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RELATIONSHIPS BETWEEN GENERALIZED WIENER INTEGRALS AND CONDITIONAL ANALYTIC FEYNMAN INTEGRALS OVER CONTINUOUS PATHS

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Cordially dedicated to Jerry Johnson

Abstract. Let $C[0,t]$ denote a generalized Wiener space, the space of real-valued continuous functions on the interval $[0,t]$, and define a random vector $Z_n\colon C[0,t]\to\mathbb{R}^{n+1}$ by

$$Z_n(x)=\biggl(x(0)+a(0),\int_0^{t_1}h(s)\,\mathrm{d}x(s)+x(0)+a(t_1),\ldots,\int_0^{t_n}h(s)\,\mathrm{d}x(s)+x(0)+a(t_n)\biggr),$$

where $a\in C[0,t]$, $h\in L_2[0,t]$, and $0<t_1<\ldots<t_n\leqslant t$ is a partition of $[0,t]$. Using simple formulas for generalized conditional Wiener integrals, given Z_n we will evaluate the generalized analytic conditional Wiener and Feynman integrals of the functions F in a Banach algebra which corresponds to Cameron-Storvick's Banach algebra $\mathcal{S}$. Finally, we express the generalized analytic conditional Feynman integral of F as a limit of the non-conditional generalized Wiener integral of a polygonal function using a change of scale transformation for which a normal density is the kernel. This result extends the existing change of scale formulas on the classical Wiener space, abstract Wiener space and the analogue of the Wiener space $C[0,t]$.

Keywords: analogue of Wiener space; analytic conditional Feynman integral; change of scale formula; conditional Wiener integral; Wiener integral

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1. Introduction

Let $C_0[0,t]$ denote the Wiener space, the space of continuous real-valued functions x on $[0,t]$ with $x(0)=0$. It has been recognized since Feynman introduced his integrals that there is a close formal analogy with Wiener integrals. One of approaches to define the Feynman integrals is using an analytic continuation in the mass parameter (i.e., the scale parameter) rather than in the time, so that we can treat not only attractive potentials but also repulsive ones. The conditional analytic Feynman integrals by the analytic continuation of the scale parameter can describe the Feynman integrals for Wiener paths which pass through a particular position at each time. To evaluate the conditional analytic Feynman integrals for the Wiener paths, it is essential to handle the scale parameter in conditional Wiener integrals. Unfortunately, the Wiener measure and Wiener measurability behave badly under change of scale and under translation (see [1], [2]) so that we need to change the scale formulas for the conditional Wiener integrals. Various kinds of change of scale formulas in [4], [14], [19], [20], [21] for ordinary Wiener integrals of bounded and unbounded functions were developed on the classical and abstract Wiener spaces in [15]. Furthermore, the second author of this paper and his coauthors in [6], [11], [18] introduced various kinds of change of scale formulas for the conditional Wiener integrals of functions defined on $C_0[0,t]$, the infinite dimensional Wiener space (see [5]) and $C[0,t]$, an analogue of the Wiener space (see [13], [17]) which is the space of real-valued continuous paths on $[0,t]$ and will be introduced in the next section.

Let $a\in C[0,t]$ and $h\in L_2[0,t]$ with $h\neq 0$ a.e. on $[0,t]$. Define the stochastic processes $X,Z\colon C[0,t]\times[0,t]\to\mathbb{R}$ by

$$X(x,s)=\int_0^s h(u)\,\mathrm{d}x(u)\quad\text{and}\quad Z(x,s)=X(x,s)+x(0)+a(s)$$

for $x\in C[0,t]$ and $s\in[0,t]$, where the integral denotes the Paley-Wiener-Zygmund integral which will be introduced in the next section (see [13]). Define random vectors $X_n\colon C[0,t]\to\mathbb{R}^n$, $X_{n+1}\colon C[0,t]\to\mathbb{R}^{n+1}$, $Z_n\colon C[0,t]\to\mathbb{R}^{n+1}$ and $Z_{n+1}\colon C[0,t]\to\mathbb{R}^{n+2}$ by

$$X_n(x)=(X(x,t_1),\dots,X(x,t_n)),$$
$$X_{n+1}(x)=(X(x,t_1),\dots,X(x,t_n),X(x,t_{n+1})),$$
$$Z_n(x)=(Z(x,t_0),\dots,Z(x,t_n))$$

and

$$Z_{n+1}(x)=(Z(x,t_0),\dots,Z(x,t_n),Z(x,t_{n+1}))$$

for $x \in C[0,t]$, where $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = t$ is a partition of $[0,t]$. On the space $C[0,t]$ the author in [7] derived a simple formula for a generalized conditional Wiener integral given the vector-valued conditioning function X_{n+1}. Using the formula with X_{n+1}, Yoo and the author in [12] evaluated a generalized analytic conditional Wiener integral of the function G_r having the form

$$G_r(x) = F(x)\Psi\Big(\int_0^t v_1(s)\,\mathrm{d}x(s), \ldots, \int_0^t v_r(s)\,\mathrm{d}x(s)\Big)$$

for F in a Banach algebra which corresponds to the Cameron-Storvick's Banach algebra $\mathcal{S}$ in [3] and for $\Psi = f + \phi$ which need not be bounded or continuous, where $f \in L_p(\mathbb{R}^r)$, $1 \leqslant p \leqslant \infty$, $\{v_1, \ldots, v_r\}$ is an orthonormal subset of $L_2[0,t]$ and ϕ is the Fourier transform of a measure of bounded variation over $\mathbb{R}^r$. They then established various kinds of change of scale formulas for the generalized analytic conditional Wiener integral of G_r with the conditioning function X_{n+1}. Further works were developed by the second author of this paper. In fact he in [9] evaluated generalized analytic conditional Wiener and Feynman integrals of the cylinder function G having the form

$$G(x) = f((e,x))\phi((e,x))$$

for $x \in C[0,t]$, where $f \in L_p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, e is a unit element in $L_2[0,t]$, that is, the L_2-norm of e is 1, and ϕ is the Fourier transform of a measure of bounded variation over $\mathbb{R}$. He then expressed the generalized analytic conditional Feynman integral of G as the limit of non-conditional generalized Wiener integrals using a change of scale transformation. In [10] he introduced a simple formula for a generalized conditional Wiener integral on $C[0,t]$ with the conditioning function X_n and then evaluated the generalized analytic conditional Wiener and Feynman integrals of G. He expressed the generalized analytic conditional Feynman integral of G as two kinds of limits of non-conditional generalized Wiener integrals of polygonal functions and of cylinder functions using a change of scale transformation. In fact, as a function of $\xi_{n+1} \in \mathbb{R}$, the normal density

$$\Big[\frac{\lambda}{2\pi[b(t) - b(t_n)]}\Big]^{1/2} \exp\Big\{-\frac{\lambda(\xi_{n+1} - \xi_n)^2}{2[b(t) - b(t_n)]}\Big\}$$

plays a role of the kernel for the transformation, where ξ_n is a real number, λ is a complex number with positive real part and b is a variance function.

On the other hand, the author in [8] introduced simple formulas for a generalized conditional Wiener integral on $C[0,t]$ with the conditioning functions Z_n and Z_{n+1}, and then evaluated the generalized conditional Wiener integrals of functions including the time integrals which are important in Feynman integration theories, in particular, the Feynman-Kac formula. Using these simple formulas with Z_n and Z_{n+1} we will evaluate the generalized analytic conditional Wiener and Feynman integrals of the functions F in a Banach algebra which corresponds to the Banach algebra $\mathcal{S}$. Finally we will express the generalized analytic conditional Feynman integral of F as limits of non-conditional generalized Wiener integrals of a polygonal function using a change of scale transformation for which a normal density is the kernel. These results extend the existing change of scale formulas in [9], [10], [11], [12], [18] on the classical and the analogue of the Wiener space $C[0,t]$. While the choice of a complete orthonormal subset of $L_2[0,t]$ used in the present transformation is independent of e in the definition of the cylinder function in [9], [10], [14], the choices of orthonormal bases of $L_2[0,t]$ in the other change of scale formulas in [9], [10], [12], [18], [21] depend on the orthonormal set $\{v_1,\dots,v_r\}$ which is used in the definition of the cylinder function. The conditioning functions X_{n+1} and Z_{n+1} contain the present positions of the generalized Wiener paths, but X_n and Z_n do not contain them. Moreover, the conditioning functions X_n and X_{n+1} do not contain the initial positions of the generalized Wiener paths, but Z_n and Z_{n+1} contain them so that the results of this paper also extend those in [4].

