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EQUIVALENT QUASI-NORMS AND ATOMIC DECOMPOSITION OF WEAK TRIEBEL-LIZORKIN SPACES

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Abstract. Recently, the weak Triebel-Lizorkin space was introduced by Grafakos and He, which includes the standard Triebel-Lizorkin space as a subset. The latter has a wide applications in aspects of analysis. In this paper, the authors firstly give equivalent quasi-norms of weak Triebel-Lizorkin spaces in terms of Peetre's maximal functions. As an application of those equivalent quasi-norms, an atomic decomposition of weak Triebel-Lizorkin spaces is given.

Keywords: weak Lebesgue space; Triebel-Lizorkin space; equivalent norm; maximal function; atom

MSC 2010: 46E35, 42B25, 42B35

1. Introduction

It is well known that homogeneous and inhomogeneous Besov and Trieble-Lizorkin spaces include many classical function spaces, such as Sobolev spaces, Bessel potential spaces, Hardy spaces, local Hardy spaces, and BMO function spaces. These spaces have been studied in detail in [6], [7], [8], [18], [19], [20], [21], [24]. They play an important role in analysis. The theory of these spaces have had a remarkable development in part due to its usefulness in applications. For instance, they appear often in the study of partial differential equations. Especially, Triebel applied them in the study of the Navier-Stokes equations in [22], [23]. In recent decades, there have been some generalizations of these spaces. Firstly, the Besov-type space $B^{s,\tau}_{p,q}$ and the Triebel-Lizorkin type space $F^{s,\tau}_{p,q}$ were studied in [5], [3], [4], [16], [30], [32].

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Their homogeneous versions were originally studied in [29], [31] in order to clarify the relation between the classical Besov spaces $\dot{B}^s_{p,q}$, Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$, and the Q_α spaces studied in [26], [27]. Another class of generalisations, variable exponent Besov and Triebel-Lizorkin spaces were introduced in [1], [2], [11], [12], [28].

Recently, He in [10] considered square function characterization of weak Hardy spaces. Then in [9] Grafakos and He discussed various maximal characterization of these spaces and stated an interpolation theorem for $H^{p,\infty}$ from initial strong H^{p_0} and H^{p_1} estimates with $p_0 < p < p_1$, and they also introduced weak Triebel-Lizorkin spaces. From their definition we can immediately see that the usual Triebel-Lizorkin space is a subset of a weak Triebel-Lizorkin space. In this paper we shall present the equivalent quasi-norms of weak Triebel-Lizorkin spaces in terms of Peetre's maximal functions in Section 2. In Section 3, we describe an atomic decomposition of these spaces. Our result is inspired by the atomic decompositions of the previously mentioned Besov type and Triebel-Lizorkin type spaces.

Throughout this paper $|S|$ denotes the Lebesgue measure and χ_S the characteristic function of a measurable set $S \subset \mathbb{R}^n$. We also use the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leqslant cb$. If $a \lesssim b$ and $b \lesssim a$ we will write $a \sim b$. C is always a positive constant but it may change from line to line.

2. The quasi-norm characterizations

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space on $\mathbb{R}^n$, $\mathcal{S}'(\mathbb{R}^n)$ being its dual space on $\mathbb{R}^n$. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{\varphi}$ or $\mathcal{F}\varphi$ denotes its Fourier transform, and $\varphi^\vee$ or $\mathcal{F}^{-1}\varphi$ denotes its inverse Fourier transform. Take $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi_0(x) \geqslant 0$ and

$$\varphi_0(x) = \begin{cases} 1, & |x| \leqslant 1, \\ 0, & |x| \geqslant 2. \end{cases}$$

Now define $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ and set $\varphi_j(x) = \varphi(2^{-j}x)$ for all $j \in \mathbb{N}$. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a resolution of unity, which means $\sum\limits_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$.

We use $L^{p,\infty}$ to denote the weak Lebesgue space, which means it is the set of all Lebesgue measurable functions f on $\mathbb{R}^n$ with the quasi-norm

$$\|f\|_{L^{p,\infty}} := \sup_{\lambda>0} |\{x \in \mathbb{R}^n \colon |f(x)| > \lambda\}|^{1/p} < \infty.$$

$L^{p,\infty}(l_q)$ is the space of all sequences $\{g_j\}$ of measurable functions on $\mathbb{R}^n$ with finite quasi-norms

$$\|\{g_j\}_{j=0}^{\infty}\|_{L^{p,\infty}(l_q)} := \biggl\| \biggl(\sum_{j=0}^{\infty} |g_j|^q\biggr)^{1/q} \biggr\|_{L^{p,\infty}}.$$

Now, the weak Triebel-Lizorkin spaces is introduced as follows.

Definition 1. Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity as above, $s\in\mathbb{R}$, $0<q\leqslant\infty$, $0<p<\infty$. The set

$$\{f\in\mathcal{S}'(\mathbb{R}^n)\colon\ \|\{2^{sj}\varphi_j^\vee * f\}_{j=0}^\infty\|_{L^{p,\infty}(l_q)}<\infty\}$$

is called the weak Triebel-Lizorkin space and denoted by $F^{s,q}_{p,\infty}(\mathbb{R}^n)$. The quasi-norm of $f\in F^{s,q}_{p,\infty}(\mathbb{R}^n)$ in this space is denoted by

$$\|f\|_{F^{s,q}_{p,\infty}} := \|\{2^{sj}\varphi_j^\vee * f\}_{j=0}^\infty\|_{L^{p,\infty}(l_q)}.$$

In [9] Grafakos and He pointed out that the weak Triebel-Lizorkin spaces are independent of the choice of the resolution of unity $\{\varphi_j\}_{j\in\mathbb{N}_0}$. In this paper we shall prove this by using Peetre maximal operators for the first time. In fact, we shall give five equivalent quasi-norms for the weak Triebel-Lizorkin spaces.

