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POINTWISE FOURIER INVERSION OF DISTRIBUTIONS ON SPHERES

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Abstract. Given a distribution T on the sphere we define, in analogy to the work of Łojasiewicz, the value of T at a point ξ of the sphere and we show that if T has the value τ at ξ, then the Fourier-Laplace series of T at ξ is Abel-summable to τ.

Keywords: distribution; sphere; Fourier-Laplace series; Abel summability

MSC 2010: 42C10, 46F12

1. Introduction

Consider the periodic distribution T with period 2π defined by

$$T(\varphi) := \lim_{\varepsilon\to 0+} \int_{\varepsilon}^{2\pi-\varepsilon} \cot(\tfrac{1}{2}t)\varphi(t)\,\mathrm{d}t$$

for all test functions φ (T is the principal value of $\cot(\frac{1}{2}t)$). Its Fourier coefficients, given by $\mathcal{F}T(k) := T(\mathrm{e}^{-\mathrm{i}kt})/2\pi$, are equal to $-\mathrm{i}$ for $k > 0$, 0 for $k = 0$ and i for $k < 0$. Hence, the Fourier series of T,

$$\sum_{k\in\mathbb{Z}} \mathcal{F}T(k)\mathrm{e}^{\mathrm{i}kt},$$

does not converge at any $t \in [-\pi, \pi]$; generally, one only reads that it converges to T in the sense of distributions. In fact it is possible to reconstruct T from $\mathcal{F}T$ using pointwise convergence only (and no test functions); the Fourier series of T is

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Abel-summable to $\cot(\frac{1}{2}t)$ at every $t \neq 0$:

$$\begin{aligned}\lim_{r\to 1_-}\sum_{k\in\mathbb{Z}} r^{|k|}\mathcal{F}T(k)\mathrm{e}^{\mathrm{i}kt} &= \lim_{r\to 1_-}(-\mathrm{i})\sum_{k=1}^{\infty}(r\mathrm{e}^{\mathrm{i}t})^k + \mathrm{i}\sum_{k=1}^{\infty}(r\mathrm{e}^{-\mathrm{i}t})^k \\ &= \lim_{r\to 1_-}(-\mathrm{i})\frac{r\mathrm{e}^{\mathrm{i}t}}{1-r\mathrm{e}^{\mathrm{i}t}} + \mathrm{i}\frac{r\mathrm{e}^{-\mathrm{i}t}}{1-r\mathrm{e}^{-\mathrm{i}t}} \\ &= \lim_{r\to 1_-}\frac{2r\sin t}{1+r^2-2r\cos t} \\ &= \cot(\tfrac{1}{2}t).\end{aligned}$$

This result is general: Walter [9], page 146, proved that if a periodic distribution T in one variable has the value τ at a point t (in the sense of Łojasiewicz), then the Fourier series of T at t is Cesàro- and hence Abel-summable to τ. A complete characterization for Fourier series and Fourier integrals on $\mathbb{R}$ was given in [8]. Note that the pointwise convergence or summability of expansions of distributions has been investigated with respect to other orthogonal systems, such as wavelets (see [5], [9], [10]).

If we want to generalize Walter's result to the spheres $\mathbb{S}^{n-1}$, $n \geqslant 2$, we must define the notion of value at a point for distributions on the sphere. In Section 2, after introducing useful notation we give a definition which is analogous to the one of Łojasiewicz, but which only uses the Laplace-Beltrami operator and its iterates instead of more general differential operators. We are then able in Section 3 to show that if T has the value τ at $\xi \in \mathbb{S}^{n-1}$, then the Fourier-Laplace series of T at ξ is Abel-summable to τ.

2. Preliminaries

We write $\mathbb{S}^{n-1}$ for the unit sphere in $\mathbb{R}^n$, $n \geqslant 2$, and σ_{n-1} for the measure on $\mathbb{S}^{n-1}$ induced by the Lebesgue measure on $\mathbb{R}^n$, so that

$$\omega_{n-1} := \int_{\mathbb{S}^{n-1}} \mathrm{d}\sigma_{n-1}(\eta) = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