2. An analogue of Wiener space

Let $\mathbb{C}$ and $\mathbb{C}_+$ denote the sets of complex numbers and complex numbers with positive real parts, respectively.

For a positive real t let $C[0,t]$ denote the space of real-valued continuous functions on the time interval $[0,t]$ with the supremum norm. For $\vec{t}=(t_0,t_1,\dots,t_n)$ with $0=t_0<t_1<\dots<t_n\leqslant t$ let $J_{\vec{t}}\colon C[0,t]\to\mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x)=(x(t_0),x(t_1),\dots,x(t_n)).$$

For B_j, $j=0,1,\dots,n$, in the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$, the subset $J_{\vec{t}}^{-1}\Big(\prod\limits_{j=0}^{n}B_j\Big)$ of $C[0,t]$ is called an interval; let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on $\mathcal{B}(\mathbb{R})$, define a pre-measure m_φ on $\mathcal{I}$ by

$$\begin{aligned}m_\varphi\Big[J_{\vec{t}}^{-1}\Big(\prod_{j=0}^{n}B_j\Big)\Big]&=\Big[\prod_{j=1}^{n}\frac{1}{2\pi(t_j-t_{j-1})}\Big]^{1/2}\\&\quad\times\int_{B_0}\int_{\prod_{j=1}^{n}B_j}\exp\Big\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\Big\}\,\mathrm{d}(u_1,\dots,u_n)\,\mathrm{d}\varphi(u_0).\end{aligned}$$

The Borel σ-algebra of $C[0,t]$, $\mathcal{B}(C[0,t])$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $C[0,t]$ such that

$w_\varphi(I) = m_\varphi(I)$ for all $I \in \mathcal{I}$. This measure w_φ is called an analogue of the Wiener measure associated with the probability measure φ (see [13], [17]). Let $\{e_j\colon j = 1, 2, \ldots\}$ be a complete orthonormal subset of $L_2[0,t]$ such that each e_j is of bounded variation. For $v \in L_2[0,t]$ and x in $C[0,t]$ let

$$(v, x) = \int_0^t v(u)\,\mathrm{d}x(u) = \lim_{n\to\infty} \sum_{j=1}^n \int_0^t \langle v, e_j\rangle e_j(u)\,\mathrm{d}x(u)$$

if the limit exists, where $\langle\cdot,\cdot\rangle$ denotes the inner product over $L_2[0,t]$. Then (v,x) is called the Paley-Wiener-Zygmund integral of v according to x; note that $(v,\cdot)$ is a mean zero Gaussian random variable with variance $\|v\|^2$ if $v \neq 0$, where $\|\cdot\|$ denotes the L^2-norm on $L_2[0,t]$ (see [13]).

Let $F\colon C[0,t] \to \mathbb{C}$ be integrable and let X be a random vector on $C[0,t]$. Then we have the conditional expectation $E[F|X]$ given X from a well-known probability theory. Furthermore, there exists a P_X-integrable function ψ on the value space of X such that $E[F|X](x) = (\psi\circ X)(x)$ for w_φ a.e. $x \in C[0,t]$, where P_X is the probability distribution of X. The function ψ is called the conditional Wiener w_φ-integral of F given X and it is also denoted by $E[F|X]$.

Let $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = t$ be a partition of $[0,t]$, where n is a fixed nonnegative integer. Let $h \in L_2[0,t]$ be of bounded variation with $h \neq 0$ a.e. on $[0,t]$. For $j = 1, \ldots, n+1$ let

$$\alpha_j = \frac{1}{\|\chi_{(t_{j-1},t_j]}h\|}\chi_{(t_{j-1},t_j]}h$$

and let V be the subspace of $L_2[0,t]$ generated by $\{\alpha_1, \ldots, \alpha_{n+1}\}$. Let $V^\perp$ be the orthogonal complement of V. Let $\mathcal{P}\colon L_2[0,t] \to V$ be the orthogonal projection given by

$$\mathcal{P}v = \sum_{j=1}^{n+1} \langle v, \alpha_j\rangle\alpha_j$$

and let $\mathcal{P}^\perp\colon L_2[0,t] \to V^\perp$ be the orthogonal projection. Let $a \in C[0,t]$ and define stochastic processes $X, Z\colon C[0,t]\times[0,t] \to \mathbb{R}$ by

$$X(x,s) = \int_0^s h(u)\,\mathrm{d}x(u) + x(0) \quad\text{and}\quad Z(x,s) = X(x,s) + a(s)$$

for $x \in C[0,t]$ and for $s \in [0,t]$. Define random vectors $X_n, Z_n\colon C[0,t] \to \mathbb{R}^{n+1}$ and $X_{n+1}, Z_{n+1}\colon C[0,t] \to \mathbb{R}^{n+2}$ by

$$X_n(x) = (X(x,t_0), X(x,t_1), \ldots, X(x,t_n)), \tag{2.1}$$

$$Z_n(x) = X_n(x) + (a(t_0), a(t_1), \ldots, a(t_n)), \tag{2.2}$$

$$X_{n+1}(x) = (X(x,t_0), X(x,t_1), \ldots, X(x,t_n), X(x,t_{n+1})), \tag{2.3}$$

and

$$Z_{n+1}(x) = X_{n+1}(x) + (a(t_0), a(t_1), \ldots, a(t_n), a(t_{n+1})) \tag{2.4}$$

for $x \in C[0,t]$. Let $b(s) = \int_0^s (h(u))^2 \, du$ for $s \in [0,t]$ and for any function f on $[0,t]$ define the polygonal function $P_{b,n+1}(f)$ of f by

$$\begin{aligned} P_{b,n+1}(f)(s) = \sum_{j=1}^{n+1} & \chi_{(t_{j-1},t_j]}(s) \Big[\frac{b(t_j) - b(s)}{b(t_j) - b(t_{j-1})} f(t_{j-1}) + \frac{b(s) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} f(t_j) \Big] \\ & + \chi_{\{0\}}(s) f(0) \end{aligned} \tag{2.5}$$

for $s \in [0,t]$, where $\chi_{(t_{j-1},t_j]}$ and $\chi_{\{0\}}$ denote the indicator functions of the interval $[0,t]$. For $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$ define the polygonal function $P_{b,n+1}(\vec{\xi}_{n+1})$ of $\vec{\xi}_{n+1}$ by (2.5), where $f(t_j)$ is replaced by ξ_j for $j = 0, 1, \ldots, n, n+1$. If $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$, then $P_{b,n}(\vec{\xi}_n)$ is interpreted as $\chi_{[0,t_n]} P_{b,n+1}(\vec{\xi}_{n+1})$ on $[0,t]$. For $x \in C[0,t]$ and for $s \in [0,t]$ let

$$A(s) = a(s) - P_{b,n+1}(a)(s), \tag{2.6}$$

$$X_{b,n+1}(x,s) = X(x,s) - P_{b,n+1}(X(x,\cdot))(s) \tag{2.7}$$

and

$$Z_{b,n+1}(x,s) = Z(x,s) - P_{b,n+1}(Z(x,\cdot))(s). \tag{2.8}$$

For $\alpha, \beta, u \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ let

$$\Psi(\lambda, u, \alpha, \beta) = \Big(\frac{\lambda}{2\pi\beta}\Big)^{1/2} \exp\Big\{-\frac{\lambda}{2\beta}(u-\alpha)^2\Big\} \quad \text{with } \beta \neq 0. \tag{2.9}$$

For a function $F\colon C[0,t] \to \mathbb{C}$ let $F_Z(x) = F(Z(x,\cdot))$ for $x \in C[0,t]$. For notational convenience we restate Theorems 6 and 7 of [8] as the following two theorems.

