

Jiří M. Tomáš

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THE GENERAL RIGIDITY RESULT FOR BUNDLES
OF A -COVELOCITIES AND A -JETS

JIŘÍ TOMÁŠ, Brno

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Abstract. Let M be an m -dimensional manifold and $A = \mathbb{D}_k^r/I = \mathbb{R} \oplus N_A$ a Weil algebra of height r . We prove that any A -covelocality $T_x^A f \in T_x^{A^*}M$, $x \in M$ is determined by its values over arbitrary $\max\{\text{width } A, m\}$ regular and under the first jet projection linearly independent elements of $T_x^A M$. Further, we prove the rigidity of the so-called universally reparametrizable Weil algebras. Applying essentially those partial results we give the proof of the general rigidity result $T^{A^*}M \simeq T^{r^*}M$ without coordinate computations, which improves and generalizes the partial result obtained in Tomáš (2009) from $m \geq k$ to all cases of m .

We also introduce the space $J^A(M, N)$ of A -jets and prove its rigidity in the sense of its coincidence with the classical jet space $J^r(M, N)$.

Keywords: r -jet; bundle functor; Weil functor; Lie group; jet group; B -admissible A -velocity

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1. PRELIMINARIES

1.1. We give a contribution to the theory of Weil functors. We start from the concepts of the r -jet, the jet space $J^r(M, N)$ and the bundle functor, all defined in [8]. By M we denote an m -dimensional manifold and by N a smooth manifold. All manifolds are supposed to be of class C^∞ . By $\mathcal{M}f_m$ we denote the category of m -dimensional manifolds with local diffeomorphisms, by $\mathcal{M}f$ the category of smooth manifolds with smooth maps and by \mathcal{FM} the category of fibered manifolds with smooth fibered maps. Further, we denote by \mathcal{FM}_m the category of fibered manifolds with m -dimensional bases with fibered maps covering local diffeomorphisms. By $P^r M$ we denote the frame bundle on M . A bundle functor defined on the category $\mathcal{M}f_m$ is said to be a natural bundle, see [8]. There is an important bundle functor

defined on the category $\mathcal{M}f_m \times \mathcal{M}f$, the so-called jet functor J^r . It assigns the space of r -jets $J^r(M, N)$ to any couple $(M, N) \in \text{Obj}(\mathcal{M}f_m \times \mathcal{M}f)$ and the map $J^r(g, h): J^r(M_1, N_1) \rightarrow J^r(M_2, N_2)$ defined by $j_x^r f \mapsto j_{f(x)}^r h \circ j_x^r f \circ (j_x^r g)^{-1}$ to any local diffeomorphism $g: M_1 \rightarrow M_2$ defined near x and any smooth map $h: N_1 \rightarrow N_2$.

There is a significant class of product-preserving bundle functors defined on the category $\mathcal{M}f$. The classical result (see [6], [5], [8], [11]) reads that the functors of this kind coincide with Weil functors. Denoting them by T^A we involve the associated Weil algebras to their notation. The restriction of a Weil functor to $\mathcal{M}f_m$ is said to be a Weil bundle.

The concept of a Weil functor goes back to 1953, when Weil defined and studied the spaces of the infinitely near points on manifolds, see [20]. This concept led to the contravariant definition of the Weil functor presented below. However, most authors studied Weil functors from the covariant point of view. The equivalence of both approaches was proved in [8]. Besides the authors of those fundamental results, there are many others studying not only the geometry of Weil functors but also their applications in various research areas like the differential invariant theory and the theory of natural operators, e.g., Kolář, Mikulski, Shurygin, Bushueva and many others (see e.g. [8], [7], [9], [12], [16], [15], [4]). However, there are also authors applying and connecting Weil functors to problems from the Lie group theory, e.g., Alonso, Muriel, Muñoz, Rodriguez, and others, see [1], [2], [13]. Further, there are papers applying Weil functors in the theory of jets (e.g., Kureš in [10], who also studies their applications in the theoretical mechanics).

From the algebraic point of view, Weil functors are studied by Bertram (see [3]), who generalizes their definition in the sense of substituting general fields and rings for reals to the definition of the associated Weil algebra. Weil functors are also applied in the research of geometrical categories, namely in the papers by Nishimura, see [14].

We recall briefly the elementary concepts concerning Weil functors from [8]. Let $\mathcal{E}(k)$ be the algebra of germs of smooth functions defined on \mathbb{R}^k with the source at zero. A Weil algebra A can be defined either as the direct sum $\mathbb{R} \oplus N_A$ of reals with the so-called nilpotent ideal N_A of all of its nilpotent elements, or as a quotient $\mathcal{E}(k)/I$ by an ideal $I \subset \mathcal{E}(k)$ of a finite codimension. A can be also defined as a quotient \mathbb{D}_k^r/J of the so-called jet algebra \mathbb{D}_k^r by its ideal J . In other words, A can be considered as the algebra of polynomials of k indeterminates of order at most r with the truncated multiplication factorized by one of its ideals. Finally, we define width $A = \dim(N_A/N_A^2)$ and height A as the minimal r for which $A = \mathbb{D}_k^r/J$. For the nilpotent ideal of \mathbb{D}_k^r , we simplify the notation from $N_{\mathbb{D}_k^r}$ to N_k^r .