Let $\Psi_0,\Psi\in\mathcal{S}(\mathbb{R}^n)$, $\varepsilon>0$, an integer $S\geqslant -1$ be such that

$$|\widehat{\Psi}_0(\xi)|>0 \quad \text{on } \{|\xi|<2\varepsilon\}, \tag{1}$$

$$|\widehat{\Psi}(\xi)|>0 \quad \text{on } \Big\{\frac{\varepsilon}{2}<|\xi|<2\varepsilon\Big\}, \tag{2}$$

and

$$D^\tau\widehat{\Psi}(0)=0 \quad \text{for all } |\tau|\leqslant S. \tag{3}$$

Here (1) and (2) are Tauberian conditions, while (3) expresses a moment conditions on Ψ.

Given a sequence of functions $\{\Psi_k\}_{k\in\mathbb{Z}}\subset\mathcal{S}(\mathbb{R}^n)$, a tempered distribution $f\in\mathcal{S}'(\mathbb{R}^n)$ and a positive number $a>0$, the classical Peetre maximal operator associated with $\{\Psi_k\}_{k\in\mathbb{Z}}$ is defined by

$$(\Psi_k^*)_a f(x) := \sup_{y\in\mathbb{R}^n}\frac{|\Psi_k * f(x+y)|}{(1+2^k|y|)^a},\quad x\in\mathbb{R}^n,\ k\in\mathbb{Z}.$$

Since $\Psi_k * f(y)$ makes sense pointwise everything is well-defined. We will often use dilates $\Psi_k(x)=2^{kn}\Psi(2^k x)$ of a fixed function $\Psi\in\mathcal{S}(\mathbb{R}^n)$, where $\Psi_0(x)$ may be given by a separate function. Also continuous dilates are needed. Let $\Psi_t := t^{-n}\Psi(t^{-1}\cdot)$. Let us recall the classical Peetre maximal operator introduced in [14]. We define $(\Psi_k^*)_a f(x)$ by

$$(\Psi_t^*)_a f(x) := \sup_{y\in\mathbb{R}^n}\frac{|\Psi_t * f(x+y)|}{(1+|y|/t)^a},\quad x\in\mathbb{R}^n,\ t>0.$$

Now we have equivalent quasi-norms on the weak Triebel-Lizorkin spaces.

Theorem 1. *Let $s < S+1$, $0 < p < \infty$, $0 < q \leqslant \infty$ and $a > d/\min\{p,q\}$. Further, let Φ_0, Φ belong to $\mathcal{S}(\mathbb{R}^n)$ and (1), (2) and (3). Then the space $F^{s,q}_{p,\infty}(\mathbb{R}^n)$ can be characterized by*

$$F^{s,q}_{p,\infty}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n)\colon\ \|f\|^{(i)}_{F^{s,q}_{p,\infty}} < \infty\}, \quad i = 1,\dots,5,$$

where

$$(4)\qquad \|f\|^{(1)}_{F^{s,q}_{p,\infty}} := \|\Phi_0 * f\|_{L^{p,\infty}} + \bigg\|\bigg(\int_0^1 t^{-sq}|\Phi_t * f(\cdot)|^q\frac{\mathrm{d}t}{t}\bigg)^{1/q}\bigg\|_{L^{p,\infty}},$$

$$(5)\qquad \|f\|^{(2)}_{F^{s,q}_{p,\infty}} := \|(\Phi_0^*)_a f\|_{L^{p,\infty}(\mathbb{R}^n)} + \bigg\|\bigg(\int_0^1 [t^{-s}(\Phi_t^*)_a f]^q\frac{\mathrm{d}t}{t}\bigg)^{1/q}\bigg\|_{L^{p,\infty}},$$

$$(6)\qquad \|f\|^{(3)}_{F^{s,q}_{p,\infty}} := \|\Phi_0 * f\|_{L^{p,\infty}} + \bigg\|\bigg(\int_0^1 t^{-sq}\int_{|z|<t}|(\Phi_t * f)(\cdot + z)|^q\,\mathrm{d}z\frac{\mathrm{d}t}{t^{n+1}}\bigg)^{1/q}\bigg\|_{L^{p,\infty}},$$

$$(7)\qquad \|f\|^{(4)}_{F^{s,q}_{p,\infty}} := \bigg\|\bigg(\sum_{k=0}^{\infty}[2^{ksq}(\Phi_k^*)_a f]^q\bigg)^{1/q}\bigg\|_{L^{p,\infty}},$$

$$(8)\qquad \|f\|^{(5)}_{F^{s,q}_{p,\infty}} := \bigg\|\bigg(\sum_{k=0}^{\infty}2^{ksq}|\Phi_k * f|^q\bigg)^{1/q}\bigg\|_{L^{p,\infty}}.$$

Furthermore, $\|\cdot\|^{(i)}_{F^{s,q}_{p,\infty}}$, $i = 1,2,\dots,5$ are equivalent.

We shall use the method from [25] to prove Theorem 1, which goes back to [15]. To do so, we need some lemmas.

Lemma 1 ([15], Lemma 1)**.** *Let $\mu,\nu \in \mathcal{S}(\mathbb{R}^n)$, $-1 \leqslant M \in \mathbb{Z}$,*

$$D^{\tau}\widehat{\mu}(0) = 0 \quad \text{for all } |\tau| \leqslant M.$$

Then for any $N > 0$ there is a constant C_N such that

$$\sup_{z\in\mathbb{R}^n}|\mu_t * \nu(z)|(1+|z|)^N \leqslant C_N t^{M+1},$$

where $\mu_t(x) = t^{-n}\mu(x/t)$ for all $0 < t \leqslant 2$.

Lemma 2 ([15], Lemma 2)**.** *Let $0 < q \leqslant \infty$, $\delta > 0$. For any sequence $\{g_j\}_0^{\infty}$ of nonnegative numbers denote*

$$G_j = \sum_{k=0}^{\infty}2^{-|k-j|\delta}g_k.$$

Then

$$\|\{G_j\}_0^\infty\|_{l_q} \leqslant C\|\{g_j\}_0^\infty\|_{l_q} \tag{9}$$

holds, where C is a constant only depending on q, δ.

Lemma 3. *Let $0 < p < \infty$, $\delta > 0$, $0 < q \leqslant \infty$. For any sequence $\{g_j\}_0^\infty$ of nonnegative measurable functions on $\mathbb{R}^n$ denote*

$$G_j(x) = \sum_{k=0}^{\infty} 2^{-|k-j|\delta} g_k(x), \quad x \in \mathbb{R}^n.$$

Then

$$\|\{G_j\}_0^\infty\|_{L^{p,\infty}(l_q)} \leqslant C_1\|\{g_j\}_0^\infty\|_{L^{p,\infty}(l_q)} \tag{10}$$

holds with some constant $C_1 = C_1(q,\delta)$.

