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# A FULL MULTIGRID METHOD FOR SEMILINEAR ELLIPTIC EQUATION

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*Abstract.* A full multigrid finite element method is proposed for semilinear elliptic equations. The main idea is to transform the solution of the semilinear problem into a series of solutions of the corresponding linear boundary value problems on the sequence of finite element spaces and semilinear problems on a very low dimensional space. The linearized boundary value problems are solved by some multigrid iterations. Besides the multigrid iteration, all other efficient numerical methods can also serve as the linear solver for solving boundary value problems. The optimality of the computational work is also proved. Compared with the existing multigrid methods which need the bounded second order derivatives of the nonlinear term, the proposed method only needs the Lipschitz continuation in some sense of the nonlinear term.

*Keywords*: semilinear elliptic problem; full multigrid method; multilevel correction; finite element method

*MSC 2010*: 65N30, 65N25, 65L15, 65B99

## 1. Introduction

The purpose of this paper is to study the multigird finite element method for semilinear elliptic problems. As we know, the multigrid and multilevel methods [2], [3], [4], [5], [8], [13], [14], [15], [20] provide optimal order algorithms for solving boundary value problems. The error bounds of the approximate solutions obtained from these efficient numerical algorithms are comparable to the theoretical bounds determined by the finite element discretization. In the past decade, some researches about multigrid method for nonlinear elliptic problem have been done to improve the efficiency

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of nonlinear elliptic problem solving, i.e. [14], [22], [21]. The Newton iteration has been adopted to linearize the nonlinear equation in these existing multigrid methods and then they need the bounded second order derivatives of the nonlinear terms. For more information, we refer to [9], [14], [22], and the references therein.

Recently, a type of multigrid method with optimal efficiency for eigenvalue problems has been proposed in [10], [11], [16], [17], [18], [19]. The aim of this paper is to present a full multigrid method for solving semilinear elliptic problems, based on the multilevel correction scheme [16], [17]. The main idea is to design a special low dimensional space to transform the solution of the semilinear problem into a series of solutions of the corresponding linear boundary value problems on the sequence of finite element spaces and semilinear problems on a very low dimensional space. For the linearized elliptic problem, it is not necessary to solve the linear boundary value problem exactly in each correction step. Here, we only do some multigrid iteration steps for the linear boundary value problems. In this new version of the multigrid method, solving a semilinear elliptic problem will not be much more difficult than the multigrid scheme for the corresponding linear boundary value problems. Compared with the existing multigrid methods for the semilinear problem, our method only needs the Lipschitz continuation in some sense of the nonlinear term.

An outline of the paper goes as follows. In Section 2, we introduce the finite element method for the semilinear elliptic problem. A type of full multigrid method for the semilinear elliptic problem is given in Section 3. In Section 4, some numerical examples are provided to illustrate the efficiency of the proposed numerical method. Some concluding remarks are given in the last section.

## 2. Discretization by finite element method

In this paper, the letters $C$ or $c$ (with or without subscripts) are used to denote constants which may be different at different places. For convenience, the symbols $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \approx y_3$ mean that $x_1 \leqslant C_1 y_1$, $x_2 \geqslant c_2 y_2$ and $c_3 x_3 \leqslant y_3 \leqslant C_3 x_3$. Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ denote a bounded convex domain with Lipschitz boundary $\partial\Omega$. We use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms $\|\cdot\|_{s,p,\Omega}$ and seminorms $|\cdot|_{s,p,\Omega}$ (see e.g. [1]). For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) \colon\ v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of trace. For simplicity, we use $\|\cdot\|_s$ to denote $\|\cdot\|_{s,2,\Omega}$ and $V$ to denote $H_0^1(\Omega)$ in the rest of the paper.

Here, we consider the following type of semilinear elliptic equation:

$$
\begin{cases} -\nabla\cdot(\mathcal{A}\nabla u) + f(x,u) = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}
$$

where $\mathcal{A} = (a_{i,j})_{d\times d}$ is a symmetric positive definite matrix with $a_{i,j} \in W^{1,\infty}$ $(i,j = 1,2,\dots,d)$, $f(x,u)$ is a nonlinear function with respect to the second variable.

The weak form of the semilinear problem (2.1) can be described as: Find $u \in V$ such that

$$a(u,v) + (f(x,u),v) = (g,v) \quad \forall\, v \in V, \tag{2.2}$$

where

$$a(u,v) = (\mathcal{A}\nabla u, \nabla v). \tag{2.3}$$

Obviously, $a(u,v)$ is bounded and coercive on $V$, i.e.,

$$a(u,v) \leqslant C_a\|u\|_{1,\Omega}\|v\|_{1,\Omega} \quad \text{and} \quad c_a\|u\|_{1,\Omega}^2 \leqslant a(u,u) \quad \forall\, u,v \in V. \tag{2.4}$$

Then we use the norm $\|w\|_a := \sqrt{a(w,w)}$ for any $w \in V$ in this paper to replace the standard norm $\|\cdot\|_1$.

In order to guarantee the existence and uniqueness of the problem (2.2), we assume the nonlinear term $f(\cdot,\cdot)$ satisfies the following assumption.

