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$\mathcal{C}^k$-REGULARITY FOR THE $\overline{\partial}$-EQUATION WITH A SUPPORT CONDITION

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Abstract. Let D be a $\mathcal{C}^d$ q-convex intersection, $d \geqslant 2$, $0 \leqslant q \leqslant n-1$, in a complex manifold X of complex dimension n, $n \geqslant 2$, and let E be a holomorphic vector bundle of rank N over X. In this paper, $\mathcal{C}^k$-estimates, $k = 2, 3, \ldots, \infty$, for solutions to the $\overline{\partial}$-equation with small loss of smoothness are obtained for E-valued $(0,s)$-forms on D when $n-q \leqslant s \leqslant n$. In addition, we solve the $\overline{\partial}$-equation with a support condition in $\mathcal{C}^k$-spaces. More precisely, we prove that for a $\overline{\partial}$-closed form f in $\mathcal{C}^k_{0,q}(X \setminus D, E)$, $1 \leqslant q \leqslant n-2$, $n \geqslant 3$, with compact support and for ε with $0 < \varepsilon < 1$ there exists a form u in $\mathcal{C}^{k-\varepsilon}_{0,q-1}(X \setminus D, E)$ with compact support such that $\overline{\partial} u = f$ in $X \setminus \overline{D}$. Applications are given for a separation theorem of Andreotti-Vesentini type in $\mathcal{C}^k$-setting and for the solvability of the $\overline{\partial}$-equation for currents.

Keywords: $\overline{\partial}$-equation; q-convexity; $\mathcal{C}^k$-estimate

MSC 2010: 32F10, 32W05

1. Background and the main results

The $\mathcal{C}^k$-solvability of the $\overline{\partial}$-equation is a central theme in the theory of several complex variables, it was studied by Lieb and Range in [9] for strictly pseudoconvex domains in $\mathbb{C}^n$ and by Michel in [10] for the piecewise smooth case. A few years ago, Barkatou and Khidr in [3] proved that if f is a $\overline{\partial}$-closed continuous $(0,s)$-form, $n-q \leqslant s \leqslant n$, on a $\mathcal{C}^d$, $d \geqslant 2$, q-convex intersection Ω in $\mathbb{C}^n$, $0 \leqslant q \leqslant n-1$, then there exists a continuous $(0,s-1)$-form u on Ω such that $\overline{\partial} u = f$ in Ω. Moreover, if f is in $\mathcal{C}^k_{0,s}(\overline{\Omega})$, $k = 2, 3, \ldots, \infty$, and if $0 < \varepsilon < 1$, then u is in $\mathcal{C}^{k-\varepsilon}_{0,s-1}(\overline{\Omega})$ and for each $0 < \varepsilon < 1$ there is a constant $C_{k,\varepsilon} > 0$ such that $\|u\|_{k-\varepsilon,\overline{\Omega}} \leqslant C_{k,\varepsilon} \|f\|_{k,\overline{\Omega}}$. The q-concave case is also settled in [7].

The solvability of the $\overline{\partial}$-problem with a support condition was initiated by Andreotti and Hill, see [1], [2], in terms of the Dolbeault $\overline{\partial}$-cohomology groups. In [8],

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Laurent-Thiébaut and Leiterer proved that if E is a holomorphic vector bundle over a complex manifold X of complex dimension n and Ω is an open set in X (not necessarily relatively compact in X) with smooth and compact boundary such that X is a q-convex extension of $\overline{\Omega}$, $1 \leqslant q \leqslant n-1$, then, for every $\overline{\partial}$-closed form f in $\mathcal{C}^k_{0,q}(X \setminus \Omega, E)$, $k = 0, 1, \ldots, \infty$, with compact support, there exists a form u in $\mathcal{C}^{k+1/2}_{0,q-1}(X \setminus \Omega, E)$ with compact support such that $\overline{\partial} u = f$ in $X \setminus \overline{\Omega}$.

If Ω is a relatively compact domain with Lipschitz boundary and satisfying a convexity condition called $\log \delta$-pseudoconvex in an n-dimensional Kähler manifold X, Brinkschulte in [4] proved that for every $\overline{\partial}$-closed form f in $\mathcal{C}^\infty_{r,s}(X, E)$, $0 \leqslant r \leqslant n$, $1 \leqslant s \leqslant n-1$, with compact support in $\overline{\Omega}$ there exists a form u in $\mathcal{C}^\infty_{r,s-1}(X, E)$ supported in $\overline{\Omega}$ such that $\overline{\partial} u = f$ in X. Moreover, she proved that the range of the $\overline{\partial}$-operator acting on the subspace of those forms in $\mathcal{C}^\infty_{r,n-1}(X, E)$ with compact support in $\overline{\Omega}$ is closed.