Proof. By Lemma 2, (10) follows immediately from (9). □

Lemma 4 ([6], Theorem 2.6)**.** *Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity and let $R \in \mathbb{N}$. Then there exist functions θ_0, $\theta \in \mathcal{S}(\mathbb{R}^n)$ satisfying*

$$\begin{gathered}
\operatorname{supp}\theta_0, \quad \operatorname{supp}\theta \subseteq \{x \in \mathbb{R}^n\colon\ |x| \leqslant 1\},\\
|\widehat{\theta}_0(\xi)| > 0 \quad \text{on } \{|\xi| < 2\varepsilon\},\\
|\widehat{\theta}(\xi)| > 0 \quad \text{on } \Big\{\frac{\varepsilon}{2} < |\xi| < 2\varepsilon\Big\},\\
\int_{\mathbb{R}^n} x^\gamma \theta(x)\,\mathrm{d}x = 0 \quad \text{for } 0 < |\gamma| \leqslant R
\end{gathered}$$

such that

$$\widehat{\theta}_0(\xi)\widehat{\psi}_0(\xi) + \sum_{j=1}^{\infty} \widehat{\theta}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^n,$$

where the functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ are defined via $\widehat{\psi}_0(\xi) = \varphi_0(\xi)/\widehat{\theta}_0(\xi)$ and $\widehat{\psi}(\xi) = \varphi_1(2\xi)/\widehat{\theta}(\xi)$.

Let $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$. Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator $\mathcal{M}$ is defined by

$$\mathcal{M}f(x) := \sup_{r>0} r^{-n}\int_{B(x,r)} |f(y)|\,\mathrm{d}y, \quad x \in \mathbb{R}^n,$$

and $\mathcal{M}_t f = (\mathcal{M}|f|^t)^{1/t}$ for any $0 < t \leqslant 1$, where $B(x,r) := \{y \in \mathbb{R}^n\colon\ |x-y| < r\}$.

Lemma 5 ([10], Proposition 4)**.** *Let $1 < p < \infty$ and $1 < r \leqslant \infty$. Then there exists a positive constant C such that for all sequences $\{f_j\}_{j=1}^{\infty}$ of locally integrable functions on $\mathbb{R}^n$,*

$$\Big\|\Big(\sum_{j=1}^{\infty}|\mathcal{M}f_j|^r\Big)^{1/r}\Big\|_{L^{p,\infty}} \leqslant C\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{1/r}\Big\|_{L^{p,\infty}}.$$

This immediately yields

$$\Big\|\Big(\sum_{j=1}^{\infty}|\mathcal{M}_t f_j|^r\Big)^{1/r}\Big\|_{L^{p,\infty}} \leqslant C\Big\|\Big(\sum_{j=1}^{\infty}|f_j|^r\Big)^{1/r}\Big\|_{L^{p,\infty}},$$

for $0 < t < \min\{1, p, q\}$.

Remark. Although Proposition 4 in [10] applies only for $1 < r < \infty$, the result also holds for the case $r = \infty$. Indeed, since $|f_j| \leqslant \sup\limits_{j\geqslant 1}|f_j|$, we have $\mathcal{M}|f_j| \leqslant \mathcal{M}\Big(\sup\limits_{j\geqslant 1}|f_j|\Big)$. Thus we obtain $\sup\limits_{j\geqslant 1}\mathcal{M}|f_j| \leqslant \mathcal{M}\Big(\sup\limits_{j\geqslant 1}|f_j|\Big)$.

P r o o f of Theorem 1. We divide the total proof into four steps. First, we prove the equivalence of (4) and (5). The next step is to build the bridge between (5) and (7) and to change from the system (Φ_0, Φ) to a system (Ψ_0, Ψ). The equivalence of (7) and (8) goes parallel to (4) and (5). Indeed, Definition 1 can be seen as a special case of (8). Finally, we prove that (5) is equivalent to the rest. In the following, we consider the case $q \in (0, \infty)$. For $q = \infty$, we only use the usual modification.

Step 1. We are going to prove that for every $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\|f\|^{(2)}_{F^{s,q}_{p,\infty}} \lesssim \|f\|^{(1)}_{F^{s,q}_{p,\infty}} \lesssim \|f\|^{(2)}_{F^{s,q}_{p,\infty}}.$$

From Lemmas 1 and 4, we have that, see [25], for $r < \min\{p, q\}$, $N \in \mathbb{N}$, there exists a positive constant C such that for $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\Big(\int_1^2 |2^{ls}(\Phi^*_{2^{-l}t}f)_a(x)|^q \frac{\mathrm{d}t}{t}\Big)^{r/q} \leqslant C\sum_{k\in l+\mathbb{N}_0} 2^{(l-k)(Nr-n+rs)}2^{krs}\mathcal{M} \times\Big[\Big(\int_1^2 |((\Phi_k)_t * f)(\cdot)|^q \frac{\mathrm{d}t}{t}\Big)^{r/q}\Big](x).$$

Now taking $n/a < r = p_0 < \min\{p, q\}$, $N > \max\{0, -s\} + a$ and putting $\delta = N + s - n/r > 0$, we obtain for $l \in \mathbb{N}$

$$\Big(\int_1^2 |2^{ls}(\Phi^*_{2^{-l}t}f)_a(x)|^q \frac{\mathrm{d}t}{t}\Big)^{r/q} \lesssim \sum_{k\in l+\mathbb{N}_0} 2^{-\delta r|l-k|}2^{krs}\mathcal{M} \times\Big[\Big(\int_1^2 |((\Phi_k)_t * f)(\cdot)|^q \frac{\mathrm{d}t}{t}\Big)^{r/q}\Big](x).$$