**Assumption A.** The nonlinear function $f(x,\cdot)$ satisfies the convexity and Lipschitz continuous conditions as follows:

$$\begin{cases} (f(x,w) - f(x,v), w - v) \geqslant 0 \quad \forall\, w \in V, \quad \forall\, v \in V, \\ (f(x,w) - f(x,v), \varphi) \leqslant C_f\|w - v\|_0\|\varphi\|_1 \quad \forall\, w \in V, \quad \forall\, v \in V, \quad \forall\, \varphi \in V. \end{cases} \tag{2.5}$$

Now, we introduce the finite element method for the semilinear elliptic problem (2.2). First we generate a shape regular decomposition of the computing domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ into triangles or rectangles for $d = 2$, tetrahedrons or hexahedrons for $d = 3$ (cf. [6], [7]). The mesh diameter $h$ describes the maximum diameter of all cells $K \in \mathcal{T}_h$. Based on the mesh $\mathcal{T}_h$, we construct the finite element space $V_h \subset V$. For simplicity, we set $V_h$ as the linear finite element space which is defined as

$$V_h = \{v_h \in C(\Omega)\colon\ v_h|_K \in \mathcal{P}_1 \quad \forall K \in \mathcal{T}_h\} \cap H_0^1(\Omega), \tag{2.6}$$

where $\mathcal{P}_1$ denotes the linear function space.

The standard finite element scheme for semilinear equation (2.2) is: Find $\bar{u}_h \in V_h$ such that

$$a(\bar{u}_h, v_h) + (f(x,\bar{u}_h), v_h) = (g, v_h) \quad \forall\, v_h \in V_h. \tag{2.7}$$

Denote a linearized operator $L\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$(Lw, v) = (\mathcal{A}\nabla w, \nabla v) \quad \forall\, w \in V,\ \forall\, v \in V.$$

In order to deduce the global a priori error estimates, we introduce $\eta_a(V_h)$ as follows:

$$\eta_a(V_h) = \sup_{f \in L^2(\Omega), \|f\|_0 = 1} \inf_{v_h \in V_h} \|L^{-1} f - v_h\|_a.$$

It is easy to know that $\eta_a(V_h) \to 0$ as $h \to 0$ (cf. [6], [7]).

In order to measure the error for the finite element approximations, we denote

$$\delta_h(u) = \inf_{v_h \in V_h} \|u - v_h\|_a.$$

From [14], we can give the following error estimates.

**Lemma 2.1.** *When Assumption* A *is satisfied, equations* (2.2) *and* (2.7) *are uniquely solvable and the following estimates hold:*

$$\|u - \bar{u}_h\|_a \leqslant (1 + C\eta_a(V_h))\delta_h(u), \tag{2.8}$$

$$\|u - \bar{u}_h\|_0 \lesssim \eta_a(V_h)\|u - \bar{u}_h\|_a. \tag{2.9}$$

Proof. From Theorem 6.1 in [14], we know that problems (2.2) and (2.7) are uniquely solvable. Now, it is time to prove the error estimates. For this aim, we define the finite element projection operator $P_h$ by the equation

$$a(P_h w, v_h) = a(w, v_h) \quad \forall\, w \in V,\ \forall\, v_h \in V_h.$$

It is easy to know that $\|u - P_h u\|_a = \delta_h(u)$ and $\|u - P_h u\|_0 \lesssim \eta_a(V_h)\|u - P_h u\|_a$. Let us define $w_h = P_h u - \bar{u}_h$ in this proof. From (2.2), (2.5), and (2.7), we have

$$\begin{aligned} a(P_h u - \bar{u}_h, w_h) &\leqslant a(P_h u - \bar{u}_h, w_h) + (f(x, P_h u) - f(x, \bar{u}_h), w_h) \\ &= a(P_h u, w_h) + (f(x, P_h u), w_h) - (g, w_h) \\ &= a(P_h u - u, w_h) + (f(x, P_h u) - f(x, u), w_h) \\ &= (f(x, P_h u) - f(x, u), w_h) \\ &\leqslant C_f \|u - P_h u\|_0 \|w_h\|_a. \end{aligned}$$

Then the following inequalities hold:

$$\|P_h u - \bar{u}_h\|_a \leqslant C_f \|u - P_h u\|_0 \leqslant C_f \eta_a(V_h)\|u - P_h u\|_a. \tag{2.10}$$

Combining (2.10) and the triangle inequality leads to the estimates

$$\begin{aligned} \|u - \bar{u}_h\|_a &\leqslant \|u - P_h u\|_a + \|P_h u - \bar{u}_h\|_a \\ &\leqslant \delta_h(u) + C_f \eta_a(V_h)\|u - P_h u\|_a \\ &\leqslant (1 + C_f \eta_a(V_h))\delta_h(u), \end{aligned} \tag{2.11}$$

which is the desired result (2.8). From (2.10) and the triangle inequality, we have

$$\begin{aligned} \|u - \bar{u}_h\|_0 &\leqslant \|u - P_h u\|_0 + \|P_h u - \bar{u}_h\|_0 \leqslant \|u - P_h u\|_0 + C\|P_h u - \bar{u}_h\|_a \\ &\leqslant C\eta_a(V_h)\|u - P_h u\|_a + C_f \eta_a(V_h)\|u - P_h u\|_a \\ &\leqslant (C + C_f)\eta_a(V_h)\|u - P_h u\|_a \leqslant (C + C_f)\eta_a(V_h)\|u - \bar{u}_h\|_a. \end{aligned}$$