When Ω is a completely strictly q-convex domain, $0 \leqslant q \leqslant n-1$, with smooth boundary in a complex manifold X of complex dimension n, analogous results to those of [4] have been obtained by Sambou in [13] for $\mathbb{C}$-valued (r,s)-forms with compact support in $\overline{\Omega}$, where $0 \leqslant r \leqslant n$ and $1 \leqslant s \leqslant q$. In addition, for all s such that $1 \leqslant s \leqslant q+1$, the author proved that the range of the $\overline{\partial}$-operator acting on the subspace of $\mathcal{C}^\infty$-$(r, s-1)$-forms with compact support in $\overline{\Omega}$ is closed. Further, he proved that the $\overline{\partial}$-equation is solvable on such domains for extensible currents of bidegree $(n, n-s)$ for all s such that $n-q \leqslant s \leqslant n$. Furthermore, he studied the case for strictly q-concave domains in [14].

In [12], Ricard proved weaker $\mathcal{C}^k$-estimates than those obtained by Barkatou and Khidr in [3] but for general q-convex wedges. Morover, she solved the $\overline{\partial}$-equation for E-valued $(0,s)$-forms of class $\mathcal{C}^\infty$ and with compact support in the complement of q-convex wedge in a complex manifold. This result enabled her to generalize the Andreotti Vesentini separation theorem for E-valued $(0,s)$-forms of class $\mathcal{C}^\infty$ to the complements of q-convex wedges in complex manifolds for some bidegree.

Let Ω be a bounded domain in a complex manifold X of complex dimension n, $n \geqslant 2$, and let E be a holomorphic Hermitian vector bundle of rank N over X. We fix the following notation. For all $1 \leqslant s \leqslant n$ and $l \in \mathbb{Z}^+$, we denote by $\mathcal{C}^l_{0,s}(\Omega, E)$ the Fréchet space of all E-valued $(0,s)$-forms with coefficients of class $\mathcal{C}^l$ on Ω with the topology of uniform convergence of forms and all their derivatives on compact subsets of Ω. Let K be a compact subset of Ω and $\mathcal{D}^{0,s}_K(\Omega, E)$ the closed subspace of $\mathcal{C}^l_{0,s}(\Omega, E)$ of forms with support in K endowed with the induced topology from the topology on $\mathcal{C}^l_{0,s}(\Omega, E)$. Let $\mathcal{D}^{0,s}(\Omega, E)$ be the linear subspace of $\mathcal{C}^l_{0,s}(\Omega, E)$ of all forms with compact support equipped with the strict inductive limit topology defined by the Fréchet spaces $\mathcal{D}^{0,s}_{K_i}(\Omega, E)$, where K_i are compact subsets of Ω such that $K_i \subset K^\circ_{i+1}$ and $\bigcup_i K_i = \Omega$. Then $\mathcal{D}^{0,s}(\Omega, E) = \bigcup_i \mathcal{D}^{0,s}_{K_i}(\Omega, E)$.

Further, for all $1 \leqslant s \leqslant n$ and $l, \beta \in \mathbb{R}^+$, we denote by $\mathcal{C}^l_{0,s}(\overline{\Omega}, E)$ the Banach space of all E-valued $(0,s)$-forms on $\overline{\Omega}$ which have continuous derivatives up to $[l]$ on $\overline{\Omega}$ satisfying Hölder condition of order $l-[l]$; the symbol $[l]$ is the integral part of l. The associated cohomologies groups are denoted by $H^l_{0,s}(\Omega, E)$. The corresponding norm is denoted by $\|\cdot\|_{l,\Omega}$. The subspace of all $\overline{\partial}$-closed forms in $\mathcal{C}^l_{r,s}(\overline{\Omega}, E)$ is denoted by $Z^l_{0,s}(\overline{\Omega}, E)$, and $E^{\beta\to l}_{0,s}(\overline{\Omega}, E)$ is the subspace of those forms f in $Z^l_{0,s}(\overline{\Omega}, E)$ such that $f = \overline{\partial} u$ for some u in $\mathcal{C}^\beta_{0,s-1}(\overline{\Omega}, E)$.