Then we apply Lemma 3 in $L^{p/r,\infty}(l_{q/r})$, which yields

$$\left\|\left\{\left(\int_1^2 |2^{ls}(\Phi^*_{2^{-l}t}f)_a(x)|^q \frac{\mathrm{d}t}{t}\right)^{r/q}\right\}_{l\in\mathbb{N}}\right\|_{L^{p/r,\infty}(l_{q/r})} \lesssim \left\|\left\{\mathcal{M}\left[\left(\int_1^2 |2^{ks}((\Phi_k)_t * f)(\cdot)|^q \frac{\mathrm{d}t}{t}\right)^{r/q}\right]\right\}_{l\in\mathbb{N}}\right\|_{L^{p/r,\infty}(l_{q/r})}.$$

Next, using Lemma 5, we obtain

$$\begin{aligned}\left\|\left\{\left(\int_1^2 |2^{ls}(\Phi^*_{2^{-l}t}f)_a(x)|^q \frac{\mathrm{d}t}{t}\right)^{r/q}\right\}_{l\in\mathbb{N}}\right\|_{L^{p/r,\infty}(l_{q/r})} &\lesssim \left\|\left\{\mathcal{M}\left[\left(\int_1^2 |2^{ks}((\Phi_k)_t * f)(\cdot)|^q \frac{\mathrm{d}t}{t}\right)^{r/q}\right]\right\}_{l\in\mathbb{N}}\right\|_{L^{p/r,\infty}(l_{q/r})}\\ &\lesssim \left\|\left\{\left(\int_1^2 |2^{ks}((\Phi_k)_t * f)(\cdot)|^q \frac{\mathrm{d}t}{t}\right)^{r/q}\right\}_{k\in\mathbb{N}}\right\|_{L^{p/r,\infty}(l_{q/r})}\\ &= \left\|\left\{\left(\int_1^2 |2^{ks}((\Phi_k)_t * f)(\cdot)|^q \frac{\mathrm{d}t}{t}\right)^{1/q}\right\}_{k\in\mathbb{N}}\right\|^r_{L^{p,\infty}(l_q)}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\left\|\left(\int_0^1 |\lambda^{-s}(\Phi^*_\lambda f)_a(\cdot)|^q \frac{\mathrm{d}\lambda}{\lambda}\right)^{1/q}\right\|_{L^{p,\infty}} &\approx \left\|\left(\sum_{l=1}^\infty \int_1^2 |2^{ls}(\Phi^*_{2^{-l}t}f)_a(\cdot)|^q \frac{\mathrm{d}t}{t}\right)^{1/q}\right\|_{L^{p,\infty}}\\ &\lesssim \left\|\left\{\left(\int_1^2 |2^{ls}\Phi_{2^{-l}t} * f(\cdot)|^q \frac{\mathrm{d}t}{t}\right)^{1/q}\right\}_{l\in\mathbb{N}}\right\|_{L^{p,\infty}(l_q)}\\ &\approx \left\|\left(\int_0^1 |\lambda^{-s}\Phi_\lambda * f(\cdot)|^q \frac{\mathrm{d}\lambda}{\lambda}\right)^{1/q}\right\|_{L^{p,\infty}}.\end{aligned}$$

This proves $\|f\|^{(2)}_{F^{s,q}_{p,\infty}} \lesssim \|f\|^{(1)}_{F^{s,q}_{p,\infty}}$. Since the reverse inequality is trivial, this finishes Step 1.

Step 2. Let $\Psi_0, \Psi \in \mathcal{S}(\mathbb{R}^n)$ be functions satisfying (1), (2) and (3).

First, we are going to prove for all $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\|f\|^{(4)}_{F^{s,q}_{p,\infty}(\mathbb{R}^n,\Psi)} \lesssim \|f\|^{(2)}_{F^{s,q}_{p,\infty}(\mathbb{R}^n,\Phi)}. \tag{11}$$

Again from Lemmas 4 and 1, we have that, see [25], if we let $\delta = \min\{1, S+1-s\}$, there exists a positive constant C such that for any $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$2^{ls}(\Psi_l^* f)_a(x) \leqslant C \sum_{k\in\mathbb{N}_0} 2^{-|k-l|\delta} 2^{ks} (\Phi_{2^{-k}t}^* f)_a(x) \tag{12}$$

for all $x \in \mathbb{R}^n$ and all $t \in [1,2]$.

Suppose first that $q \geqslant 1$. Then we take on both sides $(\int_1^2 |\cdot|^q \,\mathrm{d}t/t)^{1/q}$, which gives

$$2^{ls}(\Psi_l^* f)_a(x) \lesssim \sum_{k\in\mathbb{N}_0} 2^{-|k-l|q} 2^{ks} \left(\int_1^2 |(\Phi_{2^{-k}t}^* f)_a(x)|^q \frac{\mathrm{d}t}{t}\right)^{1/q}.$$

Applying Lemma 3 we obtain that

$$\|\{2^{ls}(\Psi_l^* f)_a\}_{l\in\mathbb{N}}\|_{L^{p,\infty}(l_q)} \lesssim \left\|\left(\sum_{k=1}^{\infty} 2^{ksq} |(\Phi_{2^{-k}t}^* f)_a(x)|^q \frac{\mathrm{d}t}{t}\right)^{1/q}\right\|_{L^{p,\infty}},$$

which gives the desired result.

In case $q < 1$ we argue as follows. The quantity $\left(\int_1^2 |\cdot|^q \,\mathrm{d}t/t\right)^{1/q}$ is not longer a norm. This gives

$$(2^{ls}(\Psi_l^* f)_a(x))^q \lesssim \sum_{k\in\mathbb{N}_0} 2^{-q|k-l|q} 2^{ksq} \int_1^2 |(\Phi_{2^{-k}t}^* f)_a(x)|^q \frac{\mathrm{d}t}{t}.$$

Notice that the right-hand side is nothing else than a convolution $(\gamma * \alpha(\cdot))_l$ of the sequences

$$\gamma_k = 2^{-|k|\delta q} \quad \text{and} \quad \alpha(\cdot)_k = 2^{ksq} \int_1^2 |(\Phi_{2^{-k}t}^* f)_a(x)|^q \frac{\mathrm{d}t}{t}.$$

Now we apply the l_1-norm to both sides and get for all $x \in \mathbb{R}^n$

$$\|2^{ls}(\Psi_l^* f)_a(x)\|_{l_q}^q \leqslant \|\gamma\|_{l_1} |\alpha(\cdot)\|_{l_1} \lesssim \sum_{k=1}^{\infty} 2^{ksq} \int_1^2 |(\Phi_{2^{-k}t}^* f)_a(x)|^q \frac{\mathrm{d}t}{t}.$$

We take the power of both sides and apply the $L^{p,\infty}(\mathbb{R}^n)$-norm. This gives (11).

