Yuping Zeng; Feng Wang

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Applications of Mathematics, Vol. 62 (2017), No. 3, 243–267

Persistent URL: `http://dml.cz/dmlcz/146780`

A POSTERIORI ERROR ESTIMATES FOR A DISCONTINUOUS GALERKIN APPROXIMATION OF STEKLOV EIGENVALUE PROBLEMS

Yuping Zeng, Meizhou, Feng Wang, Nanjing

Received April 12, 2016. First published April 30, 2017.

Abstract. We derive a residual-based a posteriori error estimator for a discontinuous Galerkin approximation of the Steklov eigenvalue problem. Moreover, we prove the reliability and efficiency of the error estimator. Numerical results are provided to verify our theoretical findings.

Keywords: discontinuous Galerkin method; Steklov eigenvalue problem; a posteriori error estimate

MSC 2010: 65N15, 65N25, 65N30

1. Introduction

Steklov eigenvalue problems arise in many applications from mechanics and engineering science. For example, such spectral problems are found in the study of surface waves [12], in the stability of mechanical oscillators immersed in a viscous fluid [21], in the vibration modes of a structure in contact with an incompressible fluid [13]. Moreover, in one dimensional case, they have been applied to the study of vibrations of a pendulum [1] and eigenoscillations of mechanical systems [28].

Finite element methods are the most commonly used numerical methods for solving eigenvalue problems, for more details in this subject area please see [10] and references therein. In particular, the conforming finite element methods for Steklov eigenvalue problems are mature subjects, see [5], [7], [17] for a priori error estimates, [8], [14] for a posteriori error analysis, and [36], [46] for multigrid solvers. We also

This work was supported by NSFC under grants 11526097, 11371199, 11301275, 11371198 and 11601188, the opening fund of Jiangsu Key Lab for NSLSCS (Grant No. 201402), the Doctoral fund of Ministry of Education of China (Grant No. 20123207120001).

DOI: 10.21136/AM.2017.0115-16

refer the reader to [4], [34], [48] for the a priori estimates of nonconforming finite element approximations, [40] for the corresponding a posteriori error estimates, and [27] for multigrid methods. We should mention that, in [34], the authors considered problems in nonconvex domains. In contrast to the standard finite elements above, to our best knowledge, there exists no work on the discontinuous Galerkin (DG) method for Steklov eigenvalue problems. This paper shall make a first effort in this direction. More precisely, we will give a residual-based reliable and efficient a posteriori error estimator of DG methods for such problems, this is the main contribution of our work.

In recent years, DG methods have gained much interest due to their ease of treatment of highly unstructured meshes and inhomogeneous boundary conditions. Moreover they are suitable for hp-adaptive computation. A posteriori error estimates for DG approximations of source problems have been extensively explored in the literature, such as residual type [31], [32], equilibrated fluxes type [2], [16], gradient recovery type [41], and other error estimates measured in mesh dependent energy norms [11], [24], [38], [39], [43], [44], [47]. Moreover, the authors in [15], [29], [33] have further studied the convergence of adaptive DG methods. However, a posteriori analysis of DG method for eigenvalue problems is still very rare, for the standard Laplace eigenvalue problem we refer the reader to [26], [49]. In this work, we will further study the symmetric interior penalty discontinuous Galerkin methods (IPDG) for the Steklov eigenvalue problem. The main difficulty of the theoretical analysis stems from the complexity of bilinear forms of the IPDG method and its nonconformity. For addressing this problem, we rewrite the IPDG method in a nonconsistent way by introducing a lifting operator, and then decompose the error into a conforming and nonconforming parts that are estimated separately. Note that these techniques have been applied for source problems [15] and Laplace eigenvalue problems [26].

The rest of our paper is structured as follows. In Section 2, we first introduce the model problem and then describe the IPDG method. In Section 3, we present a residual-based a posteriori error estimator and prove its reliability and efficiency. Finally, some numerical tests demonstrating our theoretical results are provided in Section 4.

2. Model problem and discontinuous Galerkin methods

We consider the following Steklov eigenvalue model problem:

$$
\begin{aligned}
-\Delta u + u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \mathbf{n}} &= \lambda u \quad \text{on } \Gamma.
\end{aligned}
\tag{2.1}
$$

Here $\Omega \subset \mathbb{R}^2$ denotes a bounded polygonal domain and $\Gamma = \partial\Omega$. The fraction $\partial u/\partial \mathbf{n}$ represents the outward normal derivative.

We first introduce some notation. Given a polygonal domain $\mathcal{D} \subset \mathbb{R}^2$, $H^s(\mathcal{D})$ ($s \geqslant 0$) denotes the standard Sobolev space, equipped with the norm $\|\cdot\|_{s,\mathcal{D}}$ and seminorm $|\cdot|_{s,\mathcal{D}}$. When $s = 0$, $H^0(\mathcal{D})$ is the standard $L^2(\mathcal{D})$ space, and the inner product in $L^2(\mathcal{D})$ is denoted by $(\cdot,\cdot)_D$. For convenience, we set $V = H^1(\Omega)$.

It is well known that the weak formulation of the Steklov eigenvalue problem is to find $(\lambda, u) \in \mathbb{R} \times V$ with $\|u\|_b = 1$ satisfying

$$a(u,v) = \lambda b(u,v) \quad \forall\, v \in V, \tag{2.2}$$

where

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\,\mathrm{d}x,$$
$$b(u,v) = \int_\Gamma uv\,\mathrm{d}s, \quad \|u\|_b = (b(u,u))^{1/2}.$$

The corresponding source problem associated with (2.2) reads: Find $u \in V$ such that

$$a(u,v) = b(f,v) \quad \forall\, v \in V. \tag{2.3}$$

We now recall the following regularity estimates for the above source problem (see [17], [13]).

Lemma 2.1. *If $f \in L^2(\partial\Omega)$, the solution u of the source problem* (2.3) *satisfies $u \in H^{1+r/2}(\Omega)$ with $r \in (\frac{1}{2}, 1]$ and*

$$\|u\|_{1+r/2} \lesssim \|f\|_b. \tag{2.4}$$

For the case that $f \in H^{1/2}(\partial\Omega)$, we have $u \in H^{1+r}(\Omega)$ and

$$\|u\|_{1+r} \lesssim \|f\|_{1/2,\partial\Omega}. \tag{2.5}$$

From [10], we know that the eigenvalue problem (2.1) has a countable infinite set of positive eigenvalues, which are ordered increasingly by $\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_j \leqslant \ldots \to \infty$. The corresponding space of all eigenfunctions associated with λ_j is denoted by $M(\lambda_j)$. Set $\widehat{M}(\lambda_j) = \{v\colon\, v \in M(\lambda_j), \|v\|_b = 1\}$. From Lemma 2.1, we assume that the eigenfunctions $u_j \in M(\lambda_j)$ satisfy $u_j \in H^{1+r}(\Omega)$ with $r \in (\frac{1}{2}, 1]$.

Consider a conforming shape-regular family of meshes $\mathcal{T}_h$ that partition the domain Ω into triangles $\{T\}$. Let h_T be the diameter of an element T and set $h = \max\limits_{T \in \mathcal{T}_h} h_T$. The set of interior edges and the set of edges on Γ are denoted by

$\mathcal{E}_h^I$ and $\mathcal{E}_h^\Gamma$, respectively. Then the set of all edges is $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^\Gamma$. The set of three edges of an element T is denoted by $\mathcal{E}_h^T$, i.e. $\mathcal{E}_h^T = \{e \in \mathcal{E}_h \colon e \subset \partial T\}$. We also use h_e to denote the length of the edge $e \in \mathcal{E}_h$. We use $\mathbb{P}_k(\mathcal{D})$ to denote the space of polynomials of degree at most k on $\mathcal{D}$. Moreover, each edge $e \in \mathcal{E}_h$ is associated with a fixed unit normal vector $\mathbf{n}$. To simplify the notation, we will use $a \lesssim b$ to indicate that $a \leqslant Cb$ with C being a constant independent of the mesh size h.