This is the desired result (2.9) and the proof is complete. □

## 3. Full multigrid method for semilinear elliptic equation

In this section, a full multigrid method for semilinear problems is proposed based on the multilevel correction scheme in [16] and [17]. The key point is to transform the solution of the semilinear problem into a series of solutions of the corresponding linear boundary value problems on the sequence of finite element spaces and semilinear problems on a very low dimensional space. In order to carry out the multigrid method, we first generate a coarse mesh $\mathcal{T}_H$ with the mesh size $H$ and define the linear finite element space $V_H$ on the mesh $\mathcal{T}_H$. Then a sequence of triangulations $\mathcal{T}_{h_k}$ of $\Omega \subset \mathcal{R}^d$ is determined as follows. Suppose $\mathcal{T}_{h_1}$ (produced from $\mathcal{T}_H$ by regular refinements) is given and let $\mathcal{T}_{h_k}$ be obtained from $\mathcal{T}_{h_{k-1}}$ via one regular refinement step (producing $\beta^d$ subelements) such that

$$h_k = \frac{1}{\beta} h_{k-1}, \quad k = 2, \ldots, n, \tag{3.1}$$

where the positive number $\beta$ denotes the refinement index and is larger than 1 (always equals 2). Based on this sequence of meshes, we construct the corresponding nested linear finite element spaces such that

$$V_H \subseteq V_{h_1} \subset V_{h_2} \subset \ldots \subset V_{h_n}. \tag{3.2}$$

Due to the convexity of the domain $\Omega$, the sequence of finite element spaces $V_{h_1} \subset V_{h_2} \subset \ldots \subset V_{h_n}$ and the finite element space $V_H$ have the following relations of approximation accuracy:

$$\eta_a(V_{h_k}) \approx \frac{1}{\beta}\eta_a(V_{h_{k-1}}), \quad \delta_{h_k}(u) \approx \frac{1}{\beta}\delta_{h_{k-1}}(u), \quad k = 2, \ldots, n. \tag{3.3}$$

**3.1. One correction step.** In order to design the full multigrid method, first we introduce one correction step in this subsection.

Assume we have obtained an approximate solution $u_{h_k}^{(l)} \in V_{h_k}$. A correction step is designed as follows to improve the accuracy of the given approximation $u_{h_k}^{(l)}$.

**Algorithm 3.1.** One Correction Step

(1) Define the following auxiliary boundary value problem: Find $\widehat{u}_{h_k}^{(l+1)} \in V_{h_k}$ such that

$$a(\widehat{u}_{h_k}^{(l+1)}, v_{h_k}) = -(f(x, u_{h_k}^{(l)}), v_{h_k}) + (g, v_{h_k}) \quad \forall v_{h_k} \in V_{h_k}. \tag{3.4}$$

Perform $m$ multigrid iteration steps for the second order elliptic equation to obtain an approximate solution $\widetilde{u}_{h_k}^{(l+1)}$ with the error reduction rate

$$\|\widetilde{u}_{h_k}^{(l+1)} - \widehat{u}_{h_k}^{(l+1)}\|_a \leqslant \theta \|u_{h_k}^{(l)} - \widehat{u}_{h_k}^{(l+1)}\|_a, \tag{3.5}$$

where $u_{h_k}^{(l)}$ is used as the initial value for the multigrid iteration and $\theta < 1$ is a fixed constant independent of the mesh size $h_k$.

(2) Define a finite element space $V_{H,h_k} := V_H + \mathrm{span}\{\widetilde{u}_{h_k}^{(l+1)}\}$ and solve the following semilinear elliptic equation: Find $u_{h_k}^{(l+1)} \in V_{H,h_k}$ such that

$$a(u_{h_k}^{(l+1)}, v_{H,h_k}) + (f(x, u_{h_k}^{(l+1)}), v_{H,h_k}) = (g, v_{H,h_k}) \quad \forall v_{H,h_k} \in V_{H,h_k}. \tag{3.6}$$

In order to simplify the notation and summarize the above two steps, we define

$$u_{h_k}^{(l+1)} = \mathrm{SemilinearMG}(V_H, u_{h_k}^{(l)}, V_{h_k}).$$

The error estimate of Algorithm 3.1 is studied in the next theorem.

**Theorem 3.1.** *Assume the given solution $u_{h_k}^{(l)}$ has the estimate*

$$\|\bar{u}_{h_k} - u_{h_k}^{(l)}\|_0 \lesssim \eta_a(V_H) \|\bar{u}_{h_k} - u_{h_k}^{(l)}\|_a. \tag{3.7}$$

*After the one correction step defined by Algorithm* 3.1, *the resultant approximate solution $u_{h_k}^{(l+1)}$ has the estimates*

$$\|\bar{u}_{h_k} - u_{h_k}^{(l+1)}\|_a \leqslant \gamma \|\bar{u}_{h_k} - u_{h_k}^{(l)}\|_a, \tag{3.8}$$

$$\|\bar{u}_{h_k} - u_{h_k}^{(l+1)}\|_0 \leqslant C\eta_a(V_H) \|\bar{u}_{h_k} - u_{h_k}^{(l+1)}\|_a, \tag{3.9}$$

*where*

$$\gamma := (\theta + (1+\theta)C\eta_a(V_H))(1 + C\eta_a(V_H)).$$

P r o o f. From (2.5), (2.7) and (3.4), we have

$$
\begin{aligned}
a(\bar u_{h_k} - \widehat u_{h_k}^{(l+1)}, v_{h_k}) &= (f(x, u_{h_k}^{(l)}) - f(x, \bar u_{h_k}), v_{h_k}) \\
&\leqslant C_f \|\bar u_{h_k} - u_{h_k}^{(l)}\|_0 \|v_{h_k}\|_a \\
&\leqslant C\eta_a(V_H)\|\bar u_{h_k} - u_{h_k}^{(l)}\|_a \|v_{h_k}\|_a \quad \forall\, v_{h_k} \in V_{h_k}.
\end{aligned} \tag{3.10}
$$