Furthermore, by $\mathcal{D}^l_{0,s}(\Omega, E)$ we denote the Fréchet space of forms in $\mathcal{C}^l_{0,s}(X, E)$ with support in Ω and endowed with the Fréchet topology of $\mathcal{C}^l_{0,s}(\Omega, E)$. We note that if Ω is compact, then $\mathcal{D}^l_{0,s}(\Omega, E)$ is a Banach space. $\mathcal{D}^l_{0,s}(\overline{\Omega}, X, E)$ denotes the Banach space of all E-valued $(0,s)$-forms on X with support in $\overline{\Omega}$ and their restriction to $\overline{\Omega}$ being in $\mathcal{C}^l_{0,s}(\overline{\Omega}, E)$. The dual space of $\mathcal{D}^l_{0,s}(\Omega, E)$ is denoted by $\mathcal{D}'^l_{n,n-s}(\Omega, E^*)$, it is a subspace of all currents in $\mathcal{D}'_{n,n-s}(\Omega, E^\star)$ of order l on Ω. The $\overline{\partial}$-operator is defined from $\mathcal{D}'^l_{n,n-s}(\Omega, E^\star)$ into $\mathcal{D}'^l_{n,n-s+1}(\Omega, E^\star)$ as the transpose of the usual $\overline{\partial}$ operator from $\mathcal{D}^l_{0,s}(\Omega, E)$ into $\mathcal{D}^l_{0,s+1}(\Omega, E)$. Finally, we recall the notion of q-convexity.

Definition 1.1. A real-valued function ϱ of class $\mathcal{C}^2$ on a complex manifold X of complex dimension n is said to be q-convex, $0 \leqslant q \leqslant n-1$, if its Levi form L_ϱ has at least $q+1$ positive eigenvalues at every point in X. A bounded domain Ω in X is called strictly q-convex, $0 \leqslant q \leqslant n-1$, if there exist an open neighborhood $\mathbb{U}$ of $\partial\Omega$ and a $\mathcal{C}^2$ q-convex function $\varrho\colon \mathbb{U} \to \mathbb{R}$ such that $\Omega \cap \mathbb{U} = \{\zeta \in \mathbb{U}\colon \varrho(\zeta) < 0\}$.

Definition 1.2. A bounded domain Ω in an n-dimensional complex manifold X, $n \geqslant 2$, is called a $\mathcal{C}^d$, $d \geqslant 2$, q-convex intersection, $0 \leqslant q \leqslant n-1$, if there exist a bounded neighborhood U of $\overline{\Omega}$ and a finite number of real-valued $\mathcal{C}^d$ functions $\varrho_1(z), \dots, \varrho_b(z)$, $1 \leqslant b \leqslant n-1$, defined on U such that $\Omega = \{z \in U\colon \varrho_1(z) < 0, \dots, \varrho_b(z) < 0\}$ and the following conditions are fulfilled:

(1) For $1 \leqslant i_1 < i_2 < \ldots < i_l \leqslant b$ the 1-forms $d\varrho_{i_1}, \dots, d\varrho_{i_l}$ are $\mathbb{R}$-linearly independent on the set $\bigcap\limits_{j=1}^{l} \{\varrho_{i_j}(z) \leqslant 0\}$.

(2) For $1 \leqslant i_1 < i_2 < \ldots < i_l \leqslant b$, for every $z \in \bigcap\limits_{j=1}^{l} \{\varrho_{i_j}(z) \leqslant 0\}$, if we set $I = (i_1, \dots, i_l)$, there exists a linear subspace T^I_z of X of complex dimension at least $q+1$ such that for $i \in I$ the Levi forms L_{ϱ_i} restricted to T^I_z are positive definite.

Condition (2) was introduced first by Grauert in [5]. It implies that at every wedge the Levi forms of the corresponding $\{\varrho_i\}$ have their positive eigenvalues along the same directions.

Definition 1.3. Let K be a closed subset of an n-dimensional complex manifold X. We say that X is a q-convex extension of K, $1 \leqslant q \leqslant n-1$, if there exist two constants c and C such that $-\infty < c < C \leqslant \infty$ and a $\mathcal{C}^2$ q-convex function $\varrho\colon U \to (-\infty, C]$ on an open neighborhood U of $\overline{X \setminus K}$ such that $K \cap U = \{\varrho \leqslant c\}$ and the set $\{c \leqslant \varrho \leqslant t\}$ is compact for all $t < C$.