For an interior edge e shared by two elements T_1 and T_2, we denote the corresponding outward normal unit vectors by $\mathbf{n}^1$ and $\mathbf{n}^2$. Given a scalar piecewise smooth function v with $v^i = v|_{T_i}$, we define the averages and jumps by

$$\{v\} = \frac{1}{2}(v^1 + v^2), \quad [\![v]\!] = v^1\mathbf{n}^1 + v^2\mathbf{n}^2.$$

Similarly, for a vector piecewise smooth function $\mathbf{w}$ with $\mathbf{w}^i = \mathbf{w}|_{T_i}$, we define

$$\{\mathbf{w}\} = \frac{1}{2}(\mathbf{w}^1 + \mathbf{w}^2), \quad [\![\mathbf{w}]\!] = \mathbf{w}^1 \cdot \mathbf{n}^1 + \mathbf{w}^2 \cdot \mathbf{n}^2.$$

We will use the discontinuous $\mathbb{P}_k$ finite element space

$$V_h = \{v \in L^2(\Omega) \colon v|_T \in \mathbb{P}_k(T)\ \forall\, T \in \mathcal{T}_h\}.$$

The bilinear form of the symmetric IPDG method is defined by (see [9])

$$\begin{aligned} a_h(w,v) = {} & \sum_{T \in \mathcal{T}_h} \int_T (\nabla w \cdot \nabla v + wv)\,\mathrm{d}x - \sum_{e \in \mathcal{E}_h^I} \int_e \{\nabla w\} \cdot [\![v]\!]\,\mathrm{d}s \\ & - \sum_{e \in \mathcal{E}_h^I} \int_e \{\nabla v\} \cdot [\![w]\!]\,\mathrm{d}s + \sum_{e \in \mathcal{E}_h^I} \int_e \gamma [\![w]\!] \cdot [\![v]\!]\,\mathrm{d}s, \end{aligned} \tag{2.6}$$

where $\gamma = \sigma h_e^{-1}$ $(\sigma > 0)$ is the interior penalty parameter. We choose σ to be sufficiently large to have coercivity. From Remark 2.1 in [30], in the actual numerical implementations we can set $\sigma = C_I k^2$ with $C_I = 10$.

Then the discontinuous Galerkin method for solving the Steklov eigenvalue problem (2.1) is to find $(\lambda_h, u_h) \in \mathbb{R} \times V_h$ with $\|u_h\|_b = 1$ satisfying

$$a_h(u_h, v_h) = \lambda_h b(u_h, v_h) \quad \forall\, v_h \in V_h. \tag{2.7}$$

This numerical scheme has eigenvalues that can be ordered by $\lambda_{1,h} \leqslant \lambda_{2,h} \leqslant \ldots \leqslant \lambda_{j,h} \leqslant \ldots \leqslant \lambda_{N,h}$ with $N = \dim V_h$, and the corresponding eigenfunction associated with $\lambda_{j,h}$ is denoted by $u_{j,h}$.

For the a priori error analysis, we first define a DG norm $|||\cdot|||_h$ on $V(h) = V_h + V$ by

$$\text{(2.8)}\quad |||v|||_h = \Big(\|v\|_{0,\Omega}^2 + \sum_{T\in\mathcal{T}_h}\|\nabla v\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h^I} h_e\|\{\nabla v\}\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h^I}\|\gamma^{1/2}[\![v]\!]\|_{0,e}^2\Big)^{1/2}.$$

Moreover, similarly to [26], we introduce the following definitions to measure the distance of a discrete eigenfunction from the exact eigenspace.

Definition 2.2. For a function $x \in L^2(\Omega)$ and a finite dimensional subspace $Y \subset L^2(\Omega)$, we define

$$\text{dist}(x, Y)_b = \min_{y\in Y}\|x - y\|_b. \tag{2.9}$$

Similarly, for a function $x \in V_h$ and a finite dimensional subspace $Y \subset V$, we define

$$\text{dist}(x, Y)_{\text{DG}} = \min_{y\in Y}|||x - y|||_h. \tag{2.10}$$

The following result shows the continuity and coercivity of the bilinear form $a_h(\cdot,\cdot)$ (see e.g. [32], [33]).

Lemma 2.3. *We have*

$$|a_h(w, v)| \leqslant 2|||w|||_h|||v|||_h \quad \forall\, w, v \in V(h). \tag{2.11}$$

Moreover, there exists $\sigma^* > 0$ such that for all $\sigma > \sigma^*$

$$a_h(v, v) \geqslant C_a|||v|||_h^2 \quad \forall\, v \in V_h, \tag{2.12}$$

with $C_a > 0$ which depends only on the minimum angle of the mesh.

The following a priori error estimates can be obtained by using the standard technical tools of finite element approximation for eigenvalue problems (cf. [4], [6], [7], [10], [26], [34], [48]).

Theorem 2.4. *Let $(\lambda_{j,h}, u_{j,h})$ be the solution of (2.7), and assume that $\lambda_{j,h}$ converges to the exact eigenvalue λ_j that has multiplicity $m \geqslant 1$. Then for sufficiently small h,*

$$\text{dist}(u_{j,h}, \widehat{M}(\lambda_j))_{\text{DG}} \lesssim h^r, \tag{2.13}$$

$$\text{dist}(u_{j,h}, \widehat{M}(\lambda_j))_b \lesssim h^{3r/2}, \tag{2.14}$$

$$|\lambda_j - \lambda_{j,h}| \lesssim h^{2r}. \tag{2.15}$$

R e m a r k 2.5. To obtain the above a priori error estimates, we can resort to some results from references [9], [6], where DG methods for the standard Laplace source problems and eigenvalue problems were analyzed, respectively.

R e m a r k 2.6. We only consider triangular meshes and assume that they are conforming; how to extend the results to the meshes with hanging nodes is an interesting work which relies on some techniques developed in [3], [22]. Our results can be extended to quadrilateral meshes with minor modifications. It should be pointed out that if quadrilateral meshes were used, the theoretical results in this paper would be immediately available for meshes containing hanging nodes [26].

3. A posteriori error analysis

We first recall the lifting operator $\mathcal{L}\colon V(h) \to [V_h]^2$ which is useful in the subsequent error analysis (see e.g. [37]):

$$\int_\Omega \mathcal{L}(v)\cdot \mathbf{w}\,\mathrm{d}x = \sum_{e\in\mathcal{E}_h^I}\int_e [\![v]\!]\cdot\{\mathbf{w}\}\,\mathrm{d}s \quad \forall\,\mathbf{w}\in[V_h]^2. \tag{3.1}$$

Moreover, from [37] we can see that the lifting operator has the stability property

$$\|\mathcal{L}(v)\|_{0,\Omega}^2 \lesssim \sum_{e\in\mathcal{E}_h^I} \|h_e^{-1/2}[\![v]\!]\|_{0,e}^2. \tag{3.2}$$

Using this operator, we introduce an auxiliary bilinear form

$$\tilde a_h(\cdot,\cdot)\colon\ V(h)\times V(h)\to\mathbb{R}$$

defined by

$$\begin{aligned}\tilde a_h(w,v) = {} & \sum_{T\in\mathcal{T}_h}\int_T(\nabla w\cdot\nabla v + wv)\,\mathrm{d}x - \sum_{T\in\mathcal{T}_h}\int_T \mathcal{L}(v)\cdot\nabla w\,\mathrm{d}x \\ & - \sum_{T\in\mathcal{T}_h}\int_T\mathcal{L}(w)\cdot\nabla v\,\mathrm{d}x + \sum_{e\in\mathcal{E}_h^I}\int_e \gamma[\![w]\!]\cdot[\![v]\!]\,\mathrm{d}s.\end{aligned} \tag{3.3}$$

Since $\tilde a_h = a_h$ on $V_h\times V_h$, the DG method presented in (2.7) is equivalent to finding $(\lambda_h, u_h)\in\mathbb{R}\times V_h$ with $\|u_h\|_b = 1$ satisfying

$$\tilde a_h(u_h, v_h) = \lambda_h b(u_h, v_h) \quad \forall\, v_h\in V_h. \tag{3.4}$$

We then define another energy norm $\|\cdot\|_h$ on $V(h)$ by

$$\|v\|_h = \Big(\|v\|_{0,\Omega}^2 + \sum_{T\in\mathcal{T}_h} \|\nabla v\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h^I} \|\gamma^{1/2}[\![v]\!]\|_{0,e}^2\Big)^{1/2}. \tag{3.5}$$

It is easy to see that $\tilde{a}_h = a$ on $V \times V$, thus

$$\tilde{a}_h(v,v) = \|\nabla v|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2 = \|v\|_h^2 \quad \forall\, v \in V. \tag{3.6}$$

We will make use of the following continuity property of the bilinear form $\tilde{a}_h(\cdot,\cdot,)$ (see [37]).

Lemma 3.1. *We have*

$$|\tilde{a}_h(w,v)| \lesssim \|w\|_h \|v\|_h \quad \forall\, w,v \in V(h). \tag{3.7}$$

Moreover, we introduce the following distance derived from the energy norm $\|\cdot\|_h$.

Definition 3.2. For a function $x \in V_h$ and a finite dimensional subspace $Y \subset V$, we define

$$\operatorname{dist}(x,Y)_E = \min_{y\in Y} \|x-y\|_h. \tag{3.8}$$

We also need the following fact which shows that any discontinuous function can be approximated by a continuous one (see [18], [32]).