Combining (2.4) and (3.10) leads to

$$
\|\bar u_{h_k} - \widehat u_{h_k}^{(l+1)}\|_a \leqslant C\eta_a(V_H)\|\bar u_{h_k} - u_{h_k}^{(l)}\|_a. \tag{3.11}
$$

After performing $m$ multigrid iteration steps, due to (3.5) and (3.11), the following estimates hold:

$$
\begin{aligned}
\|\widetilde u_{h_k}^{(l+1)} - \bar u_{h_k}\|_a &\leqslant \|\widetilde u_{h_k}^{(l+1)} - \widehat u_{h_k}^{(l+1)}\|_a + \|\widehat u_{h_k}^{(l+1)} - \bar u_{h_k}\|_a \\
&\leqslant \theta\|u_{h_k}^{(l)} - \widehat u_{h_k}^{(l+1)}\|_a + \|\widehat u_{h_k}^{(l+1)} - \bar u_{h_k}\|_a \\
&\leqslant \theta\|u_{h_k}^{(l)} - \bar u_{h_k}\|_a + \theta\|\widehat u_{h_k}^{(l+1)} - \bar u_{h_k}\|_a + \|\widehat u_{h_k}^{(l+1)} - \bar u_{h_k}\|_a \\
&\leqslant \big(\theta + (1+\theta)C\eta_a(V_H)\big)\|\bar u_{h_k} - u_{h_k}^{(l)}\|_a.
\end{aligned} \tag{3.12}
$$

Note that the semilinear elliptic problem (3.6) can be regarded as a finite dimensional approximation of the semilinear elliptic problem (2.7). Let $P_{H,h_k}\colon V \to V_{H,h_k}$ denote the finite element projection operator which is defined as

$$
a(P_{H,h_k} w, v_{H,h_k}) = a(w, v_{H,h_k}) \quad \forall\, w \in V,\ \forall\, v_{H,h_k} \in V_{H,h_k}.
$$

Since $\widetilde u_{h_k}^{(l+1)} \in V_{H,h_k}$ and $V_H \subset V_{H,h_k}$, it is obvious that $\eta_a(V_{H,h_k}) \leqslant \eta_a(V_H)$ and

$$
\|\bar u_{h_k} - P_{H,h_k}\bar u_{h_k}\|_a = \inf_{v_{H,h_k}\in V_{H,h_k}} \|\bar u_{h_k} - v_{H,h_k}\|_a \leqslant \|\bar u_{h_k} - \widetilde u_{h_k}^{(l+1)}\|_a, \tag{3.13}
$$

$$
\begin{aligned}
\|\bar u_{h_k} - P_{H,h_k}\bar u_{h_k}\|_0 &\leqslant C\eta_a(V_{H,h_k})\|\bar u_{h_k} - P_{H,h_k}\bar u_{h_k}\|_a \\
&\leqslant C\eta_a(V_H)\|\bar u_{h_k} - P_{H,h_k}\bar u_{h_k}\|_a.
\end{aligned} \tag{3.14}
$$

Let us define $w_{h_k} = P_{H,h_k}\bar u_{h_k} - u_{h_k}^{(l+1)} \in V_{H,h_k}$ in this proof. Based on problems (2.7) and (3.6), the following estimates hold:

$$
\begin{aligned}
&a(P_{H,h_k}\bar u_{h_k} - u_{h_k}^{(l+1)}, w_{h_k}) \\
&\quad \leqslant a(P_{H,h_k}\bar u_{h_k} - u_{h_k}^{(l+1)}, w_{h_k}) + (f(x, P_{H,h_k}\bar u_{h_k}) - f(x, u_{h_k}^{(l+1)}), w_{h_k}) \\
&\quad = a(P_{H,h_k}\bar u_{h_k}, w_h) + (f(x, P_{H,h_k}\bar u_{h_k}), w_{h_k}) - (g, w_{h_k}) \\
&\quad = a(P_{H,h_k}\bar u_{h_k} - \bar u_{h_k}, w_{h_k}) + (f(x, P_{H,h_k}\bar u_{h_k}) - f(x, \bar u_{h_k}), w_{h_k}) \\
&\quad = (f(x, P_{H,h_k}\bar u_{h_k}) - f(x, \bar u_{h_k}), w_{h_k}) \leqslant C_f\|\bar u_{h_k} - P_{H,h_k}\bar u_{h_k}\|_0 \|w_{h_k}\|_a.
\end{aligned} \tag{3.15}
$$

From (3.14) and (3.15), we have

$$\begin{aligned}\|P_{H,h_k}\bar{u}_{h_k}-u_{h_k}^{(l+1)}\|_a &\leqslant C_f\|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\|_0\\ &\leqslant C\eta_a(V_H)\|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\|_a.\end{aligned}\tag{3.16}$$

Combining (3.13), (3.16), and the triangle inequality leads to the inequalities

$$\begin{aligned}\|\bar{u}_{h_k}-u_{h_k}^{(l+1)}\|_a &\leqslant \|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\|_a+\|P_{H,h_k}\bar{u}_{h_k}-u_{h_k}^{(l+1)}\|_a\\ &\leqslant (1+C\eta_a(V_H))\|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\|_a\\ &\leqslant (1+C\eta_a(V_H))\|\bar{u}_{h_k}-\widetilde{u}_{h_k}^{(l+1)}\|_a.\end{aligned}\tag{3.17}$$

