 2.4). *Let $A \in V(\mathcal{T})$ be a nonzero zero-divisor of $T(n,q)$. Then $r(A) \geqslant 2$ if and only if there is certain $B \in V(\mathcal{T})$ such that $\overrightarrow{N}_{\mathcal{T}}(A) \subsetneq \overrightarrow{N}_{\mathcal{T}}(B)$, where $r(A)$ refers to the rank of A.*

From the proof it follows that this lemma is also satisfied for matrices over infinite fields.

Now we prove a fact about (non)invertible matrices.

Lemma 2.5. *Assume F is a field and $n \in \mathbb{N}$. The matrix x is in $\mathcal{ZD}(\mathcal{T}_n(F)) \cup \{0\}$ if and only if it is noninvertible.*

Proof. Suppose first that x is invertible. If $xy = 0$, then we have $x^{-1}xy = y = 0$. Since $y = 0$, x cannot be a zero divisor. Analogously, it can be shown that $yx = 0$ implies $y = 0$.

Let now x be noninvertible. Then rank of x is equal to some k that is less than n. Let $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n)^{\mathrm{T}} \in \mathcal{M}_{n\times 1}(F)$. As $\operatorname{rank}(x) < n$, the equation $x\tilde{y} = 0_{n\times 1}$ has a nonzero solution $\hat{y}$. We define $y \in \mathcal{T}_n(F)$ as follows.

$$y_{ij} = \begin{cases} \hat{y}_i & \text{if } j = n,\\ 0 & \text{otherwise.} \end{cases}$$

Now we see that $xy = 0$, so x is a zero divisor. □

Using the above results we can prove Proposition 2.3.

Proof of Proposition 2.3. The proof will be given in seven steps.

Step 1. For any number $1 \leqslant i \leqslant n$ and $\alpha \in F^*$ there exist a natural number $1 \leqslant \pi(i,\alpha) \leqslant n$, an element $\beta(i,\alpha) \in F^*$ and an invertible matrix $t_{i,\alpha} \in \mathcal{T}_n(F)$ such that

$$\varphi(\alpha e_{ii}) = (\beta(i,\alpha)e_{\pi(i,\alpha)\pi(i,\alpha)})^{t_{i,\alpha}}.$$

From Corollary 2.1 we know that $\varphi(\alpha e_{ii}) = \Bigl(\beta \sum\limits_{k\in S(i)} e_{kk}\Bigr)^{t_{i,\alpha}}$ for some $\emptyset \subseteq S(i) \subseteq \{1,2,\ldots,n\}$, $\beta \in F$ and $t_{i,\alpha} \in \mathcal{T}_n(F)$. If $|S(i)| > 1$, then by Lemma 2.3 we would have $\overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(\varphi(\alpha e_{ii})) \subsetneq \overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(a) \neq \overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(0)$ for some multiple of some idempotent—a contradiction. Since $\overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(\alpha e_{ii}) \neq \mathcal{T}_n(F)$, we cannot also have $\varphi(\alpha e_{ii}) \neq 0$. Thus, the claim follows.

Step 2. For all $1 \leqslant i \leqslant n$ and $\alpha \in F^*$ we have $\pi(i,\alpha) = \pi(i,\alpha')$ and $t_{i,\alpha} = t_{i,\alpha'}$.

Let us fix i and consider for a moment $\varphi' := \mathcal{I}nn_{t_{i,\alpha}} \cdot \varphi$. Then

$$\overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(\varphi'(\alpha e_{ii})) = \bigl\{y \in \mathcal{T}_n(F)\colon\ j \geqslant \pi(i,\alpha),\ y_{\pi(i,\alpha)j} = 0\bigr\}.$$

Hence

$$\overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(\varphi'(\alpha' e_{ii})) = \left\{y \in \mathcal{T}_n(F)\colon\ j \geqslant \pi(i,\alpha),\ y_{\pi(i,\alpha)j} = 0\right\}.$$

However, from the latter it follows that $\varphi'(\alpha' e_{ii}) = \beta' e_{\pi(i,\alpha)\pi(i,\alpha)}$.