Similarly, we obtain for all $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\|f\|^{(2)}_{F^{s,q}_{p,\infty}(\mathbb{R}^n,\Phi)} \lesssim \|f\|^{(4)}_{F^{s,q}_{p,\infty}(\mathbb{R}^n,\Psi)}.$$

Step 3. Choosing $t = 1$ in Step 1 and omitting the integration over t we see immediately

$$\|f\|^{(5)}_{F^{s,q}_{p,\infty}} \lesssim \|f\|^{(4)}_{F^{s,q}_{p,\infty}} \lesssim \|f\|^{(5)}_{F^{s,q}_{p,\infty}}.$$

Step 4. We show that (5) is equivalent to (6).

First, let us prove that for any $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\|f\|^{(2)}_{F^{s,q}_{p,\infty}} \lesssim \|f\|^{(3)}_{F^{s,q}_{p,\infty}}. \tag{13}$$

From [25], for $0 < r < \min\{p,q\}$ there exists a positive constant C such that for any $f \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}\left(\int_1^2 |(\Psi^*_{2^{-l}t} f)_a(x)|^q \frac{\mathrm{d}t}{t}\right)^{r/q} &\leqslant C \sum_{k\in\mathbb{N}_0} 2^{-kNs} 2^{(k+l)n} \\ &\times \int_{\mathbb{R}^n} \frac{\left(\int_1^2 \int_{|z|<2^{-(k+l)}t} |((\Phi_{k+l})_t * f)(z+y)|^q \,\mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}}\right)^{r/q}}{(1+2^l|x-y|)^{ar}} \,\mathrm{d}y.\end{aligned}$$

If $ar > n$ then we have

$$g_l(\cdot) = \frac{2^{nl}}{(1+2^l|\cdot|)^{ar}} \in L_1(\mathbb{R}^n).$$

Thus we have

$$\begin{aligned}\left(\int_1^2 |2^{ls}(\Phi^*_{2^{-l}t} f)_a(x)|^q \frac{\mathrm{d}t}{t}\right)^{r/q} &\lesssim \sum_{k\in\mathbb{N}_0} 2^{-kNr} 2^{kn} 2^{lsr} \\ &\times \left[g_l * \left(\int_1^2 \int_{|z|<2^{-(k+l)}t} |((\Phi_{k+l})_t * f)(z+\cdot)|^q \,\mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}}\right)^{r/q}\right](x).\end{aligned}$$

Now we use the majorant property of the Hardy-Littlewood maximal operator in [17] and continue estimating

$$\begin{aligned}\left(\int_1^2 |2^{ls}(\Phi^*_{2^{-l}t} f)_a(x)|^q \frac{\mathrm{d}t}{t}\right)^{r/q} &\lesssim \sum_{k\in\mathbb{N}_0} 2^{lsr} 2^{k(-Nr+n)} \mathcal{M} \\ &\times \left[\left(\int_1^2 \int_{|z|<2^{-(k+l)}t} |((\Phi_{k+l})_t * f)(z+\cdot)|^q \,\mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}}\right)^{r/q}\right](x).\end{aligned}$$

An index shift on the right-hand side gives

$$\begin{aligned}\left(\int_1^2 |2^{ls}(\Phi^*_{2^{-l}t} f)_a(x)|^q \frac{\mathrm{d}t}{t}\right)^{r/q} &\lesssim \sum_{k\in l+\mathbb{N}_0} 2^{lsr} 2^{(k-l)(-Nr+n)} \mathcal{M} \\ &\times \left[\left(\int_1^2 \int_{|z|<2^{-k}t} |((\Phi_k)_t * f)(z+\cdot)|^q \,\mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}}\right)^{r/q}\right](x) \\ &= C \sum_{k\in l+\mathbb{N}_0} 2^{(l-k)(Nr-n+rs)} 2^{krs} \mathcal{M} \\ &\times \left[\left(\int_1^2 \int_{|z|<2^{-k}t} |((\Phi_k)_t * f)(z+\cdot)|^q \,\mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}}\right)^{r/q}\right](x).\end{aligned}$$

Using similar arguments as after (12), we obtain (13).

Second, we prove $\|f\|^{(3)}_{F^{s,q}_{p,\infty}} \lesssim \|f\|^{(2)}_{F^{s,q}_{p,\infty}}$. Since for all $t > 0$

$$\frac{1}{t^n}\int_{|z|<t} |(\Phi_t * f)(x+z)|\,\mathrm{d}z \lesssim \sup_{|z|<t} \frac{|(\Phi_t * f)(x+z)|}{(1+1/t|z|)^a} \lesssim (\Phi_t^* f)_a(x),$$

we conclude what we want. The proof is complete. □

3. Atomic decomposition

Let $\mathbb{Z}^n$ be the lattice of all points in $\mathbb{R}^n$ with integer-valued components. For $v \in \mathbb{N}_0$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, let Q_{vm} be the dyadic cube in $\mathbb{R}^n$

$$Q_{vm} = (x_1, \ldots, x_n)\colon\ m_i \leqslant 2^v x_i < m_i + 1, \quad i = 1, 2, \ldots, n.$$

If Q_{vm} is such cube in $\mathbb{R}^n$ and $c > 0$, then cQ_{vm} is the cube in $\mathbb{R}^n$ concentric with Q_{vm} with sides also parallel to coordinate axes and of length $c2^{-v}$. By χ_{vm} we denote the characteristic function of the cube cQ_{vm}. The main goal of this section is to prove an atomic decomposition result for the space $F^{s,q}_{p,\infty}$. First, we introduce the basic notation.