This is the desired result (3.8). From (3.15) and the triangle inequality, we have the estimates

$$\begin{aligned}\|\bar{u}_{h_k}-u_{h_k}^{(l+1)}\|_0 &\leqslant \|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\|_0+\|P_{H,h_k}\bar{u}_{h_k}-u_{h_k}^{(l+1)}\|_0\\ &\leqslant \|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\|_0+C\|P_{H,h_k}\bar{u}_{h_k}-u_{h_k}^{(l+1)}\|_a\\ &\leqslant C\eta_a(V_H)\|\bar{u}_{h_k}-P_{H,h_k}\bar{u}_{h_k}\|_a\\ &\leqslant C\eta_a(V_H)\|\bar{u}_{h_k}-u_{h_k}^{(l+1)}\|_a,\end{aligned}\tag{3.18}$$

which is the desired result (3.9) and the proof is complete. □

Remark 3.1. The proof of Theorem 3.1 shows that the structure of the low dimensional space $V_{H,h_k}$ plays the key role for Algorithm 3.1. This special space makes the finite element projection $P_{H,h_k}$ has both the accuracy as in (3.13) and the $L^2$-norm estimate by duality argument as in (3.14).

**3.2. Full multigrid method.** In this subsection, a full multigrid method is proposed based on the one correction step defined in Algorithm 3.1. This algorithm can reach the optimal convergence rate with the optimal computational complexity.

**Agorithm 3.2.** Full Multigrid Scheme

(1) Solve the following semilinear problem in $V_{h_1}$: Find $u_{h_1}\in V_{h_1}$ such that

$$a(u_{h_1},v_{h_1})+(f(x,u_{h_1}),v_{h_1})=(g,v_{h_1})\quad\forall\, v_{h_1}\in V_{h_1}.$$

(2) For $k=2,\dots,n$, do the following iteration:
   (a) Set $u_{h_k}^{(0)}=u_{h_{k-1}}$.
   (b) For $l=0,\dots,p-1$, do the iterations

$$u_{h_k}^{(l+1)}=\mathrm{SemilinearMG}(V_H,u_{h_k}^{(l)},V_{h_k}).$$

   (c) Define $u_{h_k}=u_{h_k}^{(p)}$.

   End Do

Finally, we obtain an approximate solution $u_{h_n}\in V_{h_n}$.

**Theorem 3.2.** *After implementing Algorithm* 3.2, *we have the error estimates for the final approximation* $u_{h_n}$

$$\|\bar{u}_{h_n} - u_{h_n}\|_a \leqslant \frac{2\gamma^p\beta}{1-\gamma^p\beta}\delta_{h_n}(u), \tag{3.19}$$

$$\|\bar{u}_{h_n} - u_{h_n}\|_0 \leqslant C\eta_a(V_H)\|\bar{u}_{h_n} - u_{h_n}\|_a, \tag{3.20}$$

*under the condition that the coarsest mesh size* $H$ *is small enough so that* $\gamma^p\beta < 1$.

Proof. From the first step of Algorithm 3.2, we have $u_{h_1} = \bar{u}_{h_1}$. Then from Lemma 2.1 and the proof of Theorem 3.1, the following estimates hold:

$$\begin{aligned}\|\bar{u}_{h_2} - u_{h_2}\|_a &= \|\bar{u}_{h_2} - u_{h_2}^{(p)}\|_a \leqslant \gamma^p\|\bar{u}_{h_2} - u_{h_2}^{(0)}\|_a \\ &= \gamma^p\|\bar{u}_{h_2} - u_{h_1}\|_a = \gamma^p\|\bar{u}_{h_2} - \bar{u}_{h_1}\|_a,\end{aligned} \tag{3.21}$$

$$\|\bar{u}_{h_2} - u_{h_2}\|_0 \leqslant C\eta_a(V_H)\|\bar{u}_{h_2} - u_{h_2}\|_a. \tag{3.22}$$

Based on (3.21), (3.22), Theorem 3.1, and a recursive argument, the final approximate solution has the error estimates

$$\begin{aligned}\|\bar{u}_{h_n} - u_{h_n}\|_a &\leqslant \gamma^p\|\bar{u}_{h_n} - u_{h_n}^{(0)}\|_a = \gamma^p\|\bar{u}_{h_n} - u_{h_{n-1}}\|_a \\ &\leqslant \gamma^p(\|\bar{u}_{h_n} - \bar{u}_{h_{n-1}}\|_a + \|\bar{u}_{h_{n-1}} - u_{h_{n-1}}\|_a) \\ &\leqslant \gamma^p\|\bar{u}_{h_n} - \bar{u}_{h_{n-1}}\|_a + \gamma^{2p}\big(\|\bar{u}_{h_{n-1}} - \bar{u}_{h_{n-2}}\|_a + \|\bar{u}_{h_{n-2}} - u_{h_{n-2}}\|_a\big) \\ &\leqslant \sum_{k=1}^{n-1}\gamma^{kp}\|\bar{u}_{h_{n-k+1}} - \bar{u}_{h_{n-k}}\|_a \\ &\leqslant \sum_{k=1}^{n-1}\gamma^{kp}(\|\bar{u}_{h_{n-k+1}} - u\|_a + \|u - \bar{u}_{h_{n-k}}\|_a) \\ &\leqslant 2\sum_{k=1}^{n-1}\gamma^{kp}\delta_{h_{n-k}}(u) \leqslant 2\sum_{k=1}^{n-1}\gamma^{kp}\beta^k\delta_{h_n}(u) \leqslant \frac{2\gamma^p\beta}{1-\gamma^p\beta}\delta_{h_n}(u),\end{aligned}$$

which is just the desired result (3.19). The second result (3.20) can be proved by an argument similar to that in the proof of Theorem 3.1 and the proof is complete. □

**Corollary 3.1.** *For the final approximation* $u_{h_n}$ *obtained by Algorithm* 3.2, *we have the estimates*

$$\|u - u_{h_n}\|_a \lesssim \delta_{h_n}(u), \tag{3.23}$$

$$\|u - u_{h_n}\|_0 \lesssim \eta_a(V_H)\delta_{h_n}(u). \tag{3.24}$$

P r o o f. This is a direct consequence of the combination of Lemma 2.1 and Theorem 3.2. □

**3.3. Estimate of the computational work.** In this subsection, we turn our attention to the estimate of computational work for the full multigrid method defined in Algorithm 3.2. It will be shown that the full multigrid method makes solving the semilinear elliptic problem almost as cheap as solving the corresponding linear boundary value problems.