We define a distance d on $\mathbb{S}^{n-1}$ by $d(\zeta,\eta) := 1-(\zeta|\eta)$, where $(\cdot|\cdot)$ is the euclidean scalar product in $\mathbb{R}^n$. A *spherical harmonic of degree l on $\mathbb{S}^{n-1}$*, $l \in \mathbb{N}_0$, is the restriction to $\mathbb{S}^{n-1}$ of a polynomial on $\mathbb{R}^n$ which is harmonic and homogeneous of degree l. We write $\mathcal{S}H_l(\mathbb{S}^{n-1})$ for the vector space of spherical harmonics of degree l; its dimension is

$$d_l^n := \dim_{\mathbb{C}} \mathcal{S}H_l(\mathbb{S}^{n-1}) = \frac{(2l+n-2)(n+l-3)!}{(n-2)!\,l!} = \frac{2l^{n-2}}{(n-2)!} + O(l^{n-3}).$$

Two spherical harmonics of different degrees are orthogonal with respect to the scalar product $(\cdot|\cdot)_2$ of $L^2(\mathbb{S}^{n-1}, \sigma_{n-1})$. If $f \in L^2(\mathbb{S}^{n-1})$ and $l \in \mathbb{N}_0$, we write $\Pi_l(f)$ for the orthogonal projection of f onto $\mathcal{SH}_l(\mathbb{S}^{n-1})$; the series

$$\sum_{l=0}^{\infty} \Pi_l(f),$$

called *Fourier-Laplace series of* f, converges to f in square mean. Given $\zeta \in \mathbb{S}^{n-1}$, the unique spherical harmonic $Z_l(\zeta, \cdot)$ of degree l such that

$$\Pi_l(f)(\zeta) = \int_{\mathbb{S}^{n-1}} Z_l(\zeta, \eta) f(\eta) \, \mathrm{d}\sigma_{n-1}(\eta)$$

is the *zonal with pole* ζ *of degree* l; it is the reproducing kernel of the Hilbert space $\mathcal{SH}_l(\mathbb{S}^{n-1})$. If f is a function defined on $\mathbb{S}^{n-1}$, we write $f\!\uparrow$ for the homogeneous function of degree 0 defined on $\mathbb{R}^n \setminus \{0\}$ by $(f\!\uparrow)(x) := f(x/\|x\|)$. Conversely, if g is a function defined on $\mathbb{R}^n \setminus \{0\}$, we denote by $g\!\downarrow$ its restriction to $\mathbb{S}^{n-1}$. We say that a function f on $\mathbb{S}^{n-1}$ is in $C^l(\mathbb{S}^{n-1})$ (where $l \in \mathbb{N}_0$) if $f\!\uparrow \in C^l(\mathbb{R}^n \setminus \{0\})$. When $f \in C^l(\mathbb{S}^{n-1})$, we can define for every multiindex $\alpha \in \mathbb{N}_0^n$ with $|\alpha| := \alpha_1 + \ldots + \alpha_n \leqslant l$,

$$D_S^\alpha f := (D^\alpha (f\!\uparrow))\!\downarrow = \Big(\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} (f\!\uparrow)\Big)\!\downarrow.$$

In this way we can obtain from the Laplacian Δ on $\mathbb{R}^n$ the *Laplace-Beltrami operator on* $\mathbb{S}^{n-1}$, Δ_S; it is self-adjoint with respect to $(\cdot|\cdot)_2$ and $\mathcal{SH}_l(\mathbb{S}^{n-1})$ is an eigenspace associated to the eigenvalue $-l(l+n-2)$ (for all this, see [1] and [4]).

We write $\mathcal{D}(\mathbb{S}^{n-1})$ for the space of functions $C^\infty(\mathbb{S}^{n-1})$ with the topology given by the family of seminorms

$$p_m(\varphi) := \sup_{|\alpha| \leqslant m} \sup_{\eta \in \mathbb{S}^{n-1}} |D_S^\alpha \varphi(\eta)|,$$

where $m \in \mathbb{N}_0$ (note that $\|\varphi\|_\infty = p_0(\varphi)$). If $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$, its Fourier-Laplace series converges to φ in this topology [2], page 265.

The dual $\mathcal{D}'(\mathbb{S}^{n-1})$ of $\mathcal{D}(\mathbb{S}^{n-1})$ is the space of *distributions* on $\mathbb{S}^{n-1}$. The *Fourier-Laplace series of a distribution* T on $\mathbb{S}^{n-1}$ is

$$\sum_{l=0}^{\infty} \Pi_l(T),$$

where for $\zeta \in \mathbb{S}^{n-1}$,

$$\Pi_l(T)(\zeta) := T[\eta \mapsto Z_l(\zeta, \eta)];$$

it converges to T in the sense of distributions [2], page 265.