Theorem 2.1. *Let F be a complex valued function on $C[0,t]$ and let F_Z be integrable over $C[0,t]$. Then for $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$*

$$E[F_Z | Z_{n+1}](\vec{\xi}_{n+1}) = E[F(Z_{b,n+1}(x,\cdot) + P_{b,n+1}(\vec{\xi}_{n+1}))],$$

where $Z_{b,n+1}$ is given by (2.8), $P_{Z_{n+1}}$ is the probability distribution of Z_{n+1} on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$ and the expectation is taken over the variable x.

Theorem 2.2. *Let F be a complex valued function on $C[0,t]$ and let F_Z be integrable over $C[0,t]$. Let P_{Z_n} be the probability distribution of Z_n on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. Then for P_{Z_n} a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$*

$$E[F_Z|Z_n](\vec{\xi}_n) = \int_{\mathbb{R}} \Psi(1, \xi_{n+1} - \xi_n, a(t) - a(t_n), b(t) - b(t_n)) \times E[F(Z_{b,n+1}(x,\cdot) + P_{b,n+1}(\vec{\xi}_{n+1}))] \, \mathrm{d}\xi_{n+1},$$

where $\vec{\xi}_{n+1} = (\xi_0, \dots, \xi_n, \xi_{n+1})$ and $Z_{b,n+1}$, Ψ are given by (2.8), (2.9), *respectively.*

Lemma 2.1. *For $\lambda > 0$ let $F_Z^\lambda(x) = F_Z(\lambda^{-1/2}x)$ and $Z_{n+1}^\lambda(x) = Z_{n+1}(\lambda^{-1/2}x)$ for $x \in C[0,t]$, where Z_{n+1} is given by* (2.4)*. Suppose that $E[F_Z^\lambda]$ exists. Then*

$$E[F_Z^\lambda|Z_{n+1}^\lambda](\vec{\xi}_{n+1}) = E[F(\lambda^{-1/2}X_{b,n+1}(x,\cdot) + A + P_{b,n+1}(\vec{\xi}_{n+1}))] \tag{2.10}$$

for $P_{Z_{n+1}^\lambda}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where A, $X_{b,n+1}$ are given by (2.6), (2.7), *respectively, and $P_{Z_{n+1}^\lambda}$ is the probability distribution of Z_{n+1}^λ on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$.*

Proof. By (2.4), the definition of the conditional Wiener w_φ-integral and Theorem 2.1 we have for $P_{Z_{n+1}^\lambda}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned} &E[F_Z^\lambda|Z_{n+1}^\lambda](\vec{\xi}_{n+1}) \\ &= E[F(\lambda^{-1/2}X(x,\cdot) + a)|\lambda^{-1/2}X_{n+1}(x) + (a(t_0), a(t_1), \dots, a(t_n), a(t_{n+1}))](\vec{\xi}_{n+1}) \\ &= E[F(\lambda^{-1/2}(X(x,\cdot) - P_{b,n+1}(X(x,\cdot))) + P_{b,n+1}(\vec{\xi}_{n+1}) + a - P_{b,n+1}(a))] \\ &= E[F(\lambda^{-1/2}X_{b,n+1}(x,\cdot) + A + P_{b,n+1}(\vec{\xi}_{n+1}))], \end{aligned}$$

which completes the proof. □

Lemma 2.2. *Under the assumptions given in Lemma 2.1 with replacing Z_{n+1} by Z_n which is given by* (2.2)*, we have for $P_{Z_n^\lambda}$ a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$*

$$E[F_Z^\lambda|Z_n^\lambda](\vec{\xi}_n) = \int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1} - \xi_n, a(t) - a(t_n), b(t) - b(t_n)) \times E[F(\lambda^{-1/2}X_{b,n+1}(x,\cdot) + A + P_{b,n+1}(\vec{\xi}_{n+1}))] \, \mathrm{d}\xi_{n+1}, \tag{2.11}$$

where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$, Ψ is given by (2.9) *and $P_{Z_n^\lambda}$ is the probability distribution of Z_n^λ on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$.*

P r o o f. By (2.2) and Theorem 2.2

$$
\begin{aligned}
&E[F_Z^\lambda|Z_n^\lambda](\vec{\xi}_n)\\
&\quad= E[F(\lambda^{-1/2}X(x,\cdot)+a)|\lambda^{-1/2}X_n(x)+(a(t_0),a(t_1),\ldots,a(t_n))](\vec{\xi}_n)\\
&\quad= E[F(\lambda^{-1/2}X(x,\cdot)+a)|X_n(x)](\lambda^{1/2}(\xi_0-a(t_0),\xi_1-a(t_1),\ldots,\xi_n-a(t_n)))\\
&\quad= \int_{\mathbb{R}} \Psi(1,\xi_{n+1},\lambda^{1/2}(\xi_n-a(t_n)),b(t)-b(t_n))E[F(\lambda^{-1/2}X_{b,n+1}(x,\cdot)\\
&\qquad\qquad + P_{b,n+1}(\xi_0-a(t_0),\xi_1-a(t_1),\ldots,\xi_n-a(t_n),\lambda^{-1/2}\xi_{n+1})+a)]\,\mathrm{d}\xi_{n+1}.
\end{aligned}
$$

Letting $u=\lambda^{-1/2}\xi_{n+1}+a(t)$ we have by the change of variable theorem

$$
\begin{aligned}
&E[F_Z^\lambda|Z_n^\lambda](\vec{\xi}_n)\\
&\quad= \int_{\mathbb{R}} \Psi(\lambda,u-a(t),\xi_n-a(t_n),b(t)-b(t_n))E[F(\lambda^{-1/2}X_{b,n+1}(x,\cdot)\\
&\qquad\qquad + P_{b,n+1}(\xi_0-a(t_0),\xi_1-a(t_1),\ldots,\xi_n-a(t_n),u-a(t))+a)]\,\mathrm{d}u\\
&\quad= \int_{\mathbb{R}} \Psi(\lambda,\xi_{n+1}-\xi_n,a(t)-a(t_n),b(t)-b(t_n))E[F(\lambda^{-1/2}X_{b,n+1}(x,\cdot)+A\\
&\qquad\qquad + P_{b,n+1}(\vec{\xi}_{n+1}))]\,\mathrm{d}\xi_{n+1},
\end{aligned}
$$

which is the desired result. □

Let $I_{F_Z}^\lambda(\vec{\xi}_{n+1})$ and $K_{F_Z}^\lambda(\vec{\xi}_n)$ be the right hand sides of (2.10) and (2.11), respectively. If $I_{F_Z}^\lambda(\vec{\xi}_{n+1})$ has an analytic extension $J_\lambda^*(F_Z)(\vec{\xi}_{n+1})$ on $\mathbb{C}_+$, then it is called the conditional analytic Wiener w_φ-integral of F_Z given Z_{n+1} with the parameter λ, and denoted by

$$E^{\mathrm{anw}_\lambda}[F_Z|Z_{n+1}](\vec{\xi}_{n+1}) = J_\lambda^*(F_Z)(\vec{\xi}_{n+1})$$

for $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$. Moreover, if for a nonzero real q, $E^{\mathrm{anw}_\lambda}[F_Z|Z_{n+1}](\vec{\xi}_{n+1})$ has a limit as λ approaches $-\mathrm{i}q$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ-integral of F_Z given Z_{n+1} with the parameter q and denoted by

$$E^{\mathrm{anf}_q}[F_Z|Z_{n+1}](\vec{\xi}_{n+1}) = \lim_{\lambda\to-\mathrm{i}q} E^{\mathrm{anw}_\lambda}[F_Z|Z_{n+1}](\vec{\xi}_{n+1}).$$