First, we define the dimension of each level finite element space as $N_k := \dim V_{h_k}$. Then we have

$$N_k \approx \Big(\frac{1}{\beta}\Big)^{d(n-k)} N_n, \quad k = 1, 2, \ldots, n. \tag{3.25}$$

The computational work for the second step in Algorithm 3.2 is different from the linear elliptic problems [3], [13], [14], [15], [20]. In this step, we need to solve the semilinear elliptic problem (3.6). Always, some type of nonlinear iteration method (fixed-point iteration or Newton type iteration) is adopted to solve this low dimensional semilinear elliptic problem. In each nonlinear iteration step, it is required to assemble the matrix on the finite element space $V_{H,h_k}$ $(k = 2, \ldots, n)$ which needs the computational work $\mathcal{O}(N_k)$. Fortunately, the matrix assembling can be carried out in the parallel way easily in the finite element space, since it has no data transfer.

**Theorem 3.3.** *Assume we use $\vartheta$ computing nodes in Algorithm 3.2; the semilinear elliptic solving in the coarse spaces $V_{H,h_k}$ $(k = 2, \ldots, n)$ and $V_{h_1}$ needs work $\mathcal{O}(M_H)$ and $\mathcal{O}(M_{h_1})$, respectively, and the work of the multigrid iteration for the boundary value problem in each level space $V_{h_k}$ is $\mathcal{O}(N_k)$ for $k = 2, 3, \ldots, n$. Let $\varpi$ denote the nonlinear iteration times when we solve the semilinear elliptic problem (3.6). Then in each computational node, the work involved in Algorithm 3.2 has the estimate*

$$\text{Total work} = \mathcal{O}\Big(\Big(1 + \frac{\varpi}{\vartheta}\Big) N_n + M_H \log N_n + M_{h_1}\Big). \tag{3.26}$$

P r o o f. We use $W_k$ to denote the work involved in each correction step on the $k$th finite element space $V_{h_k}$. From the definition of Algorithm 3.2, we have the estimate

$$W_k = \mathcal{O}\Big(N_k + M_H + \varpi \frac{N_k}{\vartheta}\Big). \tag{3.27}$$

Based on the property (3.25), iterating (3.27) leads to

$$\text{Total work} = \sum_{k=1}^{n} W_k = \mathcal{O}\Big(M_{h_1} + \sum_{k=2}^{n}\Big(N_k + M_H + \varpi\frac{N_k}{\vartheta}\Big)\Big) \tag{3.28}$$
$$= \mathcal{O}\Big(\sum_{k=2}^{n}\Big(1+\frac{\varpi}{\vartheta}\Big)N_k + (n-1)M_H + M_{h_1}\Big)$$
$$= \mathcal{O}\Big(\sum_{k=2}^{n}\Big(\frac{1}{\beta}\Big)^{d(n-k)}\Big(1+\frac{\varpi}{\vartheta}\Big)N_n + M_H\log N_n + M_{h_1}\Big)$$
$$= \mathcal{O}\Big(\Big(1+\frac{\varpi}{\vartheta}\Big)N_n + M_H\log N_n + M_{h_1}\Big).$$

This is the desired result and we have completed the proof. □

Remark 3.2. Since we always have a good enough initial solution $\widetilde{u}_{h_k}^{(l+1)}$ in the second step of Algorithm 3.1, solving the semilinear elliptic problem (3.6) never needs many nonlinear iterations. In this case, the complexity in each computational node will be $\mathcal{O}(N_n)$ provided $M_H \ll N_n$ and $M_{h_1} \leqslant N_n$. For more difficult nonlinear problems, the complexity in each computational node can also be bounded by $\mathcal{O}(N_n)$ in the parallel way with enough computational nodes.

## 4. Numerical results

In this section, four numerical experiments are presented to verify the theoretical analysis and the efficiency of Algorithm 3.2. We will check different nonlinear terms which include polynomial, exponential functions and a function only having bounded first order derivative. Furthermore, we also investigate the performance of the full multigrid method on the adaptively refined meshes. In all examples, we choose $m = 2$ and $p = 1$.

Example 4.1. We consider the following semilinear elliptic problem:

$$\begin{cases} -\Delta u + u^3 = g & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega = (0,1)^3$. We choose the right-hand side term $g$ such that the exact solution is given by

$$u = \sin(\pi x)\sin(\pi y)\sin(\pi z). \tag{4.2}$$

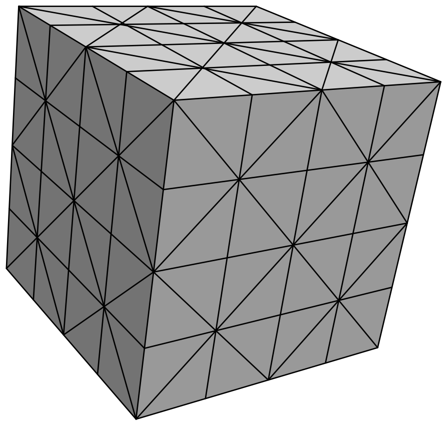

Figure 1. The initial mesh for Example 4.1.