Lemma 3.3. *For any $v \in V_h$, there exits an enrichment operator $E_h\colon\ V_h \to V_h \cap V$ such that*

$$\sum_{T\in\mathcal{T}_h} (h_T^{-2}\|v - E_h v\|_{0,T}^2 + \|\nabla(v - E_h v)\|_{0,T}^2) \lesssim \Big(\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|[\![v]\!]\|_{0,e}^2\Big). \tag{3.9}$$

Finally, we introduce the Clément or Scott-Zhang interpolation operator [20], [42].

Lemma 3.4. *Denote by U_h the conforming $\mathbb{P}_1$ finite element space. For any $\psi \in V$, there is a piecewise linear interpolation $\mathcal{J}v \in U_h$ satisfying*

$$\|\nabla(\psi - \mathcal{J}\psi)\|_{0,T} \lesssim \|\nabla\psi\|_{0,w_T} \quad \forall\, T \in \mathcal{T}_h, \tag{3.10}$$

$$\|\psi - \mathcal{J}\psi\|_{0,T} \lesssim h_T\|\nabla\psi\|_{0,w_T} \quad \forall\, T \in \mathcal{T}_h, \tag{3.11}$$

$$\|\psi - \mathcal{J}\psi\|_{0,e} \lesssim h_e^{1/2}\|\nabla\psi\|_{0,w_e} \quad \forall\, e \in \mathcal{E}_h, \tag{3.12}$$

where w_T is the set of all elements which share at least one node with T, while w_e is the set of all elements which share at least one node with e.

We then introduce our local error estimators. Let $(\lambda_{j,h}, u_{j,h})$ be an approximate eigenpair. For any edge e, we define jumps by

$$J_{j,F} = \begin{cases} [\![\nabla u_{j,h}]\!] & \forall\, e \in \mathcal{E}_h^I, \\ \lambda_{j,h} u_{j,h} - \nabla u_{j,h} \cdot \mathbf{n} & \forall\, e \in \mathcal{E}_h^\Gamma, \end{cases}$$

and

$$J_{j,U} = [\![u_{j,h}]\!] \quad \forall\, e \in \mathcal{E}_h^I.$$

Then we define the local error estimator on each element $T \in \mathcal{T}_h$ as

$$\eta_{j,T}^2 = h_T^2 \|\Delta u_{j,h} - u_{j,h}\|_{0,T}^2 + \frac{1}{2} \sum_{e \in \mathcal{E}_h^I \cap \mathcal{E}_h^T} h_e \|J_{j,F}\|_{0,e}^2 + \sum_{e \in \mathcal{E}_h^\Gamma \cap \mathcal{E}_h^T} h_e \|J_{j,F}\|_{0,e}^2 + \frac{1}{2} \sum_{e \in \mathcal{E}_h^I \cap \mathcal{E}_h^T} \|\gamma^{1/2} J_{j,U}\|_{0,e}^2. \tag{3.13}$$

3.1. Reliability. This subsection aims at proving that the error estimator is reliable.

Lemma 3.5. *Let (λ_j, u_j) and $(\lambda_{j,h}, u_{j,h})$ be the solutions of* (2.2) *and* (2.7), *respectively. For any $v \in V$, we have*

$$\int_\Gamma \lambda_j u_j (v - \mathcal{J} v)\,\mathrm{d}x - \tilde{a}_h(u_{j,h}, v - \mathcal{J} v) \lesssim (\eta_j + h^{1/2} \|\lambda_j u_j - \lambda_{j,h} u_{j,h}\|_b) \|v\|_h, \tag{3.14}$$

with $\eta_j = \Big(\sum\limits_{T \in \mathcal{T}_h} \eta_{j,T}^2\Big)^{1/2}$.

Proof. For convenience, we set

$$B = \int_\Gamma \lambda_j u_j (v - \mathcal{J} v)\,\mathrm{d}x - \tilde{a}_h(u_{j,h}, v - \mathcal{J} v). \tag{3.15}$$

Integrating by parts gives

$$\begin{aligned} B = {} & \sum_{T \in \mathcal{T}_h} \int_T (\Delta u_{j,h} - u_h)(v - \mathcal{J} v)\,\mathrm{d}x + \sum_{T \in \mathcal{T}_h} \int_T \mathcal{L}(u_{j,h}) \nabla (v - \mathcal{J} v)\,\mathrm{d}x \\ & - \sum_{e \in \mathcal{E}_h^I} \int_e [\![\nabla u_{j,h}]\!](v - \mathcal{J} v)\,\mathrm{d}s + \int_\Gamma (\lambda_j u_j - \nabla u_{j,h} \cdot \mathbf{n})(v - \mathcal{J} v)\,\mathrm{d}s \\ \equiv {} & B_1 + B_2 + B_3 + B_4. \end{aligned} \tag{3.16}$$

Using the Cauchy-Schwarz inequality and the approximation property (3.11), we have

$$
\begin{aligned}
B_1 &= \sum_{T\in\mathcal{T}_h}\int_T (\Delta u_{j,h} - u_{j,h})(v-\mathcal{J}v)\,\mathrm{d}x \\
&\lesssim \Big(\sum_{T\in\mathcal{T}_h} h_T^2\|\Delta u_{j,h} - u_{j,h}\|_{0,T}^2\Big)^{1/2}\|v\|_{1,\Omega} \\
&= \Big(\sum_{T\in\mathcal{T}_h} h_T^2\|\Delta u_{j,h} - u_{j,h}\|_{0,T}^2\Big)^{1/2}\|v\|_h,
\end{aligned}
\tag{3.17}
$$

where we have used the fact stated in (3.6).

Using the Cauchy-Schwarz inequality, the stability of the lifting operator in (3.2) and the approximation property (3.10), we can bound the term B_2 by

$$
\begin{aligned}
B_2 &\lesssim \Big(\sum_{T\in\mathcal{T}_h}\|\mathcal{L}(u_{j,h})\|_{0,T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|\nabla(v-\mathcal{J}v)\|_{0,T}^2\Big)^{1/2} \\
&\lesssim \Big(\sum_{e\in\mathcal{E}_h^I}\|h_e^{-1/2}[\![u_{j,h}]\!]\|_{0,e}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|\nabla(v-\mathcal{J}v)\|_{0,T}^2\Big)^{1/2} \\
&\lesssim \Big(\sum_{e\in\mathcal{E}_h^I}\|\gamma^{1/2}J_{j,U}\|_{0,e}^2\Big)^{1/2}\|v\|_{1,\Omega} = \Big(\sum_{e\in\mathcal{E}_h^I}\|\gamma^{1/2}J_{j,U}\|_{0,e}^2\Big)^{1/2}\|v\|_h.
\end{aligned}
\tag{3.18}
$$

Using again the Cauchy-Schwarz inequality and the approximation property (3.12), we obtain that

$$
\begin{aligned}
B_3 &= -\sum_{e\in\mathcal{E}_h^I}\int_e [\![\nabla u_{j,h}]\!](v-\mathcal{J}v)\,\mathrm{d}s \\
&\lesssim \Big(\sum_{e\in\mathcal{E}_h^I} h_e\|[\![\nabla u_{j,h}]\!]\|_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h^I} h_e^{-1}\|v-\mathcal{J}v\|_{0,e}^2\Big)^{1/2} \\
&\lesssim \Big(\sum_{e\in\mathcal{E}_h^I} h_e\|J_{j,F}\|_{0,e}^2\Big)^{1/2}\|v\|_h.
\end{aligned}
\tag{3.19}
$$

The last term B_4 can be bounded by

$$
B_4 = \int_\Gamma (\lambda_{j,h}u_{j,h} - \nabla u_{j,h}\cdot\mathbf{n})(v-\mathcal{J}v)\,\mathrm{d}s + \int_\Gamma(\lambda_j u_j - \lambda_{j,h}u_{j,h})(v-\mathcal{J}v)\,\mathrm{d}s \tag{3.20}
$$

$$\lesssim \Big(\sum_{e\in\mathcal{E}_h^\Gamma} h_e\|\lambda_{j,h}u_{j,h}-\nabla u_{j,h}\cdot\mathbf{n}\|_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h^\Gamma} h_e^{-1}\|v-\mathcal{J}v\|_{0,e}^2\Big)^{1/2}$$
$$+\Big(\sum_{e\in\mathcal{E}_h^\Gamma} h_e\|\lambda_j u_j-\lambda_{j,h}u_{j,h}\|_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h^\Gamma} h_e^{-1}\|v-\mathcal{J}v\|_{0,e}^2\Big)^{1/2}$$
$$\lesssim \Big(\sum_{e\in\mathcal{E}_h^\Gamma} h_e\|J_{j,F}\|_{0,e}^2\Big)^{1/2}\|v\|_h+\Big(\sum_{e\in\mathcal{E}_h^\Gamma} h_e\|\lambda_j u_j-\lambda_{j,h}u_{j,h}\|_{0,e}^2\Big)^{1/2}\|v\|_h.$$