To find how we can define the value of a distribution T on $\mathbb{S}^{n-1}$ at a point ζ in $\mathbb{S}^{n-1}$, we must consider the original definition on $\mathbb{R}^n$ of Łojasiewicz: a distribution S on $\mathbb{R}^n$ has the value τ at a point x_0 in $\mathbb{R}^n$ if and only if one of the following equivalent conditions is satisfied [6] on pages 15, 25, 21:

(a) $\lim_{\lambda\to 0+} S(x_0+\lambda x)=\tau$, distributionally, in a neighbourhood of x_0;

(b) $\lim_{\lambda\to 0+} S[x\mapsto \lambda^{-n}\varphi((x-x_0)/\lambda)]=\tau$ for all $\varphi\in\mathcal{D}(\mathbb{S}^{n-1})$ with $\int_{\mathbb{R}}\varphi(x)\,\mathrm{d}x=1$;

(c) there exist $\alpha\in\mathbb{N}_0^n$ and a continuous function F such that $S=D^\alpha F$ and $F(x)=\tau(x-x_0)^\alpha/\alpha!+o(\|x-x_0\|^{|\alpha|})$ in a neighbourhood of x_0.

Since there is no natural dilation on $\mathbb{S}^{n-1}$, conditions (a) and (b) are not adequate here. Condition (c) is more promising. In fact, it is heuristically quite clear: S is on a neighbourhood of x_0 the derivative D^α, up to a "negligible" term, of $\tau(x-x_0)^\alpha/\alpha!$ and $D^\alpha(\tau(x-x_0)^\alpha/\alpha!)=\tau$. However, in saying this we use the fact that the derivation of distributions on $\mathbb{R}^n$ is a generalization of the derivation of functions on $\mathbb{R}^n$: if T_f is the distribution defined by the function $f\in C^m(\mathbb{R}^n)$, then $D^\alpha T_f=T_{D^\alpha f}$ for every $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\leqslant m$, which is a consequence of the equality

$$\int_{\mathbb{R}^n}\varphi(x)D^\alpha\psi(x)\,\mathrm{d}x=(-1)^{|\alpha|}\int_{\mathbb{R}^n}D^\alpha\varphi(x)\psi(x)\,\mathrm{d}x,$$

true for all $\varphi,\psi\in\mathcal{D}(\mathbb{R}^n)$. Now, such an equality is in general false on $\mathbb{S}^{n-1}$ for the differential operators D_S^α: there is no constant c such that

$$\int_{\mathbb{S}^{n-1}}\varphi(\eta)D_S^{e_j}\psi(\eta)\,\mathrm{d}\sigma_{n-1}(\eta)=c\int_{\mathbb{S}^{n-1}}D_S^{e_j}\varphi(\eta)\psi(\eta)\,\mathrm{d}\sigma_{n-1}(\eta)$$

for all $\varphi,\psi\in\mathcal{D}(\mathbb{S}^{n-1})$, where e_j is the multiindex given by $(e_j)_l=\delta_{jl}$ (take $\varphi=1$ and $\psi(\zeta)=\zeta_j$). Instead of general D_S^α we therefore use the Laplace-Beltrami operator and its iterates, because these are self-adjoint.

There is still a point we cannot transpose without modification on $\mathbb{S}^{n-1}$: in $\mathbb{R}^n$ we have $D^\alpha(\tau(x-x_0)^\alpha/\alpha!)=\tau$ everywhere. On the contrary, there is no function $f\in C^2(\mathbb{S}^{n-1})$ such that $\Delta_S f=\tau$ if $\tau\in\mathbb{C}$, $\tau\neq 0$. We are thus led to the following.

Definition 2.1. A distribution $T\in\mathcal{D}'(\mathbb{S}^{n-1})$ *has the value* $\tau\in C$ *in* $\zeta\in\mathbb{S}^{n-1}$ if there exist $p\in\mathbb{N}_0$, $F\in C(\mathbb{S}^{n-1})$ and $f\in C^{2p}(\mathbb{S}^{n-1})$ such that

(1) in the sense of distributions, $T=\Delta_S^pF$ on a neighbourhood of ζ;

(2) $F(\eta)=f(\eta)+o[d(\zeta,\eta)^p]$ for $\eta\to\zeta$;

(3) $\Delta_S^pf(\zeta)=\tau$.