Replacing $I_{F_Z}^\lambda(\vec{\xi}_{n+1})$ by $K_{F_Z}^\lambda(\vec{\xi}_n)$, we define $E^{\mathrm{anw}_\lambda}[F_Z|Z_n](\vec{\xi}_n)$ and $E^{\mathrm{anf}_q}[F_Z|Z_n](\vec{\xi}_n)$. If $E[F_Z^\lambda]$ exists for $\lambda>0$ and has an analytic extension $J_\lambda^*(F_Z)$ on $\mathbb{C}_+$, then we call $J_\lambda^*(F_Z)$ the analytic Wiener w_φ-integral of F over $C[0,t]$ with parameter λ and denote it by

$$E^{\mathrm{anw}_\lambda}[F_Z] = J_\lambda^*(F_Z).$$

The integral $E^{\mathrm{anf}_q}[F_Z]$ is also defined by

$$E^{\mathrm{anf}_q}[F_Z] = \lim_{\lambda\to -\mathrm{i}q} E^{\mathrm{anw}_\lambda}[F_Z]$$

if it exists, where the limit is taken through $\mathbb{C}_+$.

Applying Theorem 3.5 in [13], we have the following theorem.

Theorem 2.3. *Let $\{h_1, h_2, \ldots, h_r\}$ be an orthonormal system of $L_2[0,t]$. Then $(h_1,\cdot),\ldots,(h_r,\cdot)$ are independent and each $(h_i,\cdot)$ has the standard normal distribution. Moreover, if $f\colon \mathbb{R}^r \to \mathbb{R}$ is Borel measurable, then*

$$\begin{aligned}\int_{C[0,t]} & f((h_1,x),\ldots,(h_r,x))\,\mathrm{d}\mathrm{w}_\varphi(x)\\ &\stackrel{*}{=} \Big(\frac{1}{2\pi}\Big)^{r/2}\int_{\mathbb{R}^r} f(u_1,u_2,\ldots,u_r)\exp\Big\{-\frac{1}{2}\sum_{j=1}^r u_j^2\Big\}\,\mathrm{d}(u_1,u_2,\ldots,u_r),\end{aligned}$$

where $\stackrel{*}{=}$ means that if either side exists then both sides exist and they are equal.

Proof. Let $\{e_j\colon j=1,2,\ldots\}$ be a complete orthonormal subset of $L_2[0,t]$ such that each e_j is of bounded variation. For $l=1,\ldots,r$, denote $X_l(x)=(h_l,x)$ for $x\in C[0,t]$ and let φ_{X_l} be the characteristic function of X_l. By Theorem 3.5 in [13] and the dominated convergence theorem we have for $\xi\in\mathbb{R}$

$$\begin{aligned}\varphi_{X_l}(\xi) &= \int_{C[0,t]} \exp\{\mathrm{i}\xi X_l(x)\}\,\mathrm{d}\mathrm{w}_\varphi(x)\\ &= \lim_{n\to\infty}\int_{C[0,t]}\exp\Big\{\mathrm{i}\xi\sum_{j=1}^n\langle h_l,e_j\rangle\int_0^t e_j(u)\,\mathrm{d}x(u)\Big\}\,\mathrm{d}\mathrm{w}_\varphi(x)\\ &= \lim_{n\to\infty}\exp\Big\{-\frac{\xi^2}{2}\sum_{j=1}^n\langle h_l,e_j\rangle^2\Big\} = \exp\Big\{-\frac{\xi^2}{2}\Big\}\end{aligned}$$

so that X_l has the standard normal distribution. Moreover, we have by Theorem 3.5 in [13] again

$$\begin{aligned}2+2\langle h_l,h_j\rangle &= \|h_l+h_j\|^2\\ &= \int_{C[0,t]}(h_l+h_j,x)^2\,\mathrm{d}\mathrm{w}_\varphi(x)\\ &= \int_{C[0,t]}[(h_l,x)+(h_j,x)]^2\,\mathrm{d}\mathrm{w}_\varphi(x)\\ &= 2+2\int_{C[0,t]}(h_l,x)(h_j,x)\,\mathrm{d}\mathrm{w}_\varphi(x)\end{aligned}$$

so that $\operatorname{Cov}((h_l,\cdot),(h_j,\cdot)) = \int_{C[0,t]}(h_l,x)(h_j,x)\,\mathrm{dw}_\varphi(x) = \delta_{lj}$, where δ_{lj} is the Kronecker delta function. Now $(h_1,\cdot),\dots,(h_r,\cdot)$ are independent and $((h_1,\cdot),\dots,(h_r,\cdot))$ has the multivariate normal distribution with mean zero and the covariance matrix which is the $r\times r$-identity matrix. By Theorem 4 of [16] we have the theorem. $\square$

The following lemmas are useful for proving the results in the subsequent sections (see [12]).

Lemma 2.3. *Let $v \in L_2[0,t]$. Then for w_φ a.e. $x \in C[0,t]$*

$$(v, P_{b,n+1}(X(x,\cdot))) = (\mathcal{P}M_h v, x) + (v, x(0)) = (\mathcal{P}M_h v, x),$$

where $M_h\colon L_2[0,t]\to L_2[0,t]$ is the multiplication operator defined by

$$M_h u = hu \quad \text{for } u \in L_2[0,t].$$

Lemma 2.4. *Let $v \in L_2[0,t]$, $\vec{\xi}_{n+1} = (\xi_0,\xi_1,\dots,\xi_n,\xi_{n+1}) \in \mathbb{R}^{n+2}$ and*

$$(v, P_{b,n}(\vec{\xi}_n)) = \sum_{j=1}^{n}\langle v\alpha_j,\alpha_j\rangle(\xi_j-\xi_{j-1}),$$

where $\vec{\xi}_n = (\xi_0,\xi_1,\dots,\xi_n)$. Then

$$\begin{aligned}(v, P_{b,n+1}(\vec{\xi}_{n+1})) &= \sum_{j=1}^{n+1}\langle v\alpha_j,\alpha_j\rangle(\xi_j-\xi_{j-1})\\ &= (v, P_{b,n}(\vec{\xi}_n)) + \langle v\alpha_{n+1},\alpha_{n+1}\rangle(\xi_{n+1}-\xi_n).\end{aligned}$$

Remark 2.1. (1) The multiplication operator M_h in Lemma 2.3 is well-defined because h is of bounded variation, which implies the boundedness of h. Throughout this paper M_h will denote the operator given in the lemma unless otherwise specified.

(2) For $\vec{\xi}_n = (\xi_0,\xi_1,\dots,\xi_n) \in \mathbb{R}^{n+1}$ it is possible that $P_{b,n}(\vec{\xi}_n) \notin C[0,t]$ if $\xi_n \neq 0$. In this case the symbol $(v, P_{b,n}(\vec{\xi}_n))$ does not mean the Paley-Wiener-Zygmund integral of $v \in L_2[0,t]$. It is only a formal expression for $\sum_{j=1}^{n}\langle v\alpha_j,\alpha_j\rangle(\xi_j-\xi_{j-1})$ which is given in Lemma 2.4.

3. Generalized analytic Feynman and conditional Feynman integrals

In this section we introduce the analytic Feynman, conditional Wiener and Feynman integrals of functions in a Banach algebra.

Let $\mathcal{M}(L_2[0,t])$ be the class of all complex valued Borel measures of bounded variation over $L_2[0,t]$ and let $\mathcal{S}_{\mathrm{w}_\varphi}$ be the space of all functions F which for $\sigma \in \mathcal{M}(L_2[0,t])$ have the form

$$F(x) = \int_{L_2[0,t]} \exp\{\mathrm{i}(v,x)\}\,\mathrm{d}\sigma(v) \tag{3.1}$$

for w_φ a.e. $x \in C[0,t]$. We note that $\mathcal{S}_{\mathrm{w}_\varphi}$ is a Banach algebra (see [3], [13]).