Combining the bounds of B_1–B_4 gives the desired result (3.14). We completed the proof. □

Lemma 3.6. *Let (λ_j,u_j) and $(\lambda_{j,h},u_{j,h})$ be the solutions of* (2.2) *and* (2.7), *respectively. Then we have*

$$\|u_j-u_{j,h}\|_h\lesssim \eta_j+(h^{1/2}+1)\|\lambda_j u_j-\lambda_{j,h}u_{j,h}\|_b. \tag{3.21}$$

P r o o f. The error $e_j=u_j-u_{j,h}$ can be split into two parts, i.e., $e_j=e_{j,c}+e_{j,d}$ with $e_{j,c}=u_j-E_hu_{j,h}$ and $e_{j,d}=E_hu_{j,h}-u_{j,h}$, where E_h is the enriching operator stated in Lemma 3.3. Applying the triangle inequality yields

$$\|u_j-u_{j,h}\|_h\leqslant\|e_{j,c}\|_h+\|e_{j,d}\|_h.$$

For the latter term $\|e_{j,d}\|_h$, using the result stated in (3.9) and the shape-regularity of the mesh, noting that $[\![E_hu_{j,h}]\!]=0$, we have

$$\begin{aligned}\|e_{j,d}\|_h^2&=\|E_hu_{j,h}-u_{j,h}\|_{0,\Omega}^2+\sum_{T\in\mathcal{T}_h}\|\nabla(E_hu_{j,h}-u_{j,h})\|_{0,T}^2\\&\quad+\sum_{e\in\mathcal{E}_h^I}\|\gamma^{1/2}[\![E_hu_{j,h}-u_{j,h}]\!]\|_{0,e}^2\lesssim\sum_{e\in\mathcal{E}_h^I}\|h_e^{1/2}[\![u_{j,h}]\!]\|_{0,e}^2\\&\quad+\sum_{e\in\mathcal{E}_h^I}\|h_e^{-1/2}[\![u_{j,h}]\!]\|_{0,e}^2+\sum_{e\in\mathcal{E}_h^I}\|\gamma^{1/2}[\![u_{j,h}]\!]\|_{0,e}^2\\&\lesssim\sum_{e\in\mathcal{E}_h^I}\|\gamma^{1/2}J_{j,U}\|_{0,e}^2.\end{aligned}\tag{3.22}$$

Next, we will bound the second term $\|e_{j,c}\|_h$. Noting that $e_{j,c}\in H^1(\Omega)$, from (3.6) we obtain

$$\|e_{j,c}\|_h^2=\tilde{a}_h(u_j-E_hu_{j,h},v)\tag{3.23}$$

with $v=e_{j,c}=u_j-E_hu_{j,h}$.

It follows from (3.6) that $\tilde{a}_h(u_j, v) = a(u_j, v) = \int_\Gamma \lambda_j u_j v \,\mathrm{d}x$, hence we have

$$\begin{aligned}\tilde{a}_h(u_j - E_h u_{j,h}, v) &= \int_\Gamma \lambda_j u_j v \,\mathrm{d}x - \tilde{a}_h(E_h u_{j,h}, v) \\ &= \int_\Gamma \lambda_j u_j v \,\mathrm{d}x - \tilde{a}_h(u_{j,h}, v) + \tilde{a}_h(u_{j,h} - E_h u_{j,h}, v).\end{aligned} \tag{3.24}$$

Setting $v_h = \mathcal{J}v$ in (3.4) implies that $\tilde{a}_h(u_{j,h}, \mathcal{J}v) = \int_\Gamma (\lambda_{j,h} u_{j,h}) \mathcal{J}v \,\mathrm{d}x$, thus we have

$$\begin{aligned}\tilde{a}_h(u_j - E_h u_{j,h}, v) = {}& \int_\Gamma (\lambda_j u_j - \lambda_{j,h} u_{j,h}) \mathcal{J}v \,\mathrm{d}x + \int_\Gamma \lambda_j u_j (v - \mathcal{J}v) \,\mathrm{d}x \\ & - \tilde{a}_h(u_{j,h}, v - \mathcal{J}v) + \tilde{a}_h(u_{j,h} - E_h u_{j,h}, v).\end{aligned} \tag{3.25}$$

In view of (3.12) and applying the trace theorem $\|v\|_{0,\Gamma} \lesssim \|v\|_{1,\Omega}$ for all $v \in V$, we have

$$\begin{aligned}\|\mathcal{J}v\|_{0,\Gamma} \leqslant \|v - \mathcal{J}v\|_{0,\Gamma} + \|v\|_{0,\Gamma} &\lesssim h^{1/2} \|\nabla v\|_{0,\Omega} + \|v\|_{0,\Gamma} \\ &\lesssim (h^{1/2} + 1)\|v\|_{1,\Omega} = (h^{1/2} + 1)\|v\|_h,\end{aligned} \tag{3.26}$$

which together with (3.25) yields

$$\begin{aligned}\tilde{a}_h(u_j - E_h u_{j,h}, v) &\lesssim (h^{1/2} + 1)\|\lambda_j u_j - \lambda_{j,h} u_{j,h}\|_b \|v\|_h \\ &+ \int_\Gamma \lambda_j u_j (v - \mathcal{J}v) \,\mathrm{d}x - \tilde{a}_h(u_{j,h}, v - \mathcal{J}v) + \tilde{a}_h(u_{j,h} - E_h u_{j,h}, v).\end{aligned} \tag{3.27}$$

Using the result in Lemma 3.5 and (3.7), we obtain

$$\tilde{a}_h(u_j - E_h u_{j,h}, v) \lesssim (\eta_j + (h^{1/2} + 1)\|\lambda_j u_j - \lambda_{j,h} u_{j,h}\|_b)\|v\|_h + \|e_{j,d}\|_h \|v\|_h. \tag{3.28}$$

Combining (3.22), (3.23), and (3.28) gives the desired result stated in (3.21). □

We are now in a position to show that the error estimator η_j gives an upper bound for $\mathrm{dist}(u_{j,h}, \widehat{M}(\lambda_j))_E$ up to an asymptotic high order term.

Theorem 3.7. *Let $(\lambda_{j,h}, u_{j,h})$ be the solution of (2.7), and assume that $\lambda_{j,h}$ converges to the exact eigenvalue λ_j that has multiplicity $m \geqslant 1$. For the minimizer u_j of (2.9) with $Y = \widehat{M}(\lambda_j)$ we have*

$$\mathrm{dist}(u_{j,h}, \widehat{M}(\lambda_j))_E \lesssim \eta_j + (h^{1/2} + 1)\|\lambda_j u_j - \lambda_{j,h} u_{j,h}\|_b. \tag{3.29}$$

P r o o f. Combining (3.21) and the definition in (3.8) implies the desired result.

Next, we will show that $\|\lambda_j u_j - \lambda_{j,h} u_{j,h}\|_b$ is a higher order term. To this end, we notice that

$$\|\lambda_j u_j - \lambda_{j,h} u_{j,h}\|_b \leqslant |\lambda_j - \lambda_{j,h}| \|u_{j,h}\|_b + \lambda_j \|u_j - u_{j,h}\|_b.$$

The a priori error estimates in Theorem 2.4 show that both $|\lambda_j - \lambda_{j,h}|$ and $\|u_j - u_{j,h}\|_b$ are terms of higher order than $\|u_j - u_{j,h}\|_h$. □

R e m a r k 3.8. To show $\|\lambda_j u_j - \lambda_{j,h} u_{j,h}\|_b$ is a term of higher order than $\|u_j - u_{j,h}\|_h$, we have used the a priori error estimates in Theorem 2.4 which are based on the assumptions that $u_j \in H^{1+r}(\Omega)$, stated in Lemma 2.1, and that the meshes are uniform. The extension to general regular meshes can be obtained by following some ideas used in Theorem 4.8 from [25].

Next, we will give error estimates for the eigenvalues.

Lemma 3.9. *Let* (λ_j, u_j) *and* $(\lambda_{j,h}, u_{j,h})$ *be the solutions of* (2.2) *and* (2.7), *respectively. Then*

$$a_h(u_j - u_{j,h}, u_j - u_{j,h}) = \lambda_{j,h} - \lambda_j + \lambda_j \|u_j - u_{j,h}\|_b^2 + 2\mathcal{R}_j(u_j, u_j - u_{j,h}), \tag{3.30}$$

with $\mathcal{R}_j(u_j, u_j - u_{j,h}) = a_h(u_j, u_j - u_{j,h}) - \lambda_j b(u_j, u_j - u_{j,h})$.