We give the numerical results for the approximate solutions by Algorithm 3.2. Figure 1 shows the initial triangulation. Figure 2 shows the error estimates and the CPU time in seconds. It is shown in Figure 2 that the approximate solution by Algorithm 3.2 has the optimal convergence order and the linear computational complexity which coincides with the theoretical results in Theorems 3.1, 3.2, and Corollary 3.1.

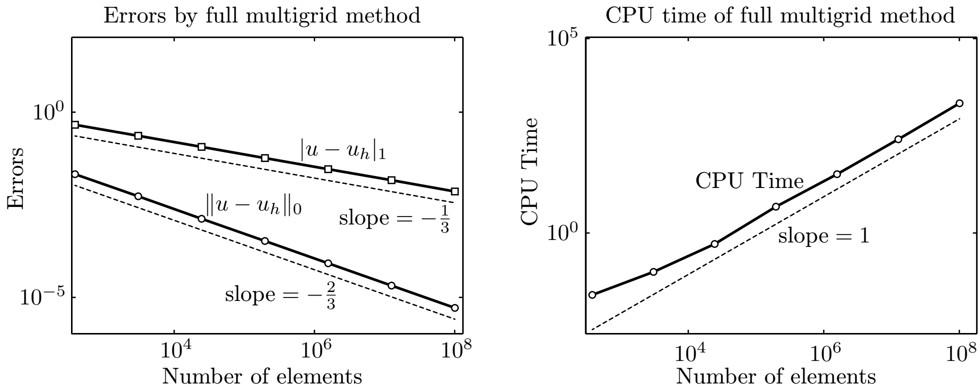


Figure 2. Errors and CPU time (in seconds) of Algorithm 3.2 for Example 4.1.

E x a m p l e 4.2. In the second example, we solve the following semilinear elliptic problem:

$$\begin{cases} -\Delta u - \mathrm{e}^{-u} = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where $\Omega = (0,1)^3$. Since the exact solution is not known, we choose an adequately accurate approximate solution on a fine enough mesh as the exact one.

Algorithm 3.2 is applied to this example. Figure 1 shows the initial mesh. Figure 3 gives the corresponding numerical results which also show the optimal convergence rate and linear computational complexity of Algorithm 3.2.

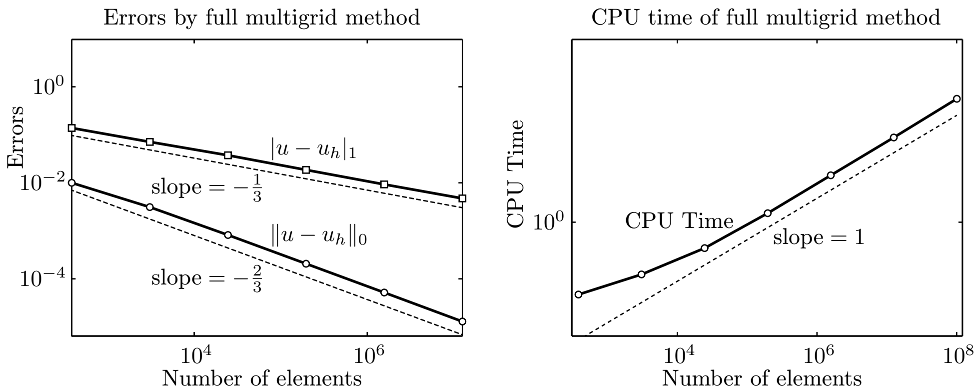


Figure 3. Errors and CPU time (in seconds) of Algorithm 3.2 for Example 4.2.

Example 4.3. In the third example, we solve the following semilinear elliptic problem:

$$
\begin{cases} -\Delta u + f(x,u) = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}
$$

with

$$
f(x,u) = \begin{cases} u^{3/2} & \text{if } u \geqslant 0, \\ -u^{3/2} & \text{if } u < 0, \end{cases} \tag{4.5}
$$

where $\Omega = (0,1)^3$. We choose the right-hand side term $g$ such that the exact solution is given by

$$
u = \sin(2\pi x)\sin(2\pi y)\sin(2\pi z). \tag{4.6}
$$

In this example, the nonlinear term $f(x,v)$ has bounded first order derivative $\partial f(x,v)/\partial v$ but unbounded second order derivative $\partial^2 f(x,v)/\partial^2 v$. Then the methods given in [9], [14] cannot be used for this example.

Algorithm 3.2 is applied to this example. Figure 1 shows the initial mesh. Figure 4 gives the corresponding numerical results which also show the optimal convergence rate and linear computational complexity of Algorithm 3.2.

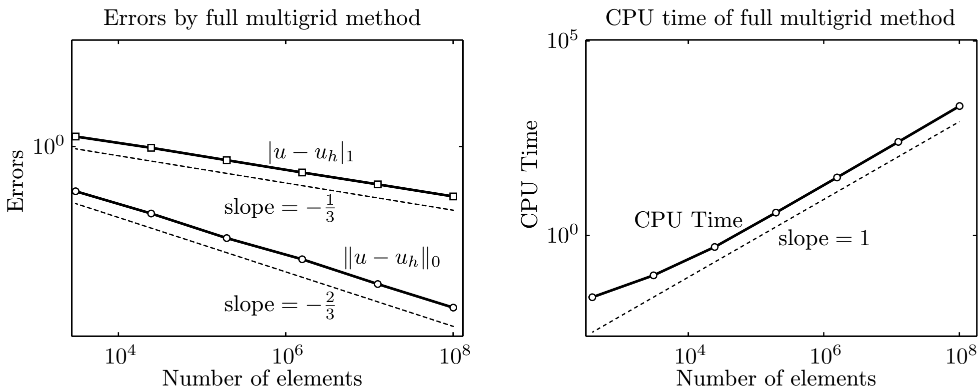


Figure 4. Errors and CPU time (in seconds) of Algorithm 3.2 for Example 4.3.