Remark 2.2. It is not difficult, using the criterion (b) above, to show that given $S\in\mathcal{D}'(\mathbb{R}^n)$, $x_0\in\mathbb{R}^n$ and $\tau\in\mathbb{C}$, if there exist $p\in\mathbb{N}_0$, $F\in C(\mathbb{R}^n)$ and $f\in C^{2p}(\mathbb{R}^n)$ such that $S=\Delta^pF$ on a neighbourhood of x_0, $F(x)=f(x)+o(\|x-x_0\|^{2p})$ for $x\to x_0$

and $\Delta^p f(x_0) = \tau$, then S has the value τ in x_0 (and this conclusion is no more true when assuming $o(\|x-x_0\|^p)$ instead of $o(\|x-x_0\|^{2p})$). The discrepancy between the exponents in this $o(\|x-x_0\|^{2p})$ and in $o[d(\zeta,\eta)^p]$ of (2) above is only superficial. Take two points $\zeta, \eta \in \mathbb{S}^{n-1}$ and let φ be the angle between ζ and η seen as vectors in $\mathbb{R}^n$. Then $d(\zeta,\eta) = 1-(\zeta|\eta) = 1-\cos(\varphi) = 2\sin^2(\varphi/2) = 2(\|\zeta-\eta\|/2)^2 = \|\zeta-\eta\|^2/2$ and $o[d(\zeta,\eta)^p] = o(\|\zeta-\eta\|^{2p})$ as $\eta \to \zeta$.

Remark 2.3. It immediately follows from the definition that if T is equal in the sense of distributions to a continuous function F on a neighbourhood of ζ, then T has the value $F(\zeta)$ in ζ.

3. Fourier inversion on the sphere

Let $T \in \mathcal{D}'(\mathbb{S}^{n-1})$. Since $\mathbb{S}^{n-1}$ is compact, T is of finite order; that is, there exist $C > 0$ and $m \in \mathbb{N}_0$ such that

$$|T(\varphi)| \leqslant C \sup_{|\alpha|\leqslant m} \sup_{\eta\in\mathbb{S}^{n-1}} |D_S^\alpha \varphi(\eta)| \tag{3.1}$$

for all $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$.

Let us now study the derivatives, with respect to η, of

$$Z_l(\zeta,\eta) = \frac{d_l^n}{\omega_{n-1}} P_l^{(n-2)/2}((\zeta|\eta)) \tag{3.2}$$

for a fixed $\zeta \in \mathbb{S}^{n-1}$, where $P_l^{(n-2)/2}$ are polynomials in one variable (see [7], Theorem 2.14, page 149). We know (see [3], page 762) that if $l \geqslant 1$,

$$D_S^{e_j} \frac{d_l^n}{\omega_{n-1}} P_l^{(n-2)/2}((\zeta|\eta)) = 2\pi \frac{d_{l-1}^{n+2}}{\omega_{n+1}} P_{l-1}^{n/2}((\zeta|\eta)) D_S^{e_j}(\zeta|\eta).$$

We get similarily for every multiindex $\alpha \neq 0$

$$D_S^\alpha[\eta \mapsto Z_l(\zeta,\eta)] = \sum_{j=1}^{|\alpha|} (2\pi)^j Q_j(\zeta,\eta) \frac{d_{l-j}^{n+2j}}{\omega_{n-1+2j}} P_{l-j}^{(n-2+2j)/2}((\zeta|\eta)), \tag{3.3}$$

where $Q_j(\zeta,\eta)$ is a linear combination of products of $D_S^\beta(\zeta|\eta)$ (with $\beta \leqslant \alpha$) which does not depend on l. Now, according to [7], Corollary 2.9, page 144,

$$|Z_l(\zeta,\eta)| \leqslant \frac{d_l^n}{\omega_{n-1}}$$

for any $\zeta, \eta \in \mathbb{S}^{n-1}$. Comparing this with (3.2), we deduce that

$$|P_l^{(n-2)/2}((\zeta|\eta))| \leqslant 1$$

for any $\zeta, \eta \in \mathbb{S}^{n-1}$. Therefore each term in the sum of (3.3) can be majorized in absolute value by

$$A_j \frac{d_{l-j}^{n+2j}}{\omega_{n-1+2j}}$$

where $A_j > 0$ does not depend on l. Moreover $d_{l-j}^{n+2j} \leqslant B_j l^{n+2j-2}$, where $B_j > 0$ does not depend on l. Put $A_0 = B_0 := 1$. Then for all $\eta \in \mathbb{S}^{n-1}$ and $\alpha \in \mathbb{N}_0^n$,

$$|D_S^\alpha Z_l(\zeta, \eta)| \leqslant (|\alpha| + 1) \max_{0 \leqslant j \leqslant |\alpha|} A_j B_j l^{n+2|\alpha|-2}.$$