We remark that if N is associated with the symbol of a Weil algebra, then it denotes its nilpotent ideal, e.g., N_A or N_k^r . On the other hand, N endowed with no index is a symbol reserved for a manifold. This convention will hold in the whole paper.

In the whole paper, we apply the covariant approach to the definition of Weil functors. This comes out from the I -factorization of germs in the following sense. Two germs $\text{germ}_0 g: \mathbb{R}_0^k \rightarrow M_x$ and $\text{germ}_0 h: \mathbb{R}_0^k \rightarrow M_x$ taking the same value x at the source $0 \in \mathbb{R}^k$ are said to be I -equivalent if and only if $\text{germ}_x \gamma \circ \text{germ}_0 g - \text{germ}_x \gamma \circ \text{germ}_0 h \in I$ for any smooth function $\gamma: M \rightarrow \mathbb{R}$ defined near x . Classes of such equivalence relation are denoted by $j^A g$ and the space of them by $J^A M$. For a smooth map $\varphi: M \rightarrow N$, there is a map $J^A \varphi: J^A M \rightarrow J^A N$ defined by the assignment $j^A g \mapsto j^A(\varphi \circ g)$. For $A = \mathbb{D}_k^r$ we have $J^A M = J_0^r(\mathbb{R}^k, M)$ and $J^A \varphi = J_0^r(\text{id}_{\mathbb{R}^k}, \varphi)$.

We remark that a Weil functor can be also defined from the contravariant point of view as a functor T^A defined by $T^A M = \text{Hom}(C^\infty(M, \mathbb{R}), A)$ on objects and by $T^A f(H)(\varphi) = H(\varphi \circ f)$ on a morphism $f: M \rightarrow N$, where $H \in T^A M$ and $\varphi \in C^\infty(N, \mathbb{R})$. There is the identification $\theta: J^A M \rightarrow T^A M$ defined by $\theta(j^A g)(\varphi) = j^A(\varphi \circ g)$ for any $\varphi \in C^\infty(M, \mathbb{R})$ defined near $g(0)$ and by $\theta(J^A f)(H)(\psi) = H(\psi \circ f)$ for any smooth $f: M \rightarrow N$, $H \in T^A M$ and an any function $\psi \in C^\infty(N, \mathbb{R})$. In what follows we will use only the notation T^A for Weil functors in spite of applying the covariant approach to them.

We remark that for $A = \mathbb{D}_k^r$ the functor $T^A = T^{\mathbb{D}_k^r}$ coincides with the well-known functor T_k^r of (k, r) -velocities defined by $T_k^r M = J_0^r(\mathbb{R}^k, M)$ on objects and by $T_k^r f(j_0^r \varphi) = j_0^r(f \circ \varphi)$ on morphisms for any $j_0^r \varphi \in T_k^r M$, see [8].

As for natural transformations $\tilde{t}_M: T^B M \rightarrow T^A M$, they bijectively correspond to homomorphisms $t: B \rightarrow A$. Further, homomorphisms $B \rightarrow A$ correspond bijectively to the so-called B -admissible A -velocities defined in [7] as follows. Let $A = \mathcal{E}(k)/I$ and $B = \mathcal{E}(p)/J$ be Weil algebras. For a smooth map $f: \mathbb{R}_0^k \rightarrow \mathbb{R}_0^p$, an A -velocity $j^A f$ is said to be B -admissible if and only if $\text{germ}_0 \varphi \in J \Rightarrow \text{germ}_0(\varphi \circ f) \in I$.

Thus every B -admissible A -velocity $j^A f$ is assigned bijectively a natural transformation $\tilde{t}_M: T^B M \rightarrow T^A M$ induced by a Weil algebra homomorphism $t = t^{j^A f}: B \rightarrow A$. The natural transformation \tilde{t}_M is defined as

$$(1.1) \quad \tilde{t}_M(j^B \varphi) = \tilde{t}_M^{j^A f}(j^B \varphi) = j^A(\varphi \circ f).$$

In other words, a Weil algebra homomorphism $t: B \rightarrow A$ together with its associated natural transformation $\tilde{t}_M: T^B M \rightarrow T^A M$ corresponds to a reparametrization of the indeterminates t_1, \dots, t_p by suitable polynomials in τ_1, \dots, τ_k , transforming the ideal J to I . In particular, isomorphisms of Weil algebras correspond to invertible reparametrizations of this kind transforming bijectively the ideals defining them, which in the case of automorphisms coincide.

Let G_k^r denote the general jet group, see [8]. We recall that its support is formed by $J_0^r(\mathbb{R}^k, \mathbb{R}^k)_0$ endowed with the multiplication defined by the composition of jets.