Further, X is said to be a generalized q-convex extension of K if for every neighborhood V of K there exists a closed subset K_0 with $\mathcal{C}^\infty$ boundary such that $K \subset K_0 \subset V$ and X is a q-convex extension of K_0.

We note that if the boundary of K is of class $\mathcal{C}^\infty$, the fact that X is a q-convex extension of K implies that X is a generalized q-convex extension of K.

The main results of this paper are formulated in the next two theorems. More precisely, we first prove the following global $\mathcal{C}^k$-existence theorem.

Theorem 1.4. *Let $D \subset\subset X$ be a $\mathcal{C}^d$, $d \geqslant 2$, q-convex intersection in a complex manifold X of complex dimension n, $n \geqslant 2$, and let E be a Hermitian holomorphic vector bundle over X. Then for every $\overline{\partial}$-closed form f in $\mathcal{C}^0_{0,s}(D,E)$, $n-q \leqslant s \leqslant n$, there exists a form g in $\mathcal{C}^0_{0,s-1}(D,E)$ such that $\overline{\partial} g = f$. Moreover, if f is in $\mathcal{C}^k_{0,s}(\overline{D},E)$, $k = 2,3,\ldots,\infty$, and if $0 < \varepsilon < 1$, then g is in $\mathcal{C}^{k-\varepsilon}_{0,s-1}(\overline{D},E)$ and there is a constant $C_{k,\varepsilon} > 0$ (independent of f) such that*

$$\|g\|_{\mathcal{C}^{k-\varepsilon}_{0,s-1}(\overline{D},E)} \leqslant C_{k,\varepsilon} \|f\|_{\mathcal{C}^k_{0,s}(\overline{D},E)}. \tag{1.1}$$

In the case $q = n-1$ (i.e. the strictly pseudoconvex case) and $X = \mathbb{C}^n$, this theorem was proved by Michel and Perotti in [11]. For the strictly q-convex case, $0 \leqslant q \leqslant n-1$, with $\mathcal{C}^\infty$ boundary, sharp $\mathcal{C}^k$ estimates were obtained by Lieb and Range in [9]. We note further that Theorem 1.4 is still valid for the particular case when $X = \mathbb{C}$, E is the trivial line bundle with the flat metric and $q = 0$, since every smooth domain in $\mathbb{C}$ is strictly pseudoconvex.

Furthermore, we prove the following $\mathcal{C}^k$-regularity with a support condition for the $\overline{\partial}$-equation on the complement of a q-convex intersection in a complex manifold.

Theorem 1.5. *Let $D \subset\subset X$ be a $\mathcal{C}^d$, $d \geqslant 2$, q-convex intersection in an n-dimensional complex manifold X, $n \geqslant 3$, and let E be a holomorphic Hermitian vector bundle over X. We assume moreover that X is a generalized q-convex extension of $\overline{D}$. If f is a $\overline{\partial}$-closed form in $\mathcal{C}^k_{0,q}(X \setminus D, E)$, $k = 2,3,\ldots,\infty$, $1 \leqslant q \leqslant n-2$, with compact support and if $0 < \varepsilon < 1$, then there exists a form u in $\mathcal{C}^{k-\varepsilon}_{0,q-1}(X \setminus D, E)$ with compact support such that $\overline{\partial} u = f$ in $X \setminus \overline{D}$.*

As an application of Theorem 1.5, we will prove a separation theorem of Andreotti-Vesentini type in $\mathcal{C}^k$-spaces and, moreover, solve the $\overline{\partial}$-equation for currents, see Theorems 4.1 and 4.2.

2. Proof of Theorem 1.4

The proof of Theorem 1.4 consists of three main steps. First, we prove the following local result. Let $\{U_j\}_{j\in I}$ be an open covering of X consisting of coordinate neighborhoods U_j with holomorphic coordinates $z_j = (z_j^1, z_j^2, \ldots, z_j^n)$ over which E is trivial. Cover ∂D by a finite number of open sets $U_1, U_2, \ldots, U_m$ of the covering $\{U_j\}_{j\in I}$ such that $U_j \cap D$ is a local q-convex intersection; moreover, we may assume that E is trivial over some coordinate neighborhoods V_j of each $\overline{U_j \cap D}$. It follows from Theorem 3.1 in [3] that there are local linear integral solution operators $T_j^s\colon \mathcal{C}^0_{0,s}(\overline{D \cap U_j}\,) \to \mathcal{C}^0_{0,s-1}(\overline{D\cap U_j}\,)$, $j = 1, \ldots, m$, such that $\overline{\partial} T_j^s f = f$ for all $\overline{\partial}$-closed forms f in $\mathcal{C}^0_{0,s}(\overline{D\cap U_j}\,)$.