Theorem 3.1. *Let a be absolutely continuous on $[0,t]$. Let $F \in \mathcal{S}_{\mathrm{w}_\varphi}$ and $\sigma \in \mathcal{M}(L_2[0,t])$ be related by (3.1). Then for $\lambda \in \mathbb{C}_+$*

$$E^{\mathrm{anw}_\lambda}[F_Z] = \int_{L_2[0,t]} \exp\Big\{-\frac{1}{2\lambda}\|M_h v\|^2\Big\}\,\mathrm{d}\sigma_a(v),$$

where $\mathrm{d}\sigma_a(v) = \exp\{\mathrm{i}(v,a)\}\,\mathrm{d}\sigma(v)$ for $v \in L_2[0,t]$. Moreover, for a nonzero real q, $E^{\mathrm{anf}_q}[F_Z]$ is given by the right hand side of the above equality after replacing λ by $-\mathrm{i}q$.

P r o o f. We note that $(v, x(0)) = 0$ for $v \in L_2[0,t]$ and $x \in C[0,t]$. Now we have for $\lambda > 0$

$$\begin{aligned}
E[F_Z^\lambda] &= \int_{L_2[0,t]} \int_{C[0,t]} \exp\{\mathrm{i}(v, \lambda^{-1/2}X(x,\cdot) + a)\}\,\mathrm{d}\mathrm{w}_\varphi(x)\,\mathrm{d}\sigma(v) \\
&= \int_{L_2[0,t]} \int_{C[0,t]} \exp\{\mathrm{i}\lambda^{-1/2}[(M_h v, x) + (v, x(0))]\}\,\mathrm{d}\mathrm{w}_\varphi(x)\,\mathrm{d}\sigma_a(v) \\
&= \int_{L_2[0,t]} \int_{C[0,t]} \exp\{\mathrm{i}\lambda^{-1/2}(M_h v, x)\}\,\mathrm{d}\mathrm{w}_\varphi(x)\,\mathrm{d}\sigma_a(v) \\
&= \int_{L_2[0,t]} \exp\Big\{-\frac{1}{2\lambda}\|M_h v\|^2\Big\}\,\mathrm{d}\sigma_a(v)
\end{aligned}$$

by Theorem 2.3, the change of variable theorem and the well known integration formula

$$\int_{\mathbb{R}} \exp\{-au^2 + \mathrm{i}bu\}\,\mathrm{d}u = \Big(\frac{\pi}{a}\Big)^{1/2} \exp\Big\{-\frac{b^2}{4a}\Big\} \tag{3.2}$$

for $a \in \mathbb{C}_+$ and $b \in \mathbb{R}$. By Morera's theorem and the dominated convergence theorem we have the theorem. □

By Theorem 27 in [8] we have the following theorem.

Theorem 3.2. *Let* Z_{n+1} *be given by* (2.4). *Under the assumptions given in* Theorem 3.1 *we have for* $\lambda \in \mathbb{C}_+$ *and a.e.* $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$E^{\mathrm{anw}_\lambda}[F_Z|Z_{n+1}](\vec{\xi}_{n+1}) \\ = \int_{L_2[0,t]} \exp\Big\{-\frac{1}{2\lambda}\|\mathcal{P}^\perp M_h v\|^2 + \mathrm{i}(v, P_{b,n+1}(\vec{\xi}_{n+1}))\Big\}\,\mathrm{d}\sigma_A(v),$$

where $\mathrm{d}\sigma_A(v) = \exp\{\mathrm{i}(v, A)\}\,\mathrm{d}\sigma(v)$ *for* $v \in L_2[0,t]$ *and* A *is given by* (2.6). *Moreover, for nonzero real* q, $E^{\mathrm{anf}_q}[F_Z|Z_{n+1}](\vec{\xi}_{n+1})$ *is given by the right hand side of the above equality after replacing* λ *by* $-\mathrm{i}q$.

Lemma 3.1. *Let* Ψ *be given by* (2.9). *Then for* $\alpha, \beta, \gamma \in \mathbb{R}$ *and* $\lambda \in \mathbb{C}_+$

$$\int_{\mathbb{R}} \Psi(\lambda, u, \alpha, \beta) \exp\{\mathrm{i}\gamma u\}\,\mathrm{d}u = \exp\Big\{-\frac{\beta}{2\lambda}\gamma^2 + \mathrm{i}\alpha\gamma\Big\} \quad \text{with } \beta \neq 0.$$

Proof. By (3.2) and the change of variable theorem

$$\begin{aligned}\int_{\mathbb{R}} \Psi(\lambda, u, \alpha, \beta) \exp\{\mathrm{i}\gamma u\}\,\mathrm{d}u &= \Big(\frac{\lambda}{2\pi\beta}\Big)^{1/2} \int_{\mathbb{R}} \exp\Big\{-\frac{\lambda(u-\alpha)^2}{2\beta} + \mathrm{i}\gamma u\Big\}\,\mathrm{d}u \\ &= \Big(\frac{\lambda}{2\pi\beta}\Big)^{1/2} \int_{\mathbb{R}} \exp\Big\{-\frac{\lambda u^2}{2\beta} + \mathrm{i}\gamma u + \mathrm{i}\alpha\gamma\Big\}\,\mathrm{d}u \\ &= \exp\Big\{-\frac{\beta}{2\lambda}\gamma^2 + \mathrm{i}\alpha\gamma\Big\},\end{aligned}$$

which completes the proof. □

Theorem 3.3. *Let* Z_n *be given by* (2.2). *Under the assumptions given in Theorem* 3.2 *we have for* $\lambda \in \mathbb{C}_+$ *and a.e.* $\vec{\xi}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned}E^{\mathrm{anw}_\lambda}[F_Z|Z_n](\vec{\xi}_n) \\ = \int_{L_2[0,t]} \exp\Big\{&-\frac{1}{2\lambda}[\|\mathcal{P}^\perp M_h v\|^2 + [b(t) - b(t_n)]\langle v\alpha_{n+1}, \alpha_{n+1}\rangle^2] \\ &+ \mathrm{i}[(v, P_{b,n}(\vec{\xi}_n)) + [a(t) - a(t_n)]\langle v\alpha_{n+1}, \alpha_{n+1}\rangle]\Big\}\,\mathrm{d}\sigma_A(v),\end{aligned}$$

where $(v, P_{b,n}(\vec{\xi}_n))$ *is given in Lemma* 2.4. *Moreover,* $E^{\mathrm{anf}_q}[F_Z|Z_n](\vec{\xi}_n)$ *is given by the right hand side of the above equality after replacing* λ *by* $-\mathrm{i}q$.

P r o o f. For $\lambda > 0$ and $\vec{\xi_n} = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$ we have by Lemma 2.4 and Theorem 3.2

$$\begin{aligned}
K_{F_Z}^{\lambda}(\vec{\xi_n}) &= \int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1} - \xi_n, a(t) - a(t_n), b(t) - b(t_n)) E[F(\lambda^{-1/2} X_{b,n+1}(x, \cdot) \\
&\qquad + A + P_{b,n+1}(\vec{\xi}_{n+1}))] \,\mathrm{d}\xi_{n+1} \\
&= \int_{L_2[0,t]} \exp\Big\{-\frac{1}{2\lambda}\|\mathcal{P}^{\perp} M_h v\|^2\Big\} \int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1} - \xi_n, a(t) - a(t_n), b(t) \\
&\qquad - b(t_n)) \exp\{\mathrm{i}(v, P_{b,n+1}(\vec{\xi}_{n+1}))\} \,\mathrm{d}\xi_{n+1} \,\mathrm{d}\sigma_A(v) \\
&= \int_{L_2[0,t]} \exp\Big\{-\frac{1}{2\lambda}\|\mathcal{P}^{\perp} M_h v\|^2 + \mathrm{i}(v, P_{b,n}(\vec{\xi_n}))\Big\} \int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1} - \xi_n, a(t) \\
&\qquad - a(t_n), b(t) - b(t_n)) \exp\{\mathrm{i}\langle v\alpha_{n+1}, \alpha_{n+1}\rangle(\xi_{n+1} - \xi_n)\} \,\mathrm{d}\xi_{n+1} \,\mathrm{d}\sigma_A(v),
\end{aligned}$$