Definition 2. Let $s \in \mathbb{R}$, $\tau \in [0,\infty)$ and $0 < q,\ p \leqslant \infty$. Then for all complex valued sequences $\lambda = \{\lambda_{vm} \in \mathbb{C}\colon\ v \in N_0,\ m \in \mathbb{Z}^n\}$ we define

$$f^{s,q}_{p,\infty} := \{\lambda\colon\ \|\lambda | f^{s,q}_{p,\infty}\| < \infty\}$$

where

$$\|\lambda | f^{s,q}_{p,\infty}\| := \bigg\| \bigg(\sum_{v=0}^{\infty}\bigg(\sum_{m\in\mathbb{Z}^n} 2^{vs}|\lambda_{vm}|\chi_{vm}\bigg)^q\bigg)^{1/q}\bigg\|_{L^{p,\infty}}.$$

We define atoms which are the building blocks for atomic decompositions.

Definition 3. Let $K, L \in \mathbb{N}_0$ and let $\gamma > 1$. A K-times continuously differentiable function $a \in C^K(\mathbb{R}^n)$ is called a $[K,L]$-atom centered at Q_{vm}, $v \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ if

$$\operatorname{supp} a \subseteq \gamma Q_{vm}, \tag{14}$$

$$|D^\alpha a(x)| \leqslant 2^{v|\alpha|} \quad \text{for } 0 \leqslant |\alpha| \leqslant K,\ x \in \mathbb{R}^n, \tag{15}$$

and if

$$\int_{\mathbb{R}^n} x^\alpha a(x)\,\mathrm{d}x = 0 \quad \text{for } 0 \leqslant |\alpha| < L \ \text{ and } \ v \geqslant 1. \tag{16}$$

If an atom a is located at Q_{vm}, that means if it fulfils (14), then we will denote it by a_{vm}. For $v = 0$ or $L = 0$ there are no moment conditions (16) required.

To prove the decomposition by atoms we need three basic lemmas. The first is Lemma 3.3 in [6], the second lemma is a Hardy-type inequality which is easy to prove and the last lemma first appeared in [13], Lemma 7.1, and in the following notation in [12], Lemma 3.7.

Lemma 6. *Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity and let $\{\varrho_{vm}\}_{v\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ be $[K,L]$-atoms. Then*

$$|\mathcal{F}^{-1}\varphi_j * \varrho_{vm}(x)| \leqslant C2^{(v-j)K}(1+2^v|x-2^{-v}m|)^{-M}$$

if $v \leqslant j$, and

$$|\mathcal{F}^{-1}\varphi_j * \varrho_{vm}(x)| \leqslant C2^{(j-v)(L+n+1)}(1+2^v|x-2^{-v}m|)^{-M}$$

if $v \geqslant j$, where M is sufficiently large.

Lemma 7. *Let $0 < a < 1$, $j \in \mathbb{Z}$ and $0 < q \leqslant \infty$. Let $\{\varepsilon_k\}$ be a sequences of positive real numbers and denote*

$$\delta_k = \sum_{j=0}^{k} a^{j-k}\varepsilon_j, \quad \eta_k = \sum_{j=k}^{\infty} a^{j-k}\varepsilon_j, \quad k \geqslant 0.$$

Then there exist a constant $C > 0$ depending only on a and q such that

$$\Bigl(\sum_{k=0}^{\infty}\delta_k^q\Bigr)^{1/q} + \Bigl(\sum_{k=0}^{\infty}\eta_k^q\Bigr)^{1/q} \leqslant C\Bigl(\sum_{k=0}^{\infty}\varepsilon_k^q\Bigr)^{1/q}.$$

Lemma 8. *Let $\lambda = \{\lambda_{vm} \in \mathbb{C}\colon\ v \in N_0,\ m \in \mathbb{Z}^n\}$. Then*

$$\sum_{m\in\mathbb{Z}^n} 2^{vs}|\lambda_{vm}|(1+2^v|x-2^{-v}m|)^{-M} \leqslant C\sum_{k=0}^{\infty}2^{(n/t-M)k}\mathcal{M}_t\Bigl(\sum_{m\in\mathbb{Z}^n}2^{vs}|\lambda_{vm}|\chi_{vm}\Bigr)(x)$$

if $v \leqslant j$, and

$$\sum_{m\in\mathbb{Z}^n} 2^{vs}|\lambda_{vm}|(1+2^j|x-2^{-v}m|)^{-M} \leqslant C2^{(v-j)n/t}\sum_{k=0}^{\infty}2^{(n/t-M)k}\mathcal{M}_t\Bigl(\sum_{m\in\mathbb{Z}^n}2^{vs}|\lambda_{vm}|\chi_{vm}\Bigr)(x)$$

if $v \geqslant j$, where $0 < t < \min(1,p,q)$ and M is sufficiently large.

Now, we come to the atomic decomposition.

Theorem 2. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leqslant \infty$, $K, L \in \mathbb{N}_0$ be such that $K > s$, $L + s + 1 > 0$. Then every $f \in F^{s,q}_{p,\infty}$ can be represented as*

$$f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{vm} \varrho_{vm} \quad \textit{converging in } \mathcal{S}'(\mathbb{R}^n) \tag{17}$$

where ϱ_{vm} are $[K, L]$-atoms and $\lambda = \{\lambda_{vm} \in \mathbb{C} \colon v \in N_0, m \in \mathbb{Z}^n\} \in f^{s,q}_{p,\infty}$. On the other hand, if $\lambda \in f^{s,q}_{p,\infty}$, ϱ_{vm} are $[K, L]$-atoms and $f = \sum\limits_{v=0}^{\infty} \sum\limits_{m \in \mathbb{Z}^n} \lambda_{vm} \varrho_{vm}$ converges in $\mathcal{S}'(\mathbb{R}^n)$, then $f \in F^{s,q}_{p,\infty}$.

Proof. Our method is essentially based on [5], Theorem 3.17, [6], Theorem 6, and [8]. We consider only $0 < q < \infty$. The case $q = \infty$ can be proved analogously with the necessary modifications. For clarity, we divide the proof into three steps.