P r o o f. Noting that $\|u_j\|_b = 1$ and $\|u_{j,h}\|_b = 1$, we have

$$\begin{aligned}
&\tilde{a}_h(u_j - u_{j,h}, u_j - u_{j,h}) \\
&\quad = \tilde{a}_h(u_j, u_j) + \tilde{a}_h(u_{j,h}, u_{j,h}) - 2\tilde{a}_h(u_j, u_{j,h}) \\
&\quad = \lambda_j \|u_j\|_b^2 + \lambda_{j,h} \|u_h\|_b^2 - 2\tilde{a}_h(u_j, u_{j,h}) \\
&\quad = \lambda_j + \lambda_{j,h} - 2\tilde{a}_h(u_j, u_{j,h}) \\
&\quad = \lambda_j + \lambda_{j,h} - \lambda_j \|u_j\|_b^2 - \lambda_j \|u_{j,h}\|_b^2 + (\lambda_j \|u_j\|_b^2 + \lambda_j \|u_{j,h}\|_b^2 \\
&\qquad - 2\lambda_j b(u_j, u_{j,h})) - 2\tilde{a}_h(u_j, u_{j,h}) + 2\lambda_j b(u_j, u_{j,h}) \\
&\quad = \lambda_{j,h} - \lambda_j + \lambda_j \|u_j - u_{j,h}\|_b^2 - 2\tilde{a}_h(u_j, u_{j,h}) + 2\lambda_j b(u_j, u_{j,h}), \\
&\quad = \lambda_{j,h} - \lambda_j + \lambda_j \|u_j - u_{j,h}\|_b^2 + 2\tilde{a}_h(u_j, u_j - u_{j,h}) - 2\lambda_j b(u_j, u_j - u_{j,h}),
\end{aligned}$$

which implies the desired result. □

Theorem 3.10. *Let* $(\lambda_{j,h}, u_{j,h})$ *be the solution of* (2.7), *and assume that* $\lambda_{j,h}$ *converge to the exact eigenvalue* λ_j *that has multiplicity* $m \geqslant 1$. *For the minimizer* u_j *of* (2.9) *with* $Y = \widehat{M}(\lambda_j)$ *and the minimizer* $\widehat{u}_j$ *of* (3.8) *with* $Y = \widehat{M}(\lambda_j)$, *we have*

$$|\lambda_j - \lambda_{j,h}| \lesssim \eta_j^2 + (h^{1/2} + 1)^2 \|\lambda_j u_j - \lambda_{j,h} u_{j,h}\|_b^2 + 2|\mathcal{R}_j(\widehat{u}_j, \widehat{u}_j - u_{j,h})|. \tag{3.31}$$

P r o o f. Since $\lambda_j\|u_j - u_{j,h}\|_b^2 > 0$, and using (3.7) for Lemma 3.9, we have

$$|\lambda_j - \lambda_{j,h}| \lesssim \operatorname{dist}(u_{j,h}, \widehat{M}(\lambda_j))_E^2 + 2|\mathcal{R}_j(\widehat{u}_j, \widehat{u}_j - u_{j,h})|.$$

It follows from Theorem 3.7 that

$$\begin{aligned}|\lambda_j - \lambda_{j,h}| &\lesssim (\eta_j + (h^{1/2}+1)\|\lambda_j u_j - \lambda_{j,h}u_{j,h}\|_b)^2 + 2|\mathcal{R}_j(\widehat{u}_j, \widehat{u}_j - u_{j,h})| \\ &\lesssim \eta_j^2 + (h^{1/2}+1)^2\|\lambda_j u_j - \lambda_{j,h}u_{j,h}\|_b^2 + 2|\mathcal{R}_j(\widehat{u}_j, \widehat{u}_j - u_{j,h})|.\end{aligned}$$

□

R e m a r k 3.11. As in Theorem 3.7, the term $(h^{1/2}+1)^2\|\lambda_j u_j - \lambda_{j,h}u_{j,h}\|_b^2$ can be seen as a higher order term. The nonconsistent term $\mathcal{R}_j(\widehat{u}_j, \widehat{u}_j - u_{j,h})$ appears since we have used the nonconforming discontinuous Galerkin methods, this term equals zero if conforming finite element methods are used. We have not proved that $\mathcal{R}_j(\widehat{u}_j, \widehat{u}_j - u_{j,h})$ is a term of higher order than $|\lambda_j - \lambda_{j,h}|$, but the numerical results in Section 4 show that η_j^2 is a reliable and efficient error estimator for the eigenvalue error $|\lambda_j - \lambda_{j,h}|$.

3.2. Efficiency. This subsection is devoted to the proof of efficiency of the error estimator, which relies on the bubble function techniques developed by Verfürth [45]. Let b_T be the standard polynomial bubble function on an element T. Similarly, the polynomial bubble function on an edge e is denoted by b_e. For an interior edge $e \in \mathcal{E}_h^I$ belonging to elements T and T', we have $b_e \in H_0^1(\overline{T} \cup \overline{T}')$. For a boundary edge $e \in \mathcal{E}_h^\Gamma$ with $e \subset \partial T$, we have $b_e|_{\partial T\setminus e} = 0$. The following standard results can be seen from [45].

Lemma 3.12. *For all polynomial functions $v \in \mathbb{P}_k(T)$,*

$$\|b_T v\|_{0,T} \lesssim \|v\|_{0,T}, \tag{3.32}$$

$$\|v\|_{0,T} \lesssim \|b_T^{1/2} v\|_{0,T}, \tag{3.33}$$

$$\|\nabla(b_T v)\|_{0,T} \lesssim h_T^{-1}\|v\|_{0,T}. \tag{3.34}$$

Similarly, for all polynomial functions $w \in \mathbb{P}_k(e)$, we have

$$\|b_e w\|_{0,e} \lesssim \|w\|_{0,e}, \tag{3.35}$$

$$\|w\|_{0,e} \lesssim \|b_e^{1/2} w\|_{0,e}. \tag{3.36}$$

Further, for each $b_e w$, there exists an extension W_b satisfying $W_b|_e = b_e w$ and

$$\|W_b\|_{0,T} \lesssim h_e^{1/2}\|w\|_{0,e}, \tag{3.37}$$

$$\|\nabla W_b\|_{0,T} \lesssim h_e^{-1/2}\|w\|_{0,e}. \tag{3.38}$$

With the aid of the above lemma, we can prove the following local bounds.

Lemma 3.13. *Let* (λ_j, u_j) *and* $(\lambda_{j,h}, u_{j,h})$ *be the solutions of* (2.2) *and* (2.7), *respectively. Then we have the following local bounds:*

(i) *For any* $T \in \mathcal{T}_h$,

$$h_T\|\Delta u_{j,h} - u_{j,h}\|_{0,T} \lesssim \|\nabla(u_{j,h} - u_j)\|_{0,T} + h_T\|u_{j,h} - u_j\|_{0,T}. \tag{3.39}$$

(ii) *Let* $e \in \mathcal{E}_h^I$ *be an interior edge shared by two elements* T *and* T'. *Then we have*

$$h_e^{1/2}\|J_{j,F}\|_{0,e} \lesssim \sum_{T\in U_e}(\|\nabla(u_{j,h} - u_j)\|_{0,T} + h_T\|u_{j,h} - u_j\|_{0,T}) \tag{3.40}$$

with $U_e = \{T, T'\}$.

(iii) *For each boundary edge* $e \in \mathcal{E}_h^\Gamma$ *with* $e \subset \partial T$, *we have*

$$\begin{aligned} h_e^{1/2}&\|J_{j,F}\|_{0,e} \\ &\lesssim (\|\nabla(u_{j,h} - u_j)\|_{0,T} + h_T\|u_{j,h} - u_j\|_{0,T} + h_e^{1/2}\|\lambda_j u_j - \lambda_{j,h}u_{j,h}\|_{0,e}). \end{aligned} \tag{3.41}$$

(iv) *For any edge* $e \in \mathcal{E}_h^I$,

$$h_e^{-1}\|[\![u_{j,h}]\!]\|_{0,e}^2 = h_e^{-1}\|[\![u_{j,h} - u_j]\!]\|_{0,e}^2. \tag{3.42}$$

P r o o f. (i) Set $v_h = \Delta u_{j,h} - u_{j,h}$, and $v_b = b_T v_h$. Noting that $-\Delta u_j + u_j = 0$ in $L^2(T)$ and integrating by parts, we obtain

$$\begin{aligned} \|b_T^{1/2}v_h\|_{0,T}^2 &= \int_T (\Delta u_{j,h} - u_{j,h})v_b \,\mathrm{d}x \\ &= \int_T (u_j - u_{j,h} + \Delta(u_{j,h} - u_j))v_b \,\mathrm{d}x \\ &= \int_T \nabla(u_j - u_{j,h})\nabla v_b \,\mathrm{d}x + \int_T (u_j - u_{j,h})v_b \,\mathrm{d}x, \end{aligned} \tag{3.43}$$

where on the last line we used the fact that $v_b = 0$ on ∂T. Then applying (3.33) and the Cauchy-Schwarz inequality yields

$$\|v_h\|_{0,T}^2 \lesssim \|\nabla(u_j - u_{j,h})\|_{0,T}\|\nabla v_b\|_{0,T} + \|u_j - u_{j,h}\|_{0,T}\|v_b\|_{0,T}. \tag{3.44}$$

Using (3.32) and (3.34), we further have

$$\|v_h\|_{0,T} \lesssim h_T^{-1}\|\nabla(u_j - u_{j,h})\|_{0,T} + \|u_j - u_{j,h}\|_{0,T}.$$

Noting that $h_T\|\Delta u_{j,h} - u_{j,h}\|_{0,T} = h_T\|v_h\|_{0,T}$, this combined with the above inequality gives (i).