where Ψ is given by (2.9). Since the Lebesgue measure on $\mathbb{R}$ is translation invariant we have by Lemma 3.1

$$\begin{aligned}
K_{F_Z}^{\lambda}(\vec{\xi_n}) &= \int_{L_2[0,t]} \exp\Big\{-\frac{1}{2\lambda}\|\mathcal{P}^{\perp} M_h v\|^2 + \mathrm{i}(v, P_{b,n}(\vec{\xi_n}))\Big\} \int_{\mathbb{R}} \Psi(\lambda, \xi_{n+1}, a(t) \\
&\qquad - a(t_n), b(t) - b(t_n)) \exp\{\mathrm{i}\langle v\alpha_{n+1}, \alpha_{n+1}\rangle \xi_{n+1}\} \,\mathrm{d}\xi_{n+1} \,\mathrm{d}\sigma_A(v) \\
&= \int_{L_2[0,t]} \exp\Big\{-\frac{1}{2\lambda}[\|\mathcal{P}^{\perp} M_h v\|^2 + [b(t) - b(t_n)]\langle v\alpha_{n+1}, \alpha_{n+1}\rangle^2] \\
&\qquad + \mathrm{i}[(v, P_{b,n}(\vec{\xi_n})) + [a(t) - a(t_n)]\langle v\alpha_{n+1}, \alpha_{n+1}\rangle]\Big\} \,\mathrm{d}\sigma_A(v).
\end{aligned}$$

By Morera's theorem and the dominated convergence theorem we have the theorem. □

Since $b(t_0) = 0$, $(v, \xi_0) = 0$ for $v \in L_2[0,t]$ and for $\xi_0 \in \mathbb{R}$, we have the following corollary.

Corollary 3.1. *Under the assumptions given in Theorem* 3.3 *with one exception* $n = 0$ *we have*

$$\begin{aligned}
E^{\mathrm{anw}_\lambda}[F_Z|Z_0](\xi_0) = \int_{L_2[0,t]} \exp\Big\{-\frac{1}{2\lambda}[\|\mathcal{P}^{\perp} M_h v\|^2 + b(t)\langle v\alpha_1, \alpha_1\rangle^2] \\
+ \mathrm{i}[a(t) - a(0)]\langle v\alpha_1, \alpha_1\rangle\Big\} \,\mathrm{d}\sigma_A(v)
\end{aligned}$$

and $E^{\mathrm{anf}_q}[F_Z|Z_0](\xi_0)$ *is given by the right hand side of the above equality after replacing* λ *by* $-\mathrm{i}q$.

4. Change of scale formulas using the polygonal function

In this section we derive change of scale formulas for the generalized conditional Wiener integrals of functions in a Banach algebra on the analogue of Wiener space using other functions in the same Banach algebra given in the previous section.

Throughout this paper let $\{e_1, e_2, \ldots\}$ be a complete orthonormal basis of $L_2[0,t]$. For $v \in L_2[0,t]$ let

$$c_j(v) = \langle v, e_j \rangle \quad \text{for } j = 1, 2, \ldots. \tag{4.1}$$

For $m \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and $x \in C[0,t]$ let

$$K_m(\lambda, x) = \exp\Big\{\frac{1-\lambda}{2} \sum_{j=1}^{m} (e_j, x)^2\Big\}. \tag{4.2}$$

Lemma 4.1. *Let K_m be given by* (4.2). *Then for $m \in \mathbb{N}$, $\lambda \in \mathbb{C}_+$ and any $v \in L_2[0,t]$*

$$E[K_m(\lambda, x) \exp\{\mathrm{i}(v, x)\}] = \lambda^{-m/2} \exp\Big\{-\frac{1}{2\lambda} B(m, \lambda, v)\Big\},$$

where

$$B(m, \lambda, v) = \sum_{j=1}^{m} [c_j(v)]^2 + \lambda\Big[\|v\|^2 - \sum_{j=1}^{m} [c_j(v)]^2\Big] \tag{4.3}$$

and the c_js are given by (4.1).

Proof. Suppose that $\{e_1, \ldots, e_m, v\}$ is linearly independent. By the Gram-Schmidt process and Theorem 2.3

$$\begin{aligned}
&E[K_m(\lambda, x) \exp\{\mathrm{i}(v, x)\}] \\
&= \int_{C[0,t]} \exp\Big\{\frac{1-\lambda}{2} \sum_{j=1}^{m} (e_j, x)^2 + \mathrm{i}(v, x)\Big\} \,\mathrm{d}w_\varphi(x) \\
&= \Big(\frac{1}{2\pi}\Big)^{(m+1)/2} \int_{\mathbb{R}^{m+1}} \exp\Big\{\frac{1-\lambda}{2} \sum_{j=1}^{m} u_j^2 + \mathrm{i} \sum_{j=1}^{m} c_j(v) u_j \\
&\qquad + \mathrm{i}\Big[\|v\|^2 - \sum_{j=1}^{m} [c_j(v)]^2\Big]^{1/2} u_{m+1} - \frac{1}{2} \sum_{j=1}^{m+1} u_j^2\Big\} \,\mathrm{d}(u_1, \ldots, u_m, u_{m+1})
\end{aligned}$$

$$= \left(\frac{1}{2\pi}\right)^{(m+1)/2} \int_{\mathbb{R}^{m+1}} \exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^{m} u_j^2 + \mathrm{i}\sum_{j=1}^{m} c_j(v)u_j + \mathrm{i}\Big[\|v\|^2 - \sum_{j=1}^{m}[c_j(v)]^2\Big]^{1/2} u_{m+1} - \frac{1}{2}u_{m+1}^2\Big\}\,\mathrm{d}(u_1,\ldots,u_m,u_{m+1})$$

$$= \lambda^{-m/2}\exp\Big\{-\frac{1}{2\lambda}\Big[\sum_{j=1}^{m}[c_j(v)]^2 + \lambda\Big[\|v\|^2 - \sum_{j=1}^{m}[c_j(v)]^2\Big]\Big]\Big\}$$

$$= \lambda^{-m/2}\exp\Big\{-\frac{1}{2\lambda}B(m,\lambda,v)\Big\}$$

by (3.2). If $\{e_1,\ldots,e_m,v\}$ is linearly dependent, then $\|v\|^2 = \sum_{j=1}^{m}[c_j(v)]^2$ so that it is not difficult to show

$$\begin{aligned} E[K_m(\lambda,x)\exp\{\mathrm{i}(v,x)\}] &= \lambda^{-m/2}\exp\Big\{-\frac{1}{2\lambda}B(m,0,v)\Big\} \\ &= \lambda^{-m/2}\exp\Big\{-\frac{1}{2\lambda}B(m,\lambda,v)\Big\}, \end{aligned}$$

which completes the proof. □

Theorem 4.1. *Let m be a fixed positive integer, let K_m be given by* (4.2) *and let $F \in \mathcal{S}_{\mathrm{w}_\varphi}$ be given by* (3.1)*. Then for $\lambda \in \mathbb{C}_+$*

$$E[K_m(\lambda,x)F_Z(x)] = \lambda^{-m/2}\int_{L_2[0,t]} \exp\Big\{-\frac{1}{2\lambda}B(m,\lambda,M_h v)\Big\}\,\mathrm{d}\sigma_a(v),$$

where B is given by (4.3) *and σ_a is given in Theorem* 3.1.