We now extend these operators to E-valued forms on $D \cap U_j$. To this end, we define the operators $T_N^s\colon\ f \in \mathcal{C}^0_{0,s}(\overline{D\cap U_j}, E) \to T_N^s f \in \mathcal{C}^0_{0,s-1}(\overline{D\cap U_j}, E)$ by $T_N^s f = \sum_{\lambda=1}^{N} T_j^s f^\lambda \omega_\lambda$, where $n - q \leqslant s \leqslant n$ and f^λ are the components of the restriction of f to $U_j \cap D$ with respect to a holomorphic orthonormal basis $\omega_1, \ldots, \omega_N$ on E_z for every $z \in U_j \cap D$. We consequently get the following local theorem.

Theorem 2.1. *Let $D \subset\subset X$ be a $\mathcal{C}^d$, $d \geqslant 2$ q-convex intersection in a complex manifold X of complex dimension n, $n \geqslant 2$, and let E be a Hermitian holomorphic vector bundle of rank N over X. Then for each $\xi \in \partial D$ there exist a local q-convex intersection D^ξ in X and bounded linear operators $\widetilde{T}^s\colon\ \mathcal{C}^0_{0,s}(\overline{D^\xi}, E) \to \mathcal{C}^0_{0,s-1}(\overline{D^\xi}, E)$ such that $\overline{\partial}\widetilde{T}^s f = f$ for every $\overline{\partial}$-closed form f in $\mathcal{C}^0_{0,s}(\overline{D^\xi}, E)$ and all s such that $n - q \leqslant s \leqslant n$. Further, if f is in $\mathcal{C}^k_{0,s}(\overline{D^\xi}, E)$, $k = 2, 3, \ldots, \infty$, and if $0 < \varepsilon < 1$, then $\widetilde{T}^s f$ is in $\mathcal{C}^{k-\varepsilon}_{0,s-1}(\overline{D^\xi}, E)$ and there is a positive constant $C_{k,\varepsilon}$ (independent of f) such that*

$$\|\widetilde{T}^s f\|_{\mathcal{C}^{k-\varepsilon}_{0,s-1}(\overline{D^\xi}, E)} \leqslant C_{k,\varepsilon} \|f\|_{\mathcal{C}^k_{0,s}(\overline{D^\xi}, E)}.$$

As in [3], via a partition of unity, the following lemma follows immediately from Theorem 2.1.

Lemma 2.2. *Let X, D and E be as in Theorem 2.1. Then there exists another slightly larger q-convex intersection $\widetilde{D} \subset\subset X$ such that $D \subset\subset \widetilde{D}$ and for every $\overline{\partial}$-closed form f in $\mathcal{C}^0_{0,s}(D, E)$, $n - q \leqslant s \leqslant n$, there exist two linear operators $H_1\colon\ f \in \mathcal{C}^0_{0,s}(D, E) \to \widetilde{f} \in \mathcal{C}^0_{0,s}(\widetilde{D}, E)$ and $H_2\colon\ f \in \mathcal{C}^0_{0,s}(D, E) \to u \in \mathcal{C}^0_{0,s-1}(D, E)$ such that*

(i) $\overline{\partial}\tilde{f} = 0$ *in* $\widetilde{D}$;

(ii) $\tilde{f} = f - \overline{\partial}u$ *in* D;

(iii) *if* f *is in* $\mathcal{C}^k_{0,s}(\overline{D}, E)$, $k = 2, 3, \ldots, \infty$, $0 < \varepsilon < 1$, *then* $\tilde{f}$ *is in* $\mathcal{C}^{k-\varepsilon}_{0,s}(\overline{\widetilde{D}}, E)$, u *is in* $\mathcal{C}^{k-\varepsilon}_{0,s-1}(\overline{D}, E)$ *and for each* $0 < \varepsilon < 1$ *there is a constant* $C_{k,\varepsilon}$ *(independent of* f*) such that*

$$\|\tilde{f}\|_{\mathcal{C}^{k-\varepsilon}_{0,s}(\overline{\widetilde{D}},E)} \leqslant C_{k,\varepsilon}\|f\|_{\mathcal{C}^k_{0,s}(\overline{D},E)},$$

$$\|u\|_{\mathcal{C}^{k-\varepsilon}_{0,s-1}(\overline{D},E)} \leqslant C_{k,\varepsilon}\|f\|_{\mathcal{C}^k_{0,s}(\overline{D},E)}.$$

If f *is* C^∞ *in* D, *then* $\tilde{f}$ *is* C^∞ *in* $\widetilde{D}$ *and* u *is* C^∞ *in* D.