Step 1. Let θ_0, θ, ψ_0 and ψ be the functions introduced in Lemma 6. We have

$$f = \theta_0 * \psi_0 * f + \sum_{v=1}^{\infty} \theta_v * \psi_v * f$$

and using the definition of the cubes Q_{vm} we obtain

$$f(x) = \sum_{m \in \mathbb{Z}^n} \int_{Q_{0m}} \psi_0 * f(y) \, \mathrm{d}y + \sum_{v=1}^{\infty} 2^{vn} \sum_{m \in \mathbb{Z}^n} \int_{Q_{vm}} \theta(2^v(x-y)) \psi_v * f(y) \, \mathrm{d}y,$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$. We define for every $v \in \mathbb{N}$ and all $m \in \mathbb{Z}^n$

$$\lambda_{vm} = C_\theta \sup_{y \in Q_{vm}} |\psi_v * f(y)|, \tag{18}$$

with

$$C_\theta = \max\Big\{ \sup_{|y| \leqslant 1} |D^\alpha \theta(y)| \colon |\alpha| \leqslant K \Big\}.$$

Define also

$$\varrho_{vm}(x) = \begin{cases} \dfrac{1}{\lambda_{vm}} 2^{vn} \displaystyle\int_{Q_{vm}} \theta(2^v(x-y)) \psi_v * f(y) \, \mathrm{d}y & \text{if } \lambda_{vm} \neq 0, \\ 0 & \text{if } \lambda_{vm} = 0. \end{cases} \tag{19}$$

Similarly, we define for every $m \in \mathbb{Z}^n$ the numbers λ_{0m} and the functions ϱ_{0m} taking in (21) and (22) $v = 0$ and replacing ψ_v and θ by ψ_0 and θ_0, respectively. Let us now check that such ϱ_{vm} are atoms in the sense of Definition 4. Note that the support and the moment conditions are clear by (18) and (19), respectively. It thus remains to check (16) in Definition 4. If $\lambda_{vm} \neq 0$, we have

$$\begin{aligned}|D^\alpha \varrho_{vm}(x)| &\leqslant \frac{2^{v(n+|\alpha|)}}{C_\theta} \int_{Q_{vm}} |D^\alpha \theta(2^v(x-y))||\psi_v * f(y)| \,\mathrm{d}y \Big(\sup_{y\in Q_{vm}} |\psi_v * f(y)|\Big)^{-1} \\ &\leqslant \frac{2^{v(n+|\alpha|)}}{C_\theta} \int_{Q_{vm}} |D^\alpha \theta(2^v(x-y))| \,\mathrm{d}y \\ &\leqslant 2^{v(n+|\alpha|)}|Q_{vm}| \leqslant 2^{v|\alpha|}.\end{aligned}$$

The modifications for the terms with $v = 0$ are obvious.

Step 2. Next, we show that there is a constant $C > 0$ such that

$$\|\lambda \mid f^{s,q}_{p,\infty}\| \leqslant C\|f\|_{F^{s,q}_{p,\infty}}.$$

For that reason, we exploit the equivalent quasi-norms given in Theorem 1 involving Peetre's maximal function. For any $x, y \in Q_{vm}$ and any $v \geqslant 0$ we have

$$\begin{aligned}(20)\qquad \sum_{m\in\mathbb{Z}^n} \lambda_{vm}\chi_{vm}(x) &= C_\theta \sum_{m\in\mathbb{Z}^n} \sup_{y\in Q_{vm}} |\psi_v * f(y)|\chi_{vm}(x) \\ &\leqslant C \sum_{m\in\mathbb{Z}^n} \sup_{|z|\leqslant C2^{-v}} \frac{|\psi_v * f(x-z)|}{(1+2^v|z|)^a}(1+2^v|z|)^a\chi_{vm}(x) \\ &\leqslant C(\psi_v^*)_a f(x) \sum_{m\in\mathbb{Z}^n} \chi_{vm}(x) \\ &= C(\psi_v^*)_a f(x),\end{aligned}$$

where we have used $\sum_{m\in\mathbb{Z}^n} \chi_{vm}(x) = 1$. This estimate and its counterpart for $v = 0$ (which can be obtained by a similar calculation) give

$$\|\lambda \mid f^{s,q}_{p,\infty}\| \leqslant C\Big\|\Big(\sum_{v=0}^{\infty}[2^{ksq}(\psi_v^*)_a f]^q\Big)^{1/q}\Big\|_{L^{p,\infty}} \leqslant C\|f\|_{F^{s,q}_{p,\infty}},$$

by Theorem 1 (by taking $a > n/\min\{p, q\}$).

Step 3. Assume that $f \in \mathcal{S}'(\mathbb{R}^n)$ can be represented by (20), with K and L satisfying $K > s$ and $L + s + 1 > 0$. We now show that $f \in F^{s,q}_{p,\infty}$ and that for some $c > 0$, $\|f \mid F^{s,q}_{p,\infty}\| \leqslant c\|\lambda \mid f^{s,q}_{p,\infty}\|$. We divide the summation (20) depending on $j \in \mathbb{N}_0$ into two parts,

$$f = \sum_{v=0}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{vm}\varrho_{vm} = \sum_{v=0}^{j}\sum_{m\in\mathbb{Z}^n} \lambda_{vm}\varrho_{vm} + \sum_{v=j+1}^{\infty}\sum_{m\in\mathbb{Z}^n} \lambda_{vm}\varrho_{vm}.$$

We have

$$\text{(21)} \qquad \Big\| \Big(\sum_{j=0}^{\infty} (2^{js} |\varphi_j^{\vee} * f|)^q \Big)^{1/q} \Big\|_{L^{p,\infty}}$$

$$= \Big\| \Big(\sum_{j=0}^{\infty} \Big| \sum_{v=0}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{js} \lambda_{vm} \varphi_j^{\vee} * \varrho_{vm} \Big|^q \Big)^{1/q} \Big\|_{L^{p,\infty}}$$

$$\leqslant C \Big\| \Big(\sum_{j=0}^{\infty} \Big| \sum_{v=0}^{j} \sum_{m\in\mathbb{Z}^n} 2^{js} \lambda_{vm} \varphi_j^{\vee} * \varrho_{vm} \Big|^q \Big)^{1/q} \Big\|_{L^{p,\infty}}$$

$$+ C \Big\| \Big(\sum_{j=0}^{\infty} \Big| \sum_{v=j+1}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{js} \lambda_{vm} \varphi_j^{\vee} * \varrho_{vm} \Big|^q \Big)^{1/q} \Big\|_{L^{p,\infty}}$$