(ii) For any interior edge $e \in \mathcal{E}_h^I$, let $w_h = [\![\nabla u_{j,h}]\!]$, $w_b = b_e w_h$. Let $W_b \in H_0^1(\overline{T} \cup \bar{T}')$ be the extension of w_b satisfying (3.37) and (3.38). Noting that $[\![\nabla u_j]\!] = 0$, we obtain

$$\begin{aligned}
\|b_e^{1/2} w_h\|_{0,e}^2 &= \int_e [\![\nabla u_{j,h}]\!] w_b \,\mathrm{d}s = \int_e [\![\nabla(u_{j,h} - u_j)]\!] w_b \,\mathrm{d}s \\
&= \sum_{T \in U_e} \left(\int_T \Delta(u_{j,h} - u_j) W_b \,\mathrm{d}x + \int_T \nabla(u_{j,h} - u_j)\nabla W_b \,\mathrm{d}x \right) \\
&= \sum_{T \in U_e} \left(\int_T (\Delta u_{j,h} + u_{j,h}) W_b \,\mathrm{d}x + \int_T \nabla(u_{j,h} - u_j)\nabla W_b \,\mathrm{d}x \right) \\
&\quad + \sum_{T \in U_e} \int_T (u_j - u_{j,h}) W_b \,\mathrm{d}x,
\end{aligned} \tag{3.45}$$

where we have used the fact that $-\Delta u_j + u_j = 0$ in $L^2(T)$. In view of (3.36), (3.37), and (3.38), we arrive at

$$\begin{aligned}
\|w_h\|_{0,e} \lesssim \sum_{T \in U_e} (h_e^{1/2}\|\Delta u_{j,h} - u_{j,h}\|_{0,T} + h_e^{-1/2}\|\nabla(u_j - u_{j,h})\|_{0,T} \\
+ h_e^{1/2}\|u_j - u_{j,h}\|_{0,T}).
\end{aligned} \tag{3.46}$$

Combining the bound for $\|\Delta u_{j,h} - u_{j,h}\|_{0,T}$ in (3.39) and the shape-regularity of the mesh yields

$$h_e^{1/2}\|[\![\nabla u_{j,h}]\!]\|_{0,e} \lesssim \sum_{T \in U_e} (\|\nabla(u_j - u_{j,h})\|_{0,T} + h_T\|u_j - u_{j,h}\|_{0,T}),$$

which gives (ii).

(iii) Set $z_h = \lambda_{j,h} u_{j,h} - \nabla u_{j,h} \cdot \mathbf{n}$, $z_b = b_e z_h$. Let Z_b be the extension of z_b which satisfies (3.37) and (3.38). Noting that $\nabla u_j \cdot \mathbf{n} - \lambda_j u_j = 0$ on Γ, $-\Delta u_j + u_j = 0$ in $L^2(T)$ and integrating by parts, we have

$$\begin{aligned}
\|b_e^{1/2} z_h\|_{0,e}^2 &= \int_e (\lambda_{j,h} u_{j,h} - \nabla u_{j,h} \cdot \mathbf{n}) z_b \,\mathrm{d}s \\
&= \int_e (\nabla u_j \cdot \mathbf{n} - \nabla_j u_{j,h} \cdot \mathbf{n}) z_b \,\mathrm{d}s - \int_e (\lambda_j u_j - \lambda_{j,h} u_{j,h}) z_b \,\mathrm{d}s
\end{aligned} \tag{3.47}$$

$$
\begin{aligned}
&= \int_T (\Delta u_j - \Delta u_{j,h}) Z_b \, \mathrm{d}x + \int_T \nabla(u_j - u_{j,h}) \nabla Z_b \, \mathrm{d}x \\
&\quad - \int_e (\lambda_j u_j - \lambda_{j,h} u_{j,h}) z_b \, \mathrm{d}s \\
&= \int_T (u_{j,h} - \Delta u_{j,h}) Z_b \, \mathrm{d}x + \int_T (u_j - u_{j,h}) Z_b \, \mathrm{d}x \\
&\quad + \int_T \nabla(u_j - u_{j,h}) \nabla Z_b \, \mathrm{d}x - \int_e (\lambda_j u_j - \lambda_{j,h} u_{j,h}) z_b \, \mathrm{d}s.
\end{aligned}
$$

In view of (3.35), (3.36), (3.37), and (3.38), we arrive at

$$
\begin{aligned}
\|z_h\|_{0,e} \lesssim (&h_e^{1/2} \|\Delta u_{j,h} - u_{j,h}\|_{0,T} + h_e^{1/2} \|u_j - u_{j,h}\|_{0,T} \\
&+ h_e^{-1/2} \|\nabla(u_j - u_{j,h})\|_{0,T} + \|\lambda_j u_j - \lambda_{j,h} u_{j,h}\|_{0,e}).
\end{aligned} \tag{3.48}
$$

Combining the bound for $\|\Delta u_{j,h} - u_{j,h}\|_{0,T}$ in (3.39) and the shape-regularity of the mesh yields

$$
h_e^{1/2} \|z_h\|_{0,e} \lesssim \|\nabla(u_j - u_{j,h})\|_{0,T} + h_T \|u_j - u_{j,h}\|_{0,T} + h_e^{1/2} \|\lambda_j u_j - \lambda_{j,h} u_{j,h}\|_{0,e}. \tag{3.49}
$$

This gives (iii).

(iv) For any edge $e \in \mathcal{E}_h^I$, we have $[\![u_j]\!] = 0$, which gives (3.42). $\square$

Summing over $T \in \mathcal{T}_h$ and $e \in \mathcal{E}_h$ and using the definition in (3.8), we obtain the following main result of this section.

Theorem 3.14. *Let $(\lambda_{j,h}, u_{j,h})$ be the solution of (2.7), and assume that $\lambda_{j,h}$ converges to the exact eigenvalue λ_j that has multiplicity $m \geqslant 1$. For the minimizer u_j of (3.8) with $Y = \widehat{M}(\lambda_j)$ we have*

$$
\eta_j \lesssim \operatorname{dist}(u_{j,h}, \widehat{M}(\lambda_j))_E + h^{1/2} \|\lambda_j u_j - \lambda_{j,h} u_{j,h}\|_b. \tag{3.50}
$$

4. Numerical experiments

In this section, we design an adaptive algorithm for the model problem (2.1) according to the a posteriori error estimator η_j. In each step of adaptive algorithms, we use the Dörfler marking strategy [23] (with parameter $\theta = 0.3$) and refine the marked triangles by the bisection algorithm [19]. All numerical tests are implemented by the linear discontinuous finite element, and the penalty parameter is set by $\sigma = 10$.

Example 4.1. We consider the problem (2.1) in the unit square domain $\Omega = (0,1)\times(0,1)$. Since the exact eigenvalue is not known, we use $\lambda_1 \approx 0.2400790858$ as an approximation of the first eigenvalue (see [48]), which can be obtained by solving a discrete eigenvalue problem with a uniform fine mesh of $h = \sqrt{2}/256$; the corresponding eigenfunction solution is used as 'exact' solution u_1 to compute the error.

We first give the a priori error $\|u_1 - u_{1,h}\|_h$ in a family of uniform meshes in Figure 1. We can observe that the numerical method has order one accuracy (see the absolute value of the slope of the line), which validates the theoretical analysis in Theorem 2.4. Moreover, we further give some numerical results on adaptive meshes. Figure 2 shows error $\|u_1 - u_{1,h}\|_h$ of the first eigenfunction in log-log coordinates, which is plotted as a function of number of degrees of freedom. The efficiency index $\|u_1 - u_{1,h}\|_h/\eta_1$ is shown in Figure 3. From the convergence history in Figure 2, we observe that the error of the function has asymptotical convergence rate $\|u_1 - u_{1,h}\|_h \approx CN^{-1/2}$, with N the number of degrees of freedom. We also show the adaptive mesh of level 14 in the computational procedure in Figure 4. In addition, in Figure 5 we give eigenvalue error $|\lambda_1 - \lambda_{1,h}|$ which has an asymptotical convergence rate $|\lambda_1 - \lambda_{1,h}| \approx CN^{-1}$.