It is a well-known fact that every $j_0^r g \in G_k^r$ determines an automorphism $t_{j_0^r g}$ of \mathbb{D}_k^r defined by the assignment

$$(1.2) \quad j_0^r \eta \mapsto j_0^r \eta \circ (j_0^r g)^{-1}$$

for any $j_0^r \eta \in \mathbb{D}_k^r$. If it is clear from the context we can write only $j_0^r g$ instead of $t_{j_0^r g}$, applying this convention particularly in (1.3), (1.4) and (1.5) below. Further, every automorphism of this kind determines the natural equivalence $\tilde{t}_{j_0^r g, M}$ of $T_k^r M$. For a Weil algebra $A = \mathbb{D}_k^r / I$ and the canonical projection homomorphism $p: \mathbb{D}_k^r \rightarrow A$, Alonso in [1] defined the subgroups G_A and $G^A \subseteq G_k^r \simeq \text{Aut}(\mathbb{D}_k^r)$ as

$$(1.3) \quad G_A = \{j_0^r g \in G_k^r: p \circ j_0^r g = p\}, \quad G^A = \{j_0^r g \in G_k^r: \text{Ker}(p \circ j_0^r g) = \text{Ker}(p)\}.$$

He also proved that G_A is a normal subgroup of G^A and the identification $G^A / G_A \simeq \text{Aut } A$. Clearly, $j_0^r g$ determines an automorphism of A and consequently a natural equivalence over $T^A M$ if and only if $j_0^r g \in G^A$. Clearly, $G_{\mathbb{D}_k^r} = \{j_0^r \text{id}_{\mathbb{D}_k^r}\}$ and $G^{\mathbb{D}_k^r} = G_k^r$.

In the rest of the paper, we will involve the Weil algebra into the notation of p from (1.3), i.e., we will write p_A instead p . Consider the left action l of G^A and the effective left action \bar{l} of $\text{Aut } A$ on $T_x^A M$ defined as

$$(1.4) \quad l(j_0^r g, j^A \varphi_x) = \bar{l}(\pi(j_0^r g), j^A \varphi_x) = \pi(j_0^r g)(j^A \varphi_x) = \tilde{p}_{A, M}(j_0^r \widehat{\varphi}_x \circ (j_0^r g)^{-1}),$$

where $\pi: G^A \rightarrow \text{Aut } A$ is the projection Lie group homomorphism and $j_0^r \widehat{\varphi}_x \in j^A \varphi_x$ is arbitrary. For a Lie subgroup $H \subseteq \text{Aut } A$ let us denote by $\text{Orb}(H, j^A \varphi_x)$ the H -orbit of $j^A \varphi_x$ with respect to \bar{l} restricted to $H \times T_x^A M$. Since every element of $\text{Aut } A$ determines a natural transformation over $T^A M$ determined by (1.4), the values of $T_x^A f$ on the whole $\text{Orb}(\text{Aut } A, j^A \varphi_x)$ are determined by $T_x^A f(j^A \varphi_x)$. This follows from the identity

$$(1.5) \quad \begin{aligned} T_x^A f(l(j_0^r g, j^A \varphi_x)) &= T_x^A f(\bar{l}(\pi(j_0^r g), j^A \varphi_x)) = (T_x^A f \circ \pi(j_0^r g))(j^A \varphi_x) \\ &= (\pi(j_0^r g) \circ T_x^A f)(j^A \varphi_x) = \bar{l}(\pi(j_0^r g), T_x^A f(j^A \varphi_x)) \\ &= l(j_0^r g, T_x^A f(j^A \varphi_x)). \end{aligned}$$

It is easy to see that $j^A \varphi_{x,1} = j^A \varphi_{x,2}$ implies $(j_0^r \varphi_{x,1})^{-1} \circ j_0^r \varphi_{x,2} \in G_A$.

1.2. Besides Weil bundles there are significant natural bundles $T^{r*} M$, the bundles of the r -th order covelocities (see (1.3)). They are defined by $T_x^{r*} M = J_x^r(M, \mathbb{R})_0$ on objects and by $T^{r*} g(j_x^r f) = j_x^r f \circ (j_x^r g)^{-1}$ on morphisms, where $j_x^r f \in T_x^{r*} M$ is arbitrary. In [17] we have presented the bundles $T^{r*} M$ in the form of $P^r M[N_m^r, l]$,

where $l: G_m^r \times N_m^r \rightarrow N_m^r$ denotes the left action of the jet group on the standard fiber determined by (1.4) and applied to the case of $A = \mathbb{D}_m^r$.

Moreover, we have generalized the space of higher-order covelocities $T^{r*}M$ to the space of A -covelocities $T^{A*}M$ consisting of the elements $T_x^A f: T_x^A M \rightarrow T_0^A \mathbb{R} \simeq N_A$. Further, for any local diffeomorphism g defined near x we have defined the so-called T^{A*} -map as

$$(1.6) \quad T^{A*}g(T_x^A f) = T_x^A f \circ (T_x^A g)^{-1}.$$

We recall the partial result from [17] obtained for such spaces and maps.

Let $A = \mathbb{D}_k^r/I$ and $m = \dim M$. Then the following holds:

For $m \geq k$, the spaces $T^{A*}M$ together with the maps defined by (1.6) form the natural bundle $PrM[N_m^r, l]$ identified with $T^{r*}M$.

For any $M \in \text{Obj}(\mathcal{M}f_