We deduce that for $0 \leqslant r < 1$ and $\zeta \in \mathbb{S}^{n-1}$ fixed the series

$$\sum_{l=0}^{\infty} r^l Z_l(\zeta, \eta)$$

converges as a function of η for the semi-norm p_m. It follows from (3.1) that

$$\begin{aligned}
\sum_{l=0}^{\infty} r^l \, \Pi_l(T)(\zeta) &= \lim_{L \to \infty} \sum_{l=0}^{L} r^l \, \Pi_l(T)(\zeta) \\
&= \lim_{L \to \infty} \sum_{l=0}^{L} r^l \, T[\eta \mapsto Z_l(\zeta, \eta)] \\
&= \lim_{L \to \infty} T\Big[\eta \mapsto \sum_{l=0}^{L} r^l Z_l(\zeta, \eta)\Big]
\end{aligned}$$

exists and is equal to

$$T\Big[\eta \mapsto \sum_{l=0}^{\infty} r^l Z_l(\zeta, \eta)\Big],$$

that is, by [7], Theorem 2.10, page 145, to

$$T\Big[\eta \mapsto \frac{1}{\omega_{n-1}} \frac{1 - r^2}{(1 - 2r(\zeta|\eta) + r^2)^{n/2}}\Big].$$

We are now ready to state our main result.

Theorem 3.1. *Let $T \in \mathcal{D}'(\mathbb{S}^{n-1})$, $\xi \in \mathbb{S}^{n-1}$ and $\tau \in C$. If T has the value τ in ξ, then*

$$\lim_{r\to 1_-} \sum_{l=0}^{\infty} r^l\, \Pi_l(T)(\xi) = \tau.$$

P r o o f. We divide it in two parts.

First part. For $x \in \mathbb{R}^n$ with $\|x\| < 1$ and $\eta \in \mathbb{S}^{n-1}$ we put

$$P(x,\eta) := \frac{1}{\omega_{n-1}} \frac{1-\|x\|^2}{\|x-\eta\|^n};$$

this is the well known *Poisson kernel*; among its many properties we will use the following two: if $f \in C(\mathbb{S}^{n-1})$ and $\zeta \in \mathbb{S}^{n-1}$,

$$\lim_{x\to\zeta, \|x\|<1} \int_{\mathbb{S}^{n-1}} f(\eta) P(x,\eta)\, \mathrm{d}\sigma_{n-1}(\eta) = f(\zeta) \tag{3.4}$$

(see [1], Theorem 1.17 page 13); and for all $x \in \mathbb{R}^n$ with $\|x\| < 1$,

$$\int_{\mathbb{S}^{n-1}} P(x,\eta)\, \mathrm{d}\sigma_{n-1}(\eta) = 1 \tag{3.5}$$

(see [1], Proposition 1.20, page 14).

If we write $x = r\zeta$ with $0 \leqslant r < 1$ and $\zeta \in \mathbb{S}^{n-1}$, we get

$$\begin{aligned} \|x-\eta\|^n &= (r\zeta - \eta | r\zeta - \eta)^{n/2} \\ &= \Big((r\zeta | r\zeta) - 2(r\zeta|\eta) + (\eta|\eta) \Big)^{n/2} \\ &= \big(r^2 - 2r(\zeta|\eta) + 1 \big)^{n/2}. \end{aligned}$$

Hence,

$$P(r\zeta,\eta) = \frac{1}{\omega_{n-1}} \frac{1-r^2}{(1-2r(\zeta|\eta)+r^2)^{n/2}}$$

and

$$\sum_{l=0}^{\infty} r^l\, \Pi_l(T)(\zeta) = T[\eta \mapsto P(r\zeta,\eta)].$$

We will calculate $D_S^{\alpha}[\eta \mapsto P(r\zeta,\eta)]$. Using the equality

$$\frac{\partial}{\partial x_j} \big(1 - 2r(\zeta|x)\|x\|^{-1} + r^2\big) = -2r \left(\zeta_j \|x\|^{-1} - (\zeta|x) x_j \|x\|^{-3}\right),$$

we find that

$$\frac{\partial}{\partial x_j} P\Big(r\zeta, \frac{x}{\|x\|}\Big) = \frac{(1-r^2)r}{\omega_{n-1}} \frac{n(\zeta_j \|x\|^{-1} - (\zeta|x) x_j \|x\|^{-3})}{(1-2r(\zeta|x)\|x\|^{-1} + r^2)^{1+n/2}}$$