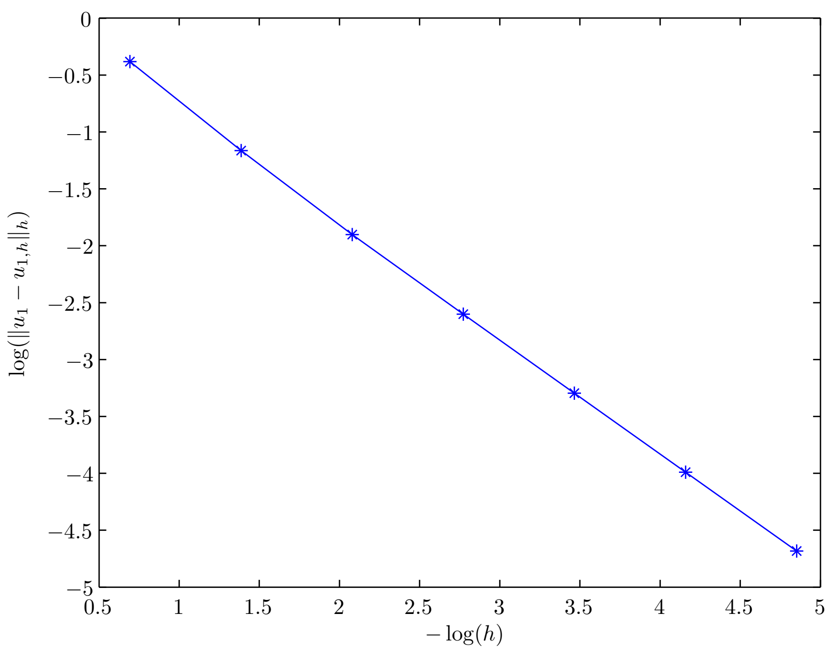

Figure 1. A priori error results of the first eigenfunction for Example 4.1.

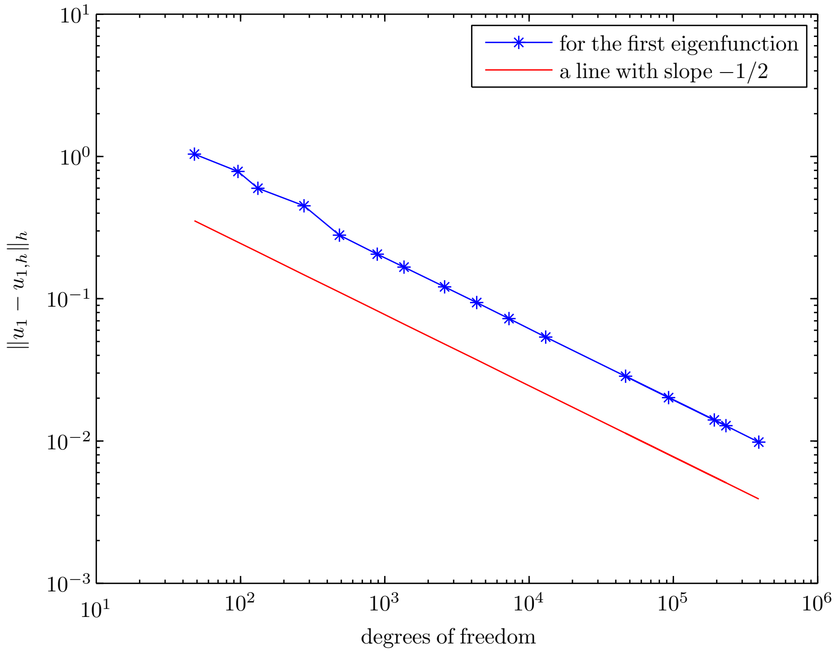

Figure 2. Convergence history of the first eigenfunction for Example 4.1.

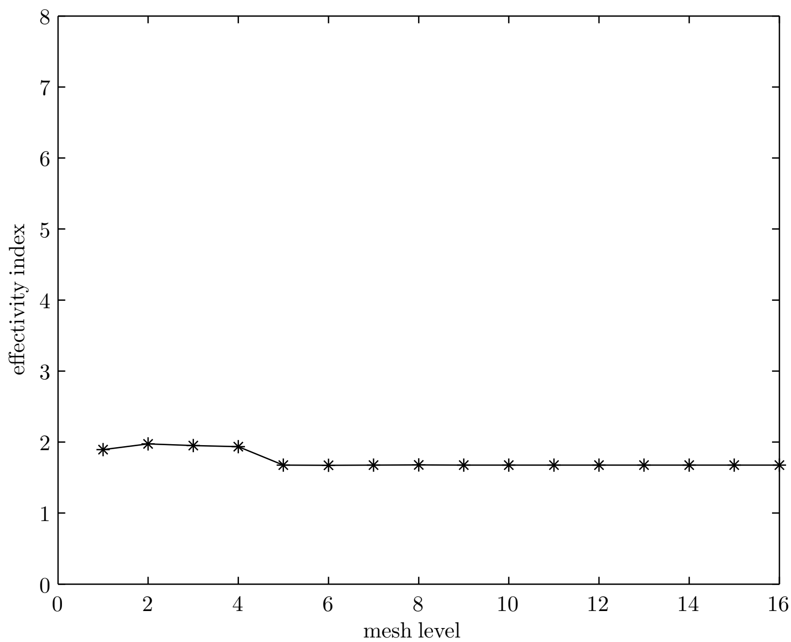

Figure 3. Efficiency index for Example 4.1.

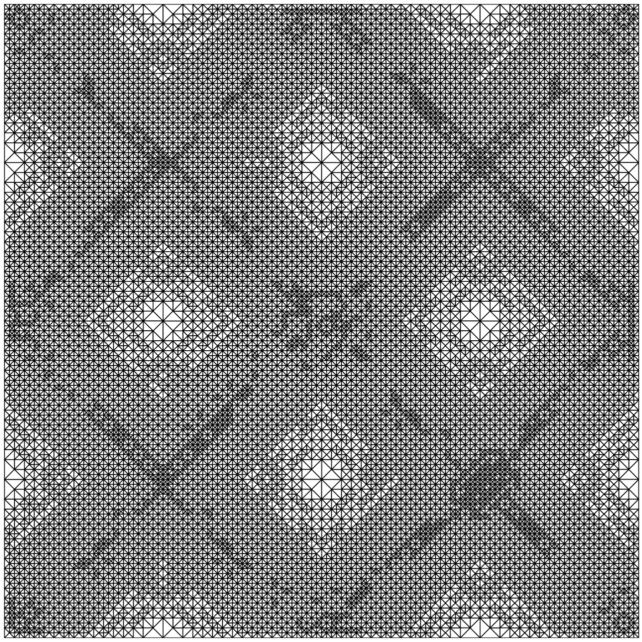

Figure 4. Adaptive mesh of level 14 for Example 4.1.

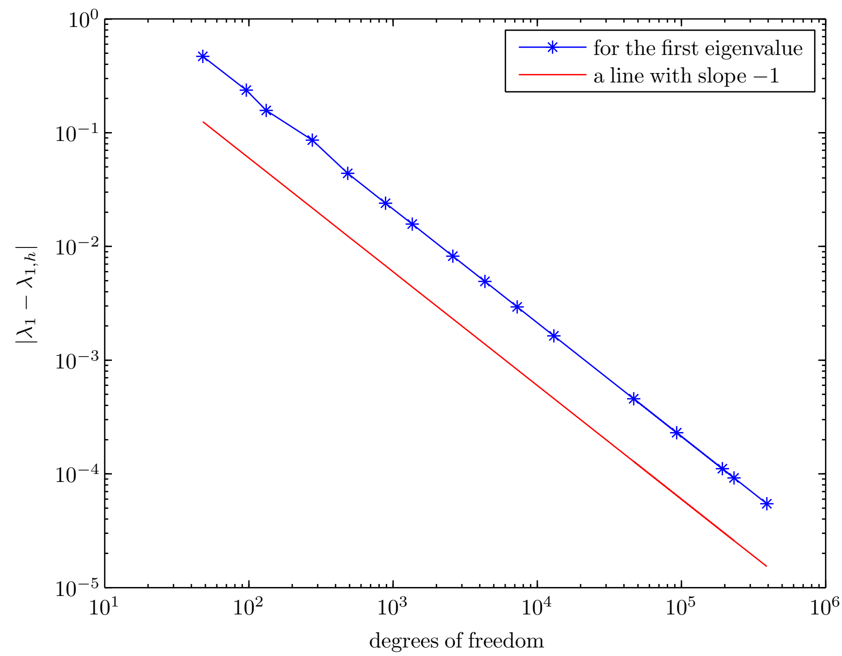

Figure 5. Convergence history of the first eigenvalue for Example 4.1.