The following lemma is a natural extension of [9], Theorem 2, to E-valued forms.

Lemma 2.3. *Let* $D \subset\subset X$ *be a* $\mathcal{C}^d$, $d \geqslant 2$, *strictly* q*-convex domain in a complex manifold* X *of complex dimension* n, $n \geqslant 2$, *and let* E *be a Hermitian holomorphic vector bundle over* X. *Then for every* $\overline{\partial}$*-closed form* f *in* $\mathcal{C}^0_{0,s}(D, E)$, $n - q \leqslant s \leqslant n$, *there exists a form* g *in* $\mathcal{C}^0_{0,s-1}(D, E)$ *such that* $\overline{\partial}g = f$. *Moreover, if* f *is in* $\mathcal{C}^k_{0,s}(\overline{D}, E)$, $k = 2, 3, \ldots, \infty$, *and if* $0 < \varepsilon < 1$, *then* g *is in* $\mathcal{C}^{k-\varepsilon}_{0,s-1}(\overline{D}, E)$ *and there is a constant* $C_{k,\varepsilon} > 0$ *(independent of* f*) such that* $\|g\|_{\mathcal{C}^{k-\varepsilon}_{0,s-1}(\overline{D},E)} \leqslant C_{k,\varepsilon}\|f\|_{\mathcal{C}^k_{0,s}(\overline{D},E)}$.

End of proof of Theorem 1.4. Let $\widetilde{D}$, $\tilde{f}$ and u be as in Lemma 2.2. By Lemma 4.3 in [3], there exists a strictly q-convex domain D' such that $D \subset\subset D' \subset\subset \widetilde{D}$. Let f be a $\overline{\partial}$-closed form in $\mathcal{C}^k_{0,s}(\overline{D}, E)$ and set $\hat{f} = \tilde{f}|_{D'}$. By Lemma 2.3, there exists a form v in $\mathcal{C}^{k-\varepsilon}_{0,s-1}(D, E)$ such that $\overline{\partial}v = \hat{f}$ in D. In view of Lemma 2.2 (ii), we then have $f = \overline{\partial}(u + v)$ in D. The form $g = u + v$ is the desired global solution that satisfies the estimates (1.1). The proof is complete. □

3. Proof of Theorem 1.5

The proof involves several steps which are detailed below. First, a simple modification of the proof of Lemma 3.2 in [8] yields the following local result.

Theorem 3.1. *Let* $D \subset\subset X$ *be a* $\mathcal{C}^d$, $d \geqslant 2$, q*-convex intersection in an* n*-dimensional complex manifold* X, $n \geqslant 3$, *with* $\{\varrho_i\}_{i=1}^b$ *and let* U *be as in Definition 1.2, such that* X *is a generalized* q*-convex extension of* $\overline{D}$. *Let* $\xi \in \partial D$ *and let* V^0 *be a neighborhood of* ξ, *then there exist a* $\delta > 0$ *and a neighborhood* V_δ *of* ξ *such that* $V_\delta \subset\subset V^0$. *Further, if* $\widehat{\varrho}_i\colon U \to \mathbb{R}$ *are* $\mathcal{C}^d$ *functions such that* $\|\varrho_i - \widehat{\varrho}_i\| < \delta$ *and* $\varrho_i \leqslant \widehat{\varrho}_i$ *on* U *for all* $i = 1, \ldots, b$, *then the domain* $\widehat{D}$ *that is defined by those functions* $\widehat{\varrho}_i$ *and satisfies condition* (1) *in Definition* 1.2 *is included in* D *and is a* $\mathcal{C}^d$ q*-convex intersection. Set* $\widehat{\Omega} = X \setminus \overline{\widehat{D}}$. *If* f *is a* $\overline{\partial}$*-closed form in* $\mathcal{C}^k_{0,q}(\widehat{\Omega}, E)$,

$k = 2, 3, \dots, \infty$, $1 \leqslant q \leqslant n-2$ and $0 < \varepsilon < 1$, with compact support, then there exists a form g in $\mathcal{C}^{k-\varepsilon}_{0,q-1}(\overline{\widehat{\Omega} \cap V_\delta}, E)$ with compact support such that $\overline{\partial} g = f$ in $\widehat{\Omega} \cap V_\delta$.