Proof. By the change of variable theorem and Lemma 4.1

$$\begin{aligned} &E[K_m(\lambda,x)F_Z(x)] \\ &= \int_{L_2[0,t]}\int_{C[0,t]} \exp\Big\{\frac{1-\lambda}{2}\sum_{j=1}^{m}(e_j,x)^2 + \mathrm{i}(v, X(x,\cdot)+a)\Big\}\,\mathrm{dw}_\varphi(x)\,\mathrm{d}\sigma(v) \\ &= \int_{L_2[0,t]}\int_{C[0,t]} \exp\Big\{\frac{1-\lambda}{2}\sum_{j=1}^{m}(e_j,x)^2 + \mathrm{i}(M_h v,x) + \mathrm{i}(v,x(0))\Big\}\,\mathrm{dw}_\varphi(x)\,\mathrm{d}\sigma_a(v) \\ &= \int_{L_2[0,t]}\int_{C[0,t]} \exp\Big\{\frac{1-\lambda}{2}\sum_{j=1}^{m}(e_j,x)^2 + \mathrm{i}(M_h v,x)\Big\}\,\mathrm{dw}_\varphi(x)\,\mathrm{d}\sigma_a(v) \\ &= \lambda^{-m/2}\int_{L_2[0,t]} \exp\Big\{-\frac{1}{2\lambda}B(m,\lambda,M_h v)\Big\}\,\mathrm{d}\sigma_a(v), \end{aligned}$$

which proves the theorem. □

Theorem 4.2. *Let A and $Z_{b,n+1}$ be given by* (2.6) *and* (2.8), *respectively, let m be a fixed positive integer and K_m be given by* (4.2). *Let $F \in \mathcal{S}_{\mathrm{w}_\varphi}$ be given by* (3.1). *Then for $\lambda \in \mathbb{C}_+$ and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$*

$$
\begin{aligned}
(4.4)\quad & E[K_m(\lambda,x)F(Z_{b,n+1}(x,\cdot)+P_{b,n+1}(\vec{\xi}_{n+1}))] \\
&= \lambda^{-m/2}\int_{L_2[0,t]}\exp\Big\{\mathrm{i}(v,P_{b,n+1}(\vec{\xi}_{n+1}))-\frac{1}{2\lambda}B(m,\lambda,\mathcal{P}^{\perp}M_h v)\Big\}\,\mathrm{d}\sigma_A(v),
\end{aligned}
$$

where B is given by (4.3) *and σ_A is given in Theorem* 3.2.

P r o o f. By Theorem 2.3, Lemma 2.3 and the change of variable theorem

$$
\begin{aligned}
&E[K_m(\lambda,x)F(Z_{b,n+1}(x,\cdot)+P_{b,n+1}(\vec{\xi}_{n+1}))] \\
&\quad= E[K_m(\lambda,x)F(X_{b,n+1}(x,\cdot)+A+P_{b,n+1}(\vec{\xi}_{n+1}))] \\
&\quad= \int_{L_2[0,t]}\exp\{\mathrm{i}(v,A+P_{b,n+1}(\vec{\xi}_{n+1}))\} \\
&\qquad\qquad\times\int_{C[0,t]}\exp\Big\{\frac{1-\lambda}{2}\sum_{j=1}^{m}(e_j,x)^2+\mathrm{i}(\mathcal{P}^{\perp}M_h v,x)\Big\}\,\mathrm{d}\mathrm{w}_\varphi(x)\,\mathrm{d}\sigma(v) \\
&\quad= \int_{L_2[0,t]}\exp\{\mathrm{i}(v,P_{b,n+1}(\vec{\xi}_{n+1}))\} \\
&\qquad\qquad\times\int_{C[0,t]}\exp\Big\{\frac{1-\lambda}{2}\sum_{j=1}^{m}(e_j,x)^2+\mathrm{i}(\mathcal{P}^{\perp}M_h v,x)\Big\}\,\mathrm{d}\mathrm{w}_\varphi(x)\,\mathrm{d}\sigma_A(v),
\end{aligned}
$$

where $X_{b,n+1}$ is given by (2.7). Now the theorem follows from Lemma 4.1. □

Theorem 4.3. *Let A and $Z_{b,n+1}$ be given by* (2.6) *and* (2.8), *respectively, let m be a fixed positive integer and K_m be given by* (4.2). *Let $F \in \mathcal{S}_{\mathrm{w}_\varphi}$ be given by* (3.1). *For $\lambda \in \mathbb{C}_+$ and $\vec{\xi}_n = (\xi_0,\xi_1,\ldots,\xi_n) \in \mathbb{R}^{n+1}$ let*

$$
\begin{aligned}
\Gamma(F,m,\lambda,\vec{\xi}_n) = \int_{\mathbb{R}} &\Psi(\lambda,\xi_{n+1}-\xi_n,a(t)-a(t_n),b(t)-b(t_n))E[K_m(\lambda,x) \\
&\times F(Z_{b,n+1}(x,\cdot)+P_{b,n+1}(\vec{\xi}_{n+1}))]\,\mathrm{d}\xi_{n+1},
\end{aligned}
$$

where $\vec{\xi}_{n+1} = (\xi_0,\xi_1,\ldots,\xi_n,\xi_{n+1})$ and Ψ is given by (2.9). *Then*

$$
\begin{aligned}
&\Gamma(F,m,\lambda,\vec{\xi}_n) \\
&\quad= \lambda^{-m/2}\int_{L_2[0,t]}\exp\Big\{-\frac{1}{2\lambda}[B(m,\lambda,\mathcal{P}^{\perp}M_h v)+[b(t)-b(t_n)]\langle v\alpha_{n+1},\alpha_{n+1}\rangle^2] \\
&\qquad\qquad+\mathrm{i}[(v,P_{b,n}(\vec{\xi}_n))+[a(t)-a(t_n)]\langle v\alpha_{n+1},\alpha_{n+1}\rangle]\Big\}\,\mathrm{d}\sigma_A(v),
\end{aligned}
$$

where B is given by (4.3) *and σ_A is given in Theorem* 3.2.

P r o o f. By (4.4) and Lemma 2.4

$$\begin{aligned}\Gamma(F,m,\lambda,\vec{\xi_n}) &= \lambda^{-m/2}\int_{L_2[0,t]}\exp\Big\{-\frac{1}{2\lambda}B(m,\lambda,\mathcal{P}^{\perp}M_h v)\Big\}\int_{\mathbb{R}}\Psi(\lambda,\xi_{n+1}-\xi_n,a(t)-a(t_n),\\ &\qquad b(t)-b(t_n))\exp\{\mathrm{i}(v,P_{b,n+1}(\vec{\xi_{n+1}}))\}\,\mathrm{d}\xi_{n+1}\,\mathrm{d}\sigma_A(v)\\ &= \lambda^{-m/2}\int_{L_2[0,t]}\exp\Big\{-\frac{1}{2\lambda}B(m,\lambda,\mathcal{P}^{\perp}M_h v)+\mathrm{i}(v,P_{b,n}(\vec{\xi_n}))\Big\}\\ &\quad\times\int_{\mathbb{R}}\Psi(\lambda,\xi_{n+1},a(t)-a(t_n),b(t)-b(t_n))\exp\{\mathrm{i}\langle v\alpha_{n+1},\alpha_{n+1}\rangle\xi_{n+1}\}\,\mathrm{d}\xi_{n+1}\,\mathrm{d}\sigma_A(v)\end{aligned}$$

since the Lebesgue measure is translation invariant on $\mathbb{R}$. By Lemma 3.1

$$\begin{aligned}\Gamma(F,m,\lambda,\vec{\xi_n}) &= \lambda^{-m/2}\int_{L_2[0,t]}\exp\Big\{-\frac{1}{2\lambda}[B(m,\lambda,\mathcal{P}^{\perp}M_h v)+[b(t)-b(t_n)]\langle v\alpha_{n+1},\alpha_{n+1}\rangle^2]\\ &\qquad +\mathrm{i}[(v,P_{b,n}(\vec{\xi_n}))+[a(t)-a(t_n)]\langle v\alpha_{n+1},\alpha_{n+1}\rangle]\Big\}\,\mathrm{d}\sigma_A(v),\end{aligned}$$

which is the desired result. □

Since $b(t_0)=0$, $(v,\xi_0)=0$ for $v\in L_2[0,t]$ and $\xi_0\in\mathbb{R}$, we have the following corollary.