and, by induction,

$$D^\alpha P\Big(r\zeta, \frac{x}{\|x\|}\Big) = \sum_{j=1}^{|\alpha|} \frac{(1-r^2)r^j}{\omega_{n-1}} \frac{R_j^\alpha(\zeta, x, \|x\|^{-1})}{(1-2r(\zeta|x)\|x\|^{-1} + r^2)^{j+n/2}},$$

where $\alpha \in \mathbb{N}_0^n$, $\alpha \neq 0$, $r > 0$, $\zeta \in \mathbb{S}^{n-1}$ and $R_j^\alpha(\zeta, x, \|x\|^{-1})$ is a polynomial in $\zeta_1, \dots, \zeta_n$, $x_1, \dots, x_n$, $\|x\|^{-1}$ which does not depend on r. Restricting it to $\mathbb{S}^{n-1}$ we deduce that

$$D_S^\alpha[\eta \mapsto P(r\zeta, \eta)] = \sum_{j=1}^{|\alpha|} \frac{(1-r^2)r^j}{\omega_{n-1}} \frac{\widetilde{R}_j^\alpha(\zeta, \eta)}{(1-2r(\zeta|\eta) + r^2)^{j+n/2}},$$

where $\widetilde{R}_j^\alpha(\zeta, \eta)$ is a polynomial in $\zeta_1, \dots, \zeta_n$, $\eta_1, \dots, \eta_n$ which does not depend on r. Let

$$M_\alpha := \max_{1 \leqslant j \leqslant |\alpha|} \sup_{\zeta, \eta \in \mathbb{S}^{n-1}} |\widetilde{R}_j^\alpha(\zeta, \eta)|$$

and observe that

$$1 - 2r(\zeta|\eta) + r^2 = (1-r)^2 + 2r(1 - (\zeta|\eta)) \geqslant 2r(1 - (\zeta|\eta)) = 2r\, d(\zeta, \eta).$$

Therefore

$$|D_S^\alpha[\eta \mapsto P(r\zeta, \eta)]| \leqslant \sum_{j=1}^{|\alpha|} \frac{(1-r^2)r^j}{\omega_{n-1}} \frac{M_\alpha}{[2r\, d(\zeta, \eta)]^{j+n/2}}. \tag{3.6}$$

Similarly, to calculate $\Delta_S^l[\eta \mapsto P(r\zeta, \eta)]$ we use the equalities

$$\sum_{j=1}^{n} \left(\zeta_j \|x\|^{-1} - (\zeta|x) x_j \|x\|^{-3}\right)^2 = \|x\|^{-2} - (\zeta|x)^2 \|x\|^{-4},$$

$$\frac{\partial}{\partial x_j}\left(1 - (\zeta|x)^2 \|x\|^{-2}\right) = -2(\zeta|x)\|x\|^{-1}\left(\zeta_j \|x\|^{-1} - (\zeta|x) x_j \|x\|^{-3}\right),$$

$$\frac{\partial}{\partial x_j}\left(\zeta_j \|x\|^{-1} - (\zeta|x) x_j \|x\|^{-3}\right) = -2\zeta_j x_j \|x\|^{-1} - (\zeta|x)\|x\|^{-3} + 3x_j^2 (\zeta|x)\|x\|^{-5},$$

$$\sum_{j=1}^{n}\left(-2\zeta_j x_j \|x\|^{-1} - (\zeta|x)\|x\|^{-3} + 3x_j^2(\zeta|x)\|x\|^{-5}\right) = (1-n)(\zeta|x)\|x\|^{-3},$$

and find, after tedious but straightforward calculations, that for every $l \in \mathbb{N}$,

$$\Delta_S^l[\eta \mapsto P(r\zeta,\eta)] = \sum_{j=1}^{2l} \frac{(1-r^2)r^j}{\omega_{n-1}} \frac{Q_j((\zeta|\eta))[1-(\zeta|\eta)^2]^{\max(j-l,0)}}{(1-2r(\zeta|\eta)+r^2)^{j+n/2}} \tag{3.7}$$

with Q_j a polynomial in one variable depending on l but not on r. But since $1-2r(\zeta|\eta)+r^2 \geqslant 2r(1-(\zeta|\eta))$, we have

$$0 \leqslant \frac{r^j[1-(\zeta|\eta)]^j}{(1-2r(\zeta|\eta)+r^2)^j} \leqslant \frac{r^j[1-(\zeta|\eta)]^j}{[2r(1-(\zeta|\eta))]^j} \leqslant \frac{1}{2^j}.$$