Lemma 3.2. *Let X, E, D and $\widehat{D}$ be given as in Theorem 3.1. Then, for all $k = 2, 3, \dots, \infty$, $1 \leqslant q \leqslant n-2$ and $n \geqslant 3$, we have*

$$E^{k-\varepsilon \to k}_{0,q}(X \setminus \widehat{D}, E) = E^{k-\varepsilon\to k}_{0,q}(X \setminus D, E) \cap Z^k_{0,q}(X \setminus \widehat{D}, E).$$

In addition, if f is in $Z^k_{0,q}(X \setminus \widehat{D}, E)$, $0 < \varepsilon < 1$, so that there exists a form u_1 in $\mathcal{C}^{k-\varepsilon}_{0,q-1}(X \setminus D, E)$ such that $\overline{\partial} u_1 = f$ on $X \setminus \overline{D}$, then there exists a form u_2 in $\mathcal{C}^{k-\varepsilon}_{0,q-1}(X \setminus \widehat{D}, E)$ such that $\overline{\partial} u_1 = f$ on $X \setminus \overline{\widehat{D}}$ and $u_1 = u_2$ on $(X \setminus D) \setminus V_\delta$.

Proof. The proof is just an adaptation of the proof of [12], Lemma 7.9. □

Using Lemmas 2.2 and 3.2 as in [6], we have the next lemma.

Lemma 3.3. *Let $D \subset\subset X$ be a $\mathcal{C}^d$, $d \geqslant 2$, q-convex intersection in an n-dimensional complex manifold X. Then there exists another slightly larger q-convex intersection $\widetilde{D} \subset\subset X$ such that $D \subset\subset \widetilde{D}$. Further, for all $k = 2, 3, \dots, \infty$ and $1 \leqslant q \leqslant n-2$, $n \geqslant 3$, the restriction homomorphisms of cohomology groups*

$$\Phi^k_q \colon H^k_{0,q}(X \setminus D, E) \to H^k_{0,q}(X \setminus \widetilde{D}, E)$$

are injective. Furthermore, if f is in $Z^k_{0,q}(X \setminus D, E)$, $0 < \varepsilon < 1$, such that there exists a form $\tilde{f}$ in $\mathcal{C}^{k-\varepsilon}_{0,q-1}(X \setminus \widetilde{D}, E)$ such that $\overline{\partial} \tilde{f} = f$ on $X \setminus \overline{\widetilde{D}}$, then there exist a neighborhood $V_{\widetilde{D}}$ of $\widetilde{D}$ and a form u in $\mathcal{C}^{k-\varepsilon}_{0,q-1}(X \setminus D, E)$ such that $\overline{\partial} u = f$ on $X \setminus \overline{D}$ and $u|_{X \setminus V_{\widetilde{D}}} = \tilde{f}$.

End of proof of Theorem 1.5. Let f be a form in $Z^k_{0,q}(X \setminus D, E)$ with compact support, $1 \leqslant q \leqslant n-2$, and let $\widetilde{D}$ be as in Lemma 3.3. Since X is a generalized q-convex extension of $\overline{D}$, there exists a strictly q-convex domain D' such that $D' \subset\subset \widetilde{D}$ and X is a q-convex extension of $\overline{D'}$. By Theorem 3.1 in [8], there exists a form g in $\mathcal{D}^{k-\varepsilon}_{0,q}(X \setminus D', E)$ such that $\overline{\partial} g = f$ in $X \setminus \overline{D'}$. Choose a non-negative $\mathcal{C}^\infty$ function ψ such that $\psi \equiv 1$ on a neighborhood of $X \setminus \widetilde{D}$ and $\psi \equiv 0$ on a neighborhood of $\overline{D'}$. Then the form $f - \overline{\partial}(\psi g)$ is zero on $X \setminus \widetilde{D}$, and hence can be trivially extended to $X \setminus D$ which contains $X \setminus \widetilde{D}$ and so it belongs to $Z^k_{0,q}(X \setminus D, E)$ with compact support in $\widetilde{D} \setminus D$, hence $f - \overline{\partial}(\psi g)$ is $\overline{\partial}$-exact on $X \setminus \widetilde{D}$. According to Lemma 3.3, there exists then a form v in $\mathcal{C}^{k-\varepsilon}_{0,q-1}(X \setminus D, E)$ with compact support in $X \setminus D$ such that $\overline{\partial} v = f - \overline{\partial}(\psi g)$ on $X \setminus \overline{D}$. The form $u = v + \psi g$ is therefore in $\mathcal{D}^{k-\varepsilon}_{0.q-1}(X \setminus D, E)$ and solves the equation $\overline{\partial} u = f$ in $X \setminus \overline{D}$. This completes the proof. □

4. Applications

Our first application is concerned with the Andreotti-Vesentini separation theorem in the $\mathcal{C}^k$-case.