Corollary 4.1. *Under the assumptions given in Theorem* 4.3 *with one exception* $n=0$ *we have*

$$\begin{aligned}\Gamma(F,m,\lambda,\xi_0)=\lambda^{-m/2}\int_{L_2[0,t]}&\exp\Big\{-\frac{1}{2\lambda}[B(m,\lambda,\mathcal{P}^{\perp}M_h v)+b(t)\langle v\alpha_1,\alpha_1\rangle^2]\\ &+\mathrm{i}[a(t)-a(0)]\langle v\alpha_1,\alpha_1\rangle\Big\}\,\mathrm{d}\sigma_A(v).\end{aligned}$$

By Parseval's identity we have for $v\in L_2[0,t]$ and $\lambda\in\mathbb{C}_+$

$$\lim_{m\to\infty}B(m,\lambda,v)=\lim_{m\to\infty}\bigg[\sum_{j=1}^{m}[c_j(v)]^2+\lambda\bigg[\|v\|^2-\sum_{j=1}^{m}[c_j(v)]^2\bigg]\bigg]=\|v\|^2,$$

where the c_js are given by (4.1). From the above equation we have the following theorem by Theorems 3.1, 3.2, 3.3, 4.1, 4.2, 4.3 and the dominated convergence theorem.

Theorem 4.4. *Let* $\lambda\in\mathbb{C}_+$, q *be a nonzero real number,* $\{\lambda_m\}_{m=1}^{\infty}$ *be a sequence in* $\mathbb{C}_+$ *converging to* $-\mathrm{i}q$ *as* m *approaches* ∞ *and let* a *be absolutely continuous on* $[0,t]$. *Let* $F\in\mathcal{S}_{\mathrm{w}_\varphi}$ *be given by* (3.1).

(1) *Under the assumptions given in Theorems* 3.1 *and* 4.1

$$E^{\mathrm{anw}_\lambda}[F_Z] = \lim_{m\to\infty} \lambda^{m/2} E[K_m(\lambda,x)F_Z(x)].$$

Moreover, $E^{\mathrm{anf}_q}[F_Z]$ *is given by the right hand side of the above equality after replacing* λ *by* λ_m*.*

(2) *Under the assumptions given in Theorems* 3.2 *and* 4.2*, we have for a.e.* $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$E^{\mathrm{anw}_\lambda}[F_Z|Z_{n+1}](\vec{\xi}_{n+1}) = \lim_{m\to\infty} \lambda^{m/2} E[K_m(\lambda,x)F(Z_{b,n+1}(x,\cdot)+P_{b,n+1}(\vec{\xi}_{n+1}))].$$

Moreover, for a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$*,* $E^{\mathrm{anf}_q}[F_Z|Z_{n+1}](\vec{\xi}_{n+1})$ *is given by the right hand side of the above equality after replacing* λ *by* λ_m*.*

(3) *Under the assumptions given in Theorems* 3.3 *and* 4.3*, we have for a.e.* $\vec{\xi}_n = (\xi_0,\xi_1,\ldots,\xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} E^{\mathrm{anw}_\lambda}[F_Z|Z_n](\vec{\xi}_n) = \lim_{m\to\infty} \lambda^{m/2} \int_{\mathbb{R}} &\Psi(\lambda,\xi_{n+1}-\xi_n,a(t)-a(t_n),b(t)-b(t_n)) \\ &\times E[K_m(\lambda,x)F(Z_{b,n+1}(x,\cdot)+P_{b,n+1}(\vec{\xi}_{n+1}))]\,\mathrm{d}\xi_{n+1}, \end{aligned}$$

where $\vec{\xi}_{n+1} = (\xi_0,\ldots,\xi_n,\xi_{n+1})$ *for* $\xi_{n+1} \in \mathbb{R}$*. Moreover, for a.e.* $\vec{\xi}_n \in \mathbb{R}^{n+1}$*,* $E^{\mathrm{anf}_q}[F_Z|Z_n](\vec{\xi}_n)$ *is given by the right hand side of the above equality after replacing* λ *by* λ_m*.*

Letting $\lambda = \varrho^{-2}$ in Theorem 4.4 we have the following change of scale formula for the generalized conditional Wiener integral on the analogue of the Wiener space using the polygonal function.

Corollary 4.2. *Let* $\varrho > 0$ *and let* $F \in \mathcal{S}_{\mathrm{w}_\varphi}$ *be given by* (3.1)*.*

(1) *Under the assumptions given in Theorems* 3.1 *and* 4.1

$$E[F(Z(\varrho x,\cdot))] = \lim_{m\to\infty} \varrho^{-m} E[K_m(\varrho^{-2},x)F(Z(x,\cdot))].$$

(2) *Under the assumptions given in Theorems* 3.2 *and* 4.2*, we have for a.e.* $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned} &E[F(Z(\varrho x,\cdot))|Z_{n+1}(\varrho x)](\vec{\xi}_{n+1}) \\ &\qquad = \lim_{m\to\infty} \varrho^{-m} E[K_m(\varrho^{-2},x)F(Z_{b,n+1}(x,\cdot)+P_{b,n+1}(\vec{\xi}_{n+1}))]. \end{aligned}$$

(3) *Under the assumptions given in Theorems* 3.3 *and* 4.3, *we have for a.e.* $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$

$$E[F(Z(\varrho x,\cdot))|Z_n(\varrho x)](\vec{\xi}_n) = \lim_{m\to\infty} \varrho^{-m} \int_{\mathbb{R}} \Psi(1,\xi_{n+1}-\xi_n, a(t)-a(t_n), \varrho^2[b(t)-b(t_n)]) \times E[K_m(\varrho^{-2},x)F(Z_{b,n+1}(x,\cdot)+P_{b,n+1}(\vec{\xi}_{n+1}))]\,\mathrm{d}\xi_{n+1},$$

where $\vec{\xi}_{n+1} = (\xi_0,\ldots,\xi_n,\xi_{n+1})$ *for* $\xi_{n+1} \in \mathbb{R}$.

Remark 4.1. (1) When $n = 0$, that is, $Z_0(x) = x(0) + a(0)$, Corollaries 3.1 and 4.1 say that $E^{\mathrm{anw}_\lambda}[F_Z|Z_0](\xi_0)$, $E^{\mathrm{anf}_q}[F_Z|Z_0](\xi_0)$ and $\Gamma(F,m,\lambda,\xi_0)$ are constant functions as functions of ξ_0 on $\mathbb{R}$. This means that all conditional integrals given $Z_0(x) = \xi_0$ in the corollaries are equal even though of the initial distribution φ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ is arbitrary.

(2) The conditioning functions X_{n+1} and Z_{n+1} contain the present positions of the generalized Wiener paths, but X_n and Z_n do not. Moreover, the conditioning functions X_n and X_{n+1} do not contain the initial positions of the generalized Wiener paths, but Z_n and Z_{n+1} contain them.

(3) If $h = 1$ and $a = 0$, then $Z_{n+1}(x) = (x(t_0), x(t_1), \ldots, x(t_n), x(t_{n+1}))$ and $Z_n(x) = (x(t_0), x(t_1), \ldots, x(t_n))$ so that the change of scale formulas for $F \in \mathcal{S}_{\mathrm{w}_\varphi}$ in this paper are exactly those in [11]. If $h = 1$, $a = 0$ and $\varphi = \delta_0$ is the Dirac measure concentrated at 0, the formulas for F in this paper are reduced to those in [18]. Moreover, if $a = 0$ and $\varphi = \delta_0$, then the change of scale formulas in [9], [10], [12] can be applied to F.

(4) All the results of this paper do not depend on a particular choice of the initial distribution φ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$.

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