Since this moment we will write $\pi(i)$ instead of $\pi(i,\alpha)$ and t_i instead of $t_{i,\alpha}$.

Note that as $\overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(e_{ii}) \neq \overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(e_{jj})$ for $i \neq j$, π is a permutation of $\{1, 2, \ldots, n\}$.

Step 3. There exists an invertible $t \in \mathcal{T}_n(F)$ such that for all $1 \leqslant i \leqslant n$ and $\alpha \in F^*$ we have

$$\varphi(\alpha e_{ii}) = (\beta(i,\alpha) e_{\pi(i)\pi(i)})^t.$$

We will construct t inductively.

Let us put $t_{(1)} = t_{\pi^{-1}(1)}$ and $\varphi_{(1)} = \mathcal{I}nn_{t_{(1)}} \cdot \varphi$. We have

$$\varphi_{(1)}(\alpha e_{\pi^{-1}(1)\pi^{-1}(1)}) = \beta(\pi^{-1}(1),\alpha) e_{11} \qquad \forall \alpha \in F^*$$

and

$$\varphi_{(1)}(\alpha e_{jj}) = (\beta(j,\alpha) e_{\pi(j)\pi(j)})^{t_j^{(1)}} \qquad \forall \alpha \in F^*,\ j \neq \pi^{-1}(1)$$
$$\text{and some } t_j^{(1)} \in \mathcal{T}_n(F).$$

Next we put $t_{(2)} = t^{(1)}_{\pi^{-1}(2)}$ and $\varphi_{(2)} = \mathcal{I}nn_{t_{(2)}} \cdot \varphi_{(1)}$.

Notice that since $e_{11} \in \overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(\varphi_{(1)}(\alpha e_{jj}))$ for all $j \neq \pi^{-1}(1)$, the matrices $\varphi_{(1)}(\alpha e_{jj})$ satisfy the condition $(\varphi_{(1)}(\alpha e_{jj}))_{1k} = 0$ for $k \geqslant 1$. Thus $(t_{(2)})_{1k} = 0$ for all $k > 1$. Hence

$$\varphi_{(2)}(\alpha e_{\pi^{-1}(1)\pi^{-1}(1)}) = \beta(\pi^{-1}(1),\alpha) e_{11} \quad \forall \alpha \in F^*$$
$$\varphi_{(2)}(\alpha e_{\pi^{-1}(2)\pi^{-1}(2)}) = \beta(\pi^{-1}(2),\alpha) e_{22} \quad \forall \alpha \in F^*$$

and

$$\varphi_{(2)}(\alpha e_{jj}) = (\beta(\pi(j),\alpha) e_{\pi(j)\pi(j)})^{t_j^{(2)}} \qquad \forall \alpha \in F^*,\ j \neq \pi^{-1}(1),\ \pi^{-1}(2)$$
$$\text{and some } t_j^{(2)} \in \mathcal{T}_n(F).$$

Performing this way we find $t = t_{(n-1)} \cdot \ldots \cdot t_{(2)} \cdot t_{(1)}$ satisfying the desired condition.

Now we will define φ_I as $\mathcal{I}nn_t \cdot \varphi$ and consider φ_I. Obviously, for every $\alpha \in F^*$ and $1 \leqslant i \leqslant n$ we have $\varphi_I(\alpha e_{ii}) = \beta(i,\alpha) e_{\pi(i)\pi(i)}$.

Step 4. For all $1 \leqslant i < j \leqslant n$ and $\alpha \in F^*$ there exists $\beta(i,j,\alpha) \in F^*$ such that $\varphi_I(\alpha e_{ij}) = \beta(i,j,\alpha) e_{\pi(i)\pi(j)}$.

This follows from the facts that $e_{ij} \in \overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(e_{kk})$ for all $k \neq j$ and $e_{ij} \in \overleftarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(e_{kk})$ for $k \neq i$.

Step 5. For all $1 \leqslant i \leqslant n$ we have $\pi(i) = i$.