Moreover $|1 \pm (\zeta|\eta)| \leqslant 2$ for any $\zeta, \eta \in \mathbb{S}^{n-1}$. We deduce

$$\begin{aligned} d(\zeta,\eta)^l|\Delta_S^l P(r\zeta,\eta)| &= |[1-(\zeta|\eta)]^l \Delta_S^l P(r\zeta,\eta)| \\ &= \Bigg| \sum_{j=1}^{l} \frac{1-r^2}{\omega_{n-1}} \frac{Q_j((\zeta|\eta))[1-(\zeta|\eta)]^{l-j}}{(1-2r(\zeta|\eta)+r^2)^{n/2}} \frac{r^j[1-(\zeta|\eta)]^j}{(1-2r(\zeta|\eta)+r^2)^j} \\ &\quad + \sum_{j=l+1}^{2l} \frac{1-r^2}{\omega_{n-1}} \frac{Q_j((\zeta|\eta))[1+(\zeta|\eta)]^{j-l}}{(1-2r(\zeta|\eta)+r^2)^{n/2}} \frac{r^j[1-(\zeta|\eta)]^j}{(1-2r(\zeta|\eta)+r^2)^j} \Bigg| \\ &\leqslant \left(\sum_{j=1}^{l} \frac{\|Q_j\|_\infty 2^{l-j}}{2^j} + \sum_{j=l+1}^{2l} \frac{\|Q_j\|_\infty 2^{j-l}}{2^j} \right) \frac{1-r^2}{\omega_{n-1}} \frac{1}{(1-2r(\zeta|\eta)+r^2)^{n/2}}, \end{aligned}$$

that is

$$d(\zeta,\eta)^l|\Delta_S^l P(r\zeta,\eta)| \leqslant C_l P(r\zeta,\eta) \tag{3.8}$$

with $C_l > 0$ a constant depending only on l. Set $C_0 := 1$, so that (3.8) is true for all $l \in \mathbb{N}_0$.

Finally, from

$$0 \leqslant \frac{(1-r^2)r^j}{(1-2r(\zeta|\eta)+r^2)^{j+n/2}} \leqslant \frac{(1-r^2)r^j}{(2r\,d(\zeta,\eta))^{j+n/2}}$$

it follows by (3.7),

$$\lim_{r\to 1_-} \int_{\eta\in\mathbb{S}^{n-1}, d(\zeta,\eta)>\delta} |\Delta_S^l P(r\zeta,\eta)| \,\mathrm{d}\sigma_{n-1}(\eta) = 0 \tag{3.9}$$

if $0 < \delta < 2$ for all $l \in \mathbb{N}_0$.

Second part. Choose $\varepsilon > 0$ arbitrary. By assumption there exist $p \in \mathbb{N}_0$, $F \in C(\mathbb{S}^{n-1})$ and $f \in C^{2p}(\mathbb{S}^{n-1})$ such that $T = \Delta_S^p F$ on a neighbourhood of ξ, $F(\eta) = f(\eta) + o[d(\xi,\eta)^p]$ as $\eta \to \xi$ and $\Delta_S^p f(\xi) = \tau$.

Hence there exists $0 < \delta_1 < 1$ such that $T = \Delta_S^p F$ on $B(\xi, 2\delta_1)$ and there exists $0 < \delta_2 < 2$ such that for all $\eta \in \mathbb{S}^{n-1}$ with $d(\xi,\eta) < \delta_2$ we have

$$|F(\eta) - f(\eta)| < \varepsilon \frac{d(\xi,\eta)^p}{4C_p}.$$

Let $\delta := \min(\delta_1, \delta_2)$. Since the support of $T - \Delta_S^p F$ is included in $\mathbb{S}^{n-1} \setminus B(\xi, 2\delta)$, there exists a constant $\widetilde{C} > 0$ such that

$$|(T - \Delta_S^p F)(\varphi)| \leqslant \widetilde{C} \sup_{|\alpha| \leqslant \widetilde{m}} \sup_{d(\eta,\xi) \geqslant \delta} |D_S^\alpha \varphi(\eta)|$$

for all $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$, where $\widetilde{m}$ is the order of the distribution $T - \Delta_S^p F$. In view of (3.6), we can then find $0 \leqslant r_1 < 1$ such that $r_1 \leqslant r < 1$ implies

$$|(T - \Delta_S^p F)[\eta \mapsto P(r\xi, \eta)]| < \frac{\varepsilon}{4}.$$

By (3.9) there exists $0 \leqslant r_2 < 1$ such that $r_2 \leqslant r < 1$ implies

$$\int_{d(\xi,\eta)>\delta} |\Delta_S^p P(r\xi,\eta)| \,\mathrm{d}\sigma_{n-1}(\eta) < \frac{\varepsilon}{4\|F\|_\infty + 4\|f\|_\infty + 1}.$$