E x a m p l e 4.2. We consider the problem (2.1) in the L-shape domain $\Omega = (0,1) \times (0,1) \setminus ([\frac{1}{2},1] \times [\frac{1}{2},1])$. Since the exact eigenvalue is not known, we use $\lambda_1 \approx 0.1829642385$ as an approximation of the first eigenvalue (see [35]), which can be obtained by solving a discrete eigenvalue problem with a uniform fine mesh of $h = \sqrt{2}/512$; the corresponding eigenfunction solution is used as 'exact' solution u_1 to compute the error.

As in the previous example, in Figure 6 we provide the convergence history of the first eigenfunction error $\|u_1 - u_{1,h}\|_h$, which converges asymptotically in the sense that $\|u_1 - u_{1,h}\|_h \approx CN^{-1/2}$. The corresponding error for the first eigenvalue $|\lambda_1 - \lambda_{1,h}|$ is shown in Figure 9. The efficiency index $\|u_1 - u_{1,h}\|_h/\eta_1$ is provided in Figure 7. We also show the adaptive mesh of the 17th level in Figure 8, from which we can see that the solution u_1 at re-entrant corner has a singularity.

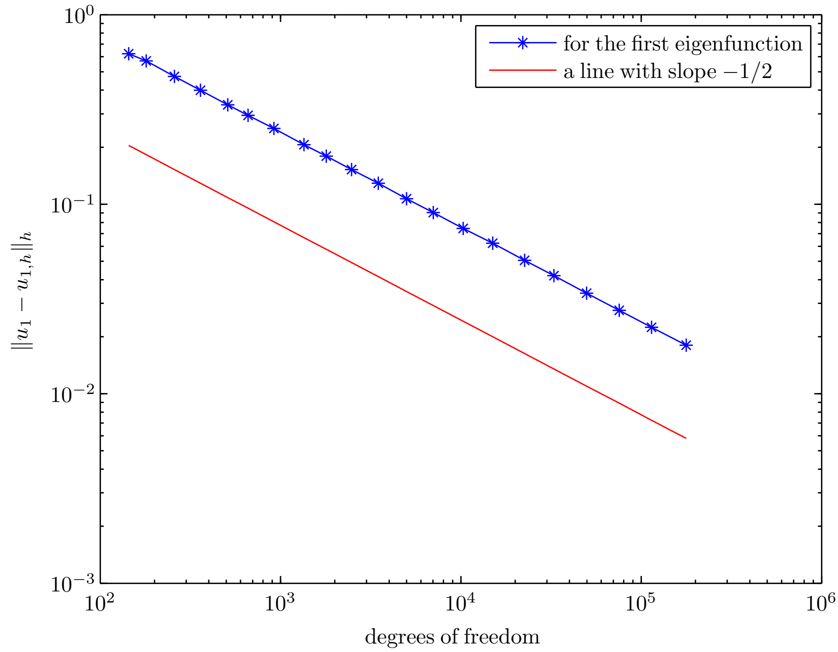

Figure 6. Convergence history of the first eigenfunction for Example 4.2.

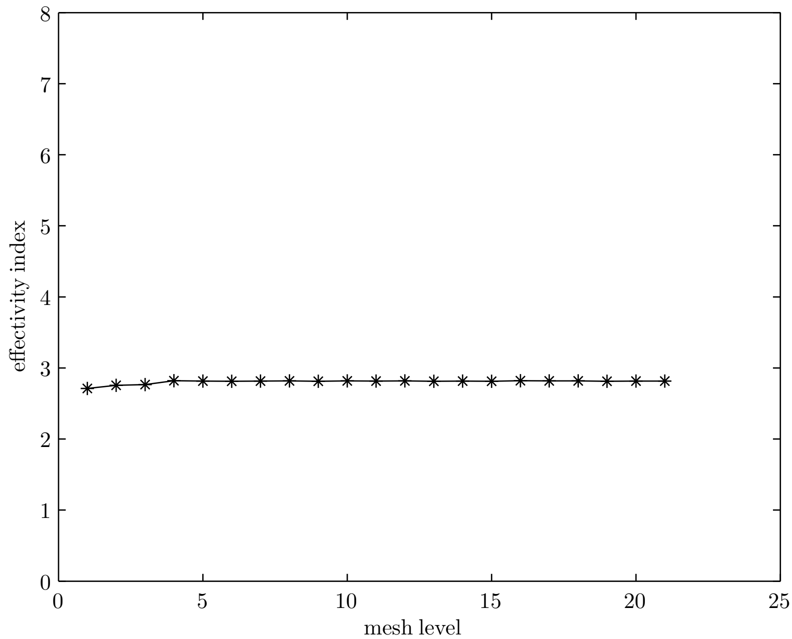

Figure 7. Efficiency index for Example 4.2.

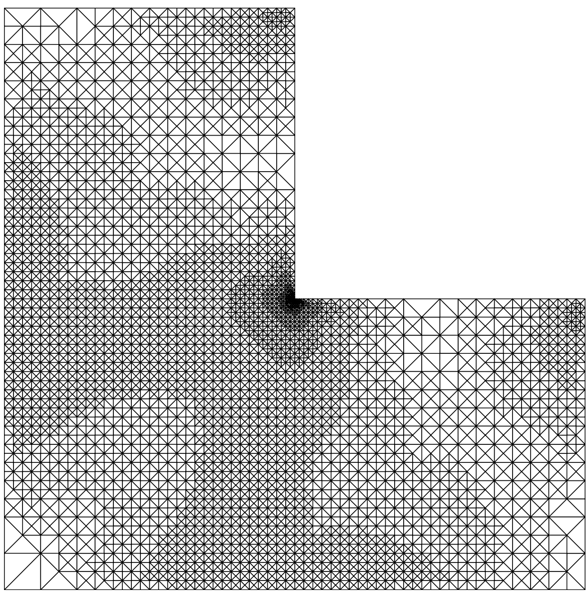
Figure 8. Adaptive mesh of level 17 for Example 4.2.

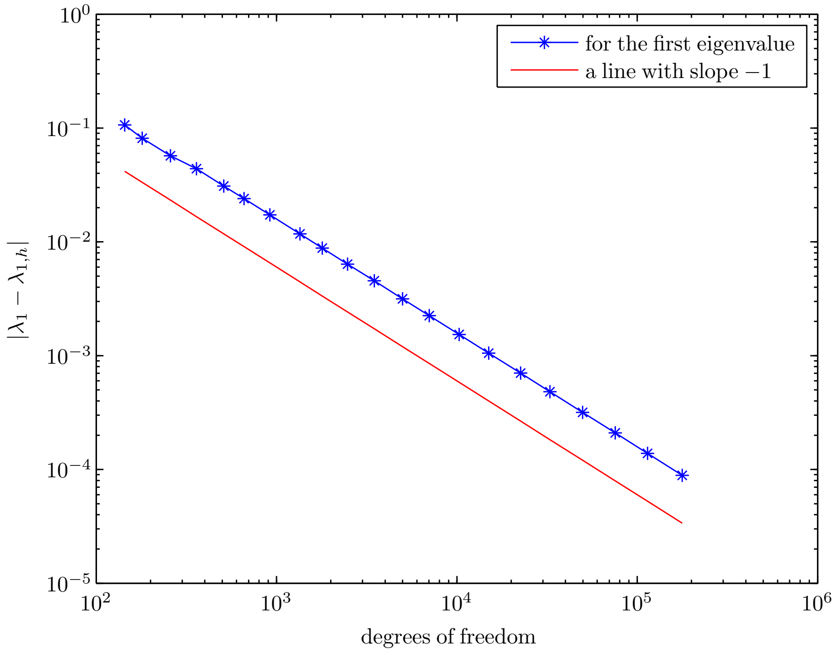

Figure 9. Convergence history of the first eigenvalue for Example 4.2.

5. Conclusions

We proposed and analyzed the interior penalty discontinuous Galerkin method for the Steklov eigenvalue problem. More precisely, we derived a residual-based a posteriori error estimator and prove its reliability and efficiency. Some numerical results are also provided to validate theoretical analysis. We have discussed the h-version of the DG method; how to extend the results to the hp-DG method will be considered in a further work. Moreover, extending the results to DG methods with general polygonal mesh and three dimensional case are more interesting and challenging.

A c k n o w l e d g e m e n t s. The authors thank two anonymous referees for their valuable comments and suggestions which helped to improve the quality of this paper.

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Authors' addresses: *Yuping Zeng*, School of Mathematics, Jiaying University, Meizhou 514015, China, e-mail: `zeng_yuping@163.com`; *Feng Wang*, Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China, e-mail: `fwang@njnu.edu.cn`.