Theorem 4.1. *Let $D \subset\subset X$ be a $\mathcal{C}^d$, $d \geqslant 2$, q-convex intersection in an n-dimensional complex manifold X, $n \geqslant 3$, with $\{\varrho_i\}_{i=1}^b$ and U as in Definition 1.2 such that X is a generalized q-convex extension of $\overline{D}$, and let E be a holomorphic vector bundle over X. Assume, moreover, that X is $(n-q)$-convex. Then the space $Z^k_{0,q}(X \setminus D, E) \cap \overline{\partial}\mathcal{C}^{k-\varepsilon}_{0,q-1}(X \setminus D, E)$ is a closed subspace of $\mathcal{C}^k_{0,q}(X \setminus D, E)$ with respect to the topology of uniform convergence of forms and all their derivatives on compact subsets of $X \setminus D$.*

Proof. Let $\{f_i\}_{i\in\mathbb{N}}$ be a sequence of forms in $Z^k_{0,q}(X\setminus D, E)\cap\overline{\partial}\mathcal{C}^{k-\varepsilon}_{0,q-1}(X\setminus D, E)$ which converges uniformly to a form f in $\mathcal{C}^k_{0,q}(X \setminus D, E)$ on compact subsets of $X\setminus D$. Let $\widetilde{D}$ be as in Lemma 3.3. Since X is a generalized q-convex extension of $\overline{D}$, there is a strictly q-convex domain D' such that $D \subset\subset D' \subset\subset \widetilde{D}$. It follows from Theorem 1.3 in [8] that there exists a form u in $\mathcal{C}^{k-\varepsilon}_{0,q-1}(X \setminus \overline{D'}, E)$ such that $\overline{\partial}u = f$ in $X \setminus \overline{D'}$. Let ψ be a $\mathcal{C}^\infty$ function such that supp $\psi \subset\subset X \setminus \overline{D'}$ and $\psi \equiv 1$ on a neighborhood of $X \setminus \widetilde{D}$. The form $f - \overline{\partial}(\psi u)$ is therefore in $Z^k_{0,q}(X \setminus D, E)$ and has compact support in $\widetilde{D} \setminus D$ and hence in $U \setminus D$. Then, by Theorem 1.5, there exists a form v in $\mathcal{C}^{k-\varepsilon}_{0,q-1}(U \setminus D, E)$ with compact support such that $\overline{\partial}v = f - \overline{\partial}(\psi u)$ in $U \setminus \overline{D}$. Extending v by zero outside $U \setminus D$ to the whole X and setting $\lambda = v + \chi u$, we then get $\lambda \in \mathcal{C}^{k-\varepsilon}_{0,q-1}(X \setminus D, E)$ and $\overline{\partial}\lambda = f$ in $X \setminus \overline{D}$. This proves the theorem. □

The second application is the following theorem that concerns the solvability of the $\overline{\partial}$-equation for E^*-valued currents.

Theorem 4.2. *Let $D \subset\subset X$ be a $\mathcal{C}^d$, $d \geqslant 2$, q-convex intersection in a complex manifold X of complex dimension n, $n \geqslant 3$, and let E be a Hermitian holomorphic vector bundle over X. Then for every $\overline{\partial}$-closed current f in $\mathcal{D}'^k_{0,q}(X \setminus \overline{D}, E^*)$, $k = 2, 3, \ldots, \infty$, $2 \leqslant q \leqslant n-1$ and $0 < \varepsilon < 1$, there exists a current g in $\mathcal{D}'^{k-\varepsilon}_{0,q-1}(X\setminus\overline{D}, E^*)$ such that $\overline{\partial}g = f$ in $X \setminus D$.*

Proof. The proof follows by using Theorem 1.5 and arguing in a manner similar to the proof of Theorem 6.2 in [4]. □

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