Finally, since $f \in C^{2p}(\mathbb{S}^{n-1})$, $\Delta_S^p f \in C(\mathbb{S}^{n-1})$, by (3.4) we get

$$\lim_{x \to \xi, \|x\|<1} \int_{\mathbb{S}^{n-1}} \Delta_S^p f(\eta)\, P(x,\eta) \,\mathrm{d}\sigma_{n-1}(\eta) = \Delta_S^p f(\xi) = \tau.$$

Therefore there exists $0 \leqslant r_3 < 1$ such that $r_3 \leqslant r < 1$ implies

$$\left| \int_{\mathbb{S}^{n-1}} \Delta_S^p f(\eta)\, P(r\xi,\eta) \,\mathrm{d}\sigma_{n-1}(\eta) - \tau \right| < \frac{\varepsilon}{4}. \tag{3.10}$$

Put $r_0 := \max(r_1, r_2, r_3)$. For all $r_0 \leqslant r < 1$ we have

$$\begin{aligned} \left| \sum_{l=0}^{\infty} r^l \,\Pi_l(T)(\xi) - \tau \right| &= |T[\eta \mapsto P(r\xi,\eta)] - \tau| \\ &\leqslant |(T - \Delta_S^p F)[\eta \mapsto P(r\xi,\eta)]| + |\Delta_S^p F[\eta \mapsto P(r\xi,\eta)] - \tau| \end{aligned}$$

$$
\begin{aligned}
&< \frac{\varepsilon}{4} + |F[\eta \mapsto \Delta_S^p P(r\xi,\eta)] - \tau| \\
&< \frac{\varepsilon}{4} + \left| \int_{\mathbb{S}^{n-1}} F(\eta)\, \Delta_S^p P(r\xi,\eta)\, \mathrm{d}\sigma_{n-1}(\eta) - \tau \right| \\
&\leqslant \frac{\varepsilon}{4} + \left| \int_{\mathbb{S}^{n-1}} (F(\eta) - f(\eta))\, \Delta_S^p P(r\xi,\eta)\, \mathrm{d}\sigma_{n-1}(\eta) \right| \\
&\quad + \left| \int_{\mathbb{S}^{n-1}} f(\eta)\, \Delta_S^p P(r\xi,\eta)\, \mathrm{d}\sigma_{n-1}(\eta) - \tau \right| \\
&\leqslant \frac{\varepsilon}{2} + \int_{d(\xi,\eta)<\delta} |F(\eta) - f(\eta)||\Delta_S^p P(r\xi,\eta)|\, \mathrm{d}\sigma_{n-1}(\eta) \\
&\quad + \int_{d(\xi,\eta)\geqslant\delta} (|F(\eta)| + |f(\eta)|)|\Delta_S^p P(r\xi,\eta)|\, \mathrm{d}\sigma_{n-1}(\eta) \\
&\leqslant \frac{\varepsilon}{2} + \int_{d(\xi,\eta)<\delta} \Big(\frac{\varepsilon}{4C_p}\Big) d(\xi,\eta)^p |\Delta_S^p P(r\xi,\eta)|\, \mathrm{d}\sigma_{n-1}(\eta) \\
&\quad + \int_{d(\xi,\eta)\geqslant\delta} (\|F\|_\infty + \|f\|_\infty)|\Delta_S^p P(r\xi,\eta)|\, \mathrm{d}\sigma_{n-1}(\eta) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}
$$

(In the fifth inequality we can use (3.10) because Δ_S is self-adjoint; in the last inequality we use (3.8) and (3.5) to majorize the integral over $B(\xi,\delta)$.) □

Remark 3.2. The theorem shows that if the value of T in ξ exists, it is unique.

Remark 3.3. The converse of the theorem is false: take $n = 2$ and T the principal value of $\cot(\frac{1}{2}t)$. Then its Fourier-Laplace series is Abel-summable to 0 at $t = 0$:

$$
\lim_{r\to 1_-} \sum_{k\in\mathbb{Z}} r^{|k|} \mathcal{F}T(k) \mathrm{e}^{\mathrm{i}k0} = \lim_{r\to 1_-} \frac{2r\sin 0}{1 + r^2 - 2r\cos 0} = \lim_{r\to 1_-} 0 = 0
$$

but T has no value at $t = 0$.

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