Note that for every $i \leqslant j$ and $\alpha \in F^*$ the matrix $\alpha e_{\pi(i)\pi(j)}$ is upper triangular. Hence, from Step 4 we conclude that $\pi(i) < \pi(j)$ for $i < j$. Clearly, there is only one permutation of $\{1, 2, \ldots, n\}$ satisfying $\pi(1) < \pi(2) < \ldots < \pi(n)$ – the identity.

In the above five steps we have proved the first claim of the proposition. In the next two we will prove the second claim.

Suppose that x is a rank one matrix satysfying (2.1), then $\varphi_I(x)$ is a rank one matrix satisfying (2.1).

From Lemma 2.5 we conclude that $x \in \mathcal{ZD}(\mathcal{T}_n(F))$. Thus, from Lemma 2.4 it follows that $\varphi_I(x)$ is a rank one matrix.

Since x satisfies (2.1), we have $e_{ii} \in \overleftarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x)$ for all $1 \leqslant i < s$ and $e_{ii} \in \overrightarrow{\mathcal{N}}_{\mathcal{T}_n(F)}(x)$ for all $t < i \leqslant n$. As $\varphi_I(x) = \beta(i,1)e_{ii}$, we get that $\varphi_I(x)$ satisfies (2.1).

Step 7. Suppose that x is a rank one matrix satysfying (2.1). Then $\varphi(x)$ is a rank one matrix satysfying (2.1).

Since $\varphi_I = \mathcal{I}nn_t \cdot \varphi$, it suffices to show that the sets of matrices satisfying (2.1) are invariant under conjugation.

Let $u \in \mathcal{T}_n(F)$ be invertible. Write x as $\left(\begin{array}{c|c} 0_{(t-1)\times(t-1)} & x' \\ \hline & x'' \end{array}\right)$, where $x' \in \mathcal{M}_{(t-1)\times(n-t+1)}(F)$, $x'' \in \mathcal{T}_{n-t+1}(F)$ and u as $\left(\begin{array}{c|c} u' & u'' \\ \hline & u''' \end{array}\right)$, where $u' \in \mathcal{T}_{t-1}(F)$, $u'' \in \mathcal{M}_{(t-1)\times(n-t+1)}(F)$, $u''' \in \mathcal{T}_{n-t+1}(F)$. Then

$$\left(\begin{array}{c|c} 0_{(t-1)\times(t-1)} & (u')^{-1}x' - (u')^{-1}u''(x'')^{u'''} \\ \hline & (x'')^{u'''} \end{array}\right). \tag{2.2}$$

Analogously, if we write x as $\left(\begin{array}{c|c} x_1' & x_1'' \\ \hline u_1' & u_1'' \end{array}\right)$, where $x_1' \in \mathcal{T}_t(F)$, $x_1'' \in \mathcal{M}_{t\times(n-t)}(F)$ and u as $\left(\begin{array}{c|c} u_1' & u_1'' \\ \hline & u_1''' \end{array}\right)$, where $u_1' \in \mathcal{T}_t(F)$, $u_1'' \in \mathcal{M}_{t\times(n-t)}(F)$, $u_1''' \in \mathcal{T}_{n-t}(F)$, then

$$x^u = \left(\begin{array}{c|c} (x_1')^{u_1'} & (u_1')^{-1}x_1'u_1'' \\ \hline & 0_{(n-t)\times(n-t)} \end{array}\right). \tag{2.3}$$

From (2.2) and (2.3) we get the claim. □

Now it is easy to prove Theorem 1.1.

P r o o f of Theorem 1.1. It suffices to prove the necessity.

From Propositions 2.1, 2.2 and 2.3 it follows that for all noninvertible $x \in \mathcal{T}_n(F)$ the matrix $\varphi(x)$ is given by formula (1.2). Moreover, as φ is a bijection and Lemma 2.5 is satisfied, we have $\varphi(\mathcal{T}_n(F) \setminus \mathcal{ZD}(\mathcal{T}_n(F))) = \mathcal{T}_n(F) \setminus \mathcal{ZD}(\mathcal{T}_n(F))$ and φ is a bijection on this set. Clearly, for all the invertible matrices x, condition (1.1) is satisfied, so we are done. $\square$

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