Example 4.4. In the last example, we solve the following semilinear elliptic problem:

$$\begin{cases} -\Delta u + u^{3/2} = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

where $\Omega = (-1,1)^3 \setminus [0,1)^3$. Due to the reentrant corner of $\Omega$, the exact solution with singularities is expected. The convergence order for the approximate solution is less than the order predicted by the theory for regular solutions. Thus, the adaptive refinement is adopted to couple with the full multigrid method described in Algorithm 3.2 (cf. [12]).

Since the exact solution is not known, we also choose an adequately accurate approximation on a fine enough mesh as the exact one. We give the numerical results of the full multigrid method in which the sequence of meshes $\mathcal{T}_{h_1}, \ldots, \mathcal{T}_{h_n}$ is produced by the adaptive refinement with the a posteriori error estimator

$$\eta^2(v,K) := h_K^2 \|\mathbb{R}_K(v)\|_{0,K}^2 + \sum_{e\in\mathcal{E}_I, e\subset\partial K} h_e \|\mathcal{J}_e(v)\|_{0,e}^2, \tag{4.8}$$

where the element residual $\mathbb{R}_K(v)$ and the jump residual $\mathcal{J}_e(v)$ are defined as follows:

$$\mathbb{R}_K(v) := g - f(x,v) - \nabla\cdot(\mathcal{A}\nabla v) \quad \text{in } K \in \mathcal{T}_{h_k}, \tag{4.9}$$

$$\mathcal{J}_e(v) := -\mathcal{A}\nabla v^+ \cdot \nu^+ - \mathcal{A}\nabla v^- \cdot \nu^- := [\mathcal{A}\nabla v]_e \cdot \nu_e \quad \text{on } e \in \mathcal{E}_I. \tag{4.10}$$

Here $\mathcal{E}_I$ denotes the set of interior faces (edges or sides) of $\mathcal{T}_{h_k}$ and $e$ is the common side of elements $K^+$ and $K^-$ with the unit outward normals $\nu^+$ and $\nu^-$, respectively, and $\nu_e = \nu^-$.

Figure 5 shows the mesh after 15 refinements and the corresponding cross section. Figure 6 shows the numerical results by Algorithm 3.2. From Figure 6, we can find that the full multigrid method can also work on the adaptive family of meshes and obtain the optimal accuracy. The full multigrid method can be coupled with the adaptive refinement naturally to produce a type of adaptive finite element method for semilinear elliptic problem, where the direct nonlinear iteration in the adaptive finite element space is not required. This can also improve the overall efficiency of the adaptive finite element method for semilinear elliptic problem solving. For more information, we refer to the paper [12].

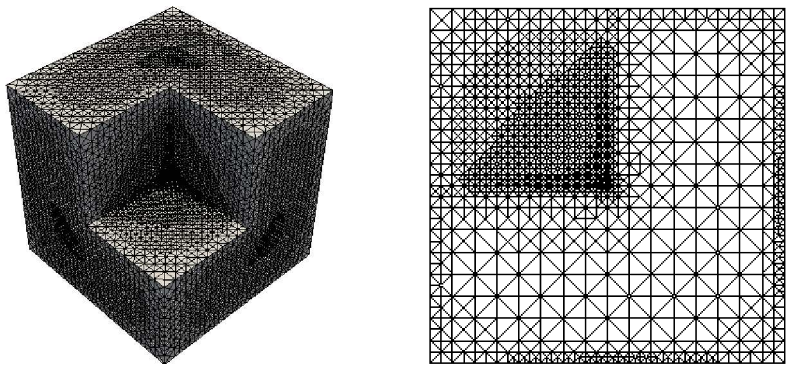

Figure 5. The triangulations after 15 adaptive refinements and the corresponding cross section for Example 4.4.

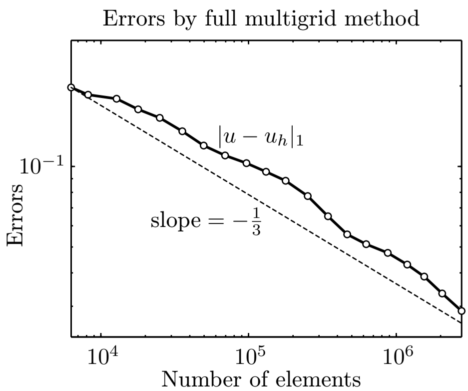


Figure 6. Errors of Algorithm 3.2 for Example 4.4.

## 5. Concluding remarks

In this paper, a full multigrid method is proposed for solving semilinear elliptic equations by the finite element method. The corresponding estimates of error and computational work are given. The main idea is to transform the solution of the semilinear problem into a series of solutions of the corresponding linear boundary value problems on the sequence of finite element spaces and semilinear problems on a very low dimensional space. Compared with the existing multigrid methods which require bounded second order derivatives of the nonlinear term, the proposed method only needs the Lipschitz continuity in some sense of the nonlinear term. Based on the full multigrid method, all existing efficient solvers for the linear elliptic problems can serve as solvers for the semilinear equations. The idea and algorithm in this paper can be extended to other nonlinear problems such as Navier-Stokes problems and phase field models.

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