$$=: \sigma_1 + \sigma_2$$

Estimation of σ_1. From Lemmas 7 and 8 we obtain

$$\text{(22)} \quad \Big(\sum_{j=0}^{\infty} \Big| \sum_{v=0}^{j} \sum_{m\in\mathbb{Z}^n} 2^{js} \lambda_{vm} \varphi_j^{\vee} * \varrho_{vm} \Big|^q \Big)^{1/q}$$

$$\lesssim \Big(\sum_{j=0}^{\infty} \Big(\sum_{v=0}^{j} \sum_{m\in\mathbb{Z}^n} 2^{(v-j)(K-s)} \sum_{m\in\mathbb{Z}^n} 2^{vs} |\lambda_{vm}| (1 + 2^v |x - 2^{-v} m|)^{-M} \Big)^q \Big)^{1/q}$$

$$\lesssim \Big(\sum_{j=0}^{\infty} \Big(\sum_{v=0}^{j} 2^{(v-j)(K-s)} \sum_{k=0}^{\infty} 2^{(n/t-M)k} \mathcal{M}_t \Big(\sum_{m\in\mathbb{Z}^n} 2^{vs} |\lambda_{vm}| \chi_{vm}(x) \Big) \Big)^q \Big)^{1/q}$$

$$\text{(23)} \quad \lesssim \Big(\sum_{j=0}^{\infty} \Big(\sum_{v=0}^{j} 2^{(v-j)(K-s)} \mathcal{M}_t \sum_{m\in\mathbb{Z}^n} 2^{vs} |\lambda_{vm}| \chi_{vm}(x) \Big)^q \Big)^{1/q}$$

where the last estimate follows by taking M sufficiently large such that $M > n/t$; from Lemma 7 we get

$$(22) \lesssim \Big(\sum_{j=0}^{\infty} \Big(\mathcal{M}_t \Big(\sum_{m\in\mathbb{Z}^n} 2^{js} |\lambda_{jm}| \chi_{jm}(x) \Big) \Big)^q \Big)^{1/q}.$$

It follows that

$$\text{(24)} \qquad \sigma_1 \lesssim \Big\| \Big(\sum_{j=0}^{\infty} \Big(\mathcal{M}_t \Big(\sum_{m\in\mathbb{Z}^n} 2^{js} |\lambda_{jm}| \chi_{jm}(\cdot) \Big) \Big)^q \Big)^{1/q} \Big\|_{L^{p,\infty}}$$

$$\lesssim \Big\| \Big(\sum_{j=0}^{\infty} \Big(\sum_{m\in\mathbb{Z}^n} 2^{js} |\lambda_{jm}| \chi_{jm}(\cdot) \Big)^q \Big)^{1/q} \Big\|_{L^{p,\infty}}$$

$$\sim \|\lambda | f^{s,q}_{p,\infty}\|$$

where we used in the last inequality the boundedness of $\mathcal{M}_t$ on $L_p(l_q)$ for $0 < t < \min(1, p, q)$.

Estimation of σ_2. Again using Lemma 7 and Lemma 9 we obtain

$$
\begin{aligned}
(25)\quad & \Big(\sum_{j=0}^{\infty}\Big|\sum_{v=j+1}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{js}\lambda_{vm}\varphi_j^{\vee} * \varrho_{vm}(x)\Big|^q\Big)^{1/q} \\
& \lesssim \Big(\sum_{j=0}^{\infty}\Big(\sum_{v=j+1}^{\infty} 2^{(j-v)(L+n+1+s)}\sum_{m\in\mathbb{Z}^n} 2^{vs}|\lambda_{vm}|(1+2^v|x-2^{-v}m|)^{-M}\Big)^q\Big)^{1/q} \\
& \lesssim \Big(\sum_{j=0}^{\infty}\Big(\sum_{v=j+1}^{\infty} 2^{(j-v)(L+n+1+s)}2^{(v-j)n/t}\sum_{k=0}^{\infty}2^{(n/t-M)k}\mathcal{M}_t \\
& \qquad\quad \times\Big(\sum_{m\in\mathbb{Z}^n} 2^{vs}|\lambda_{vm}|\chi_{vm}(x)\Big)\Big)^q\Big)^{1/q} \\
(26)\quad & \lesssim \Big(\sum_{j=0}^{\infty}\Big(\sum_{v=j+1}^{\infty} 2^{(j-v)(L+n+1+s-n/t)}\mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n} 2^{vs}|\lambda_{vm}|\chi_{vm}(x)\Big)\Big)^q\Big)^{1/q}
\end{aligned}
$$

where the last estimate follows by taking M sufficiently large such that $M > n/t$, by choosing t satisfying $0 < t < \min(p, q, 1)$ and $L+n+1+s-n/t > 0$. Then from Lemma 7 we get

$$
(25) \lesssim \Big(\sum_{j=0}^{\infty}\Big(\mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n} 2^{js}|\lambda_{jm}|\chi_{jm}(x)\Big)\Big)^q\Big)^{1/q}.
$$

It follows that

$$
\begin{aligned}
(27)\qquad \sigma_2 & \lesssim \Big\|\Big(\sum_{j=0}^{\infty}\Big(\mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n} 2^{js}|\lambda_{jm}|\chi_{jm}(\cdot)\Big)\Big)^q\Big)^{1/q}\Big\|_{L^{p,\infty}} \\
& \lesssim \Big\|\Big(\sum_{j=0}^{\infty}\Big(\sum_{m\in\mathbb{Z}^n} 2^{js}|\lambda_{jm}|\chi_{jm}(\cdot)\Big)^q\Big)^{1/q}\Big\|_{L^{p,\infty}} \\
& \sim \|\lambda \mid f^{s,q}_{p,\infty}\|
\end{aligned}
$$

where we used in the last inequality the boundedness of $\mathcal{M}_t$ on $L_p(l_q)$ for $0 < t < \min(1, p, q)$. Now, by (21), (24), (27) we get

$$
\|f\|_{F^{s,q}_{p,\infty}} \leqslant C\|\lambda \mid f^{s,q}_{p,\infty}\|.
$$

The proof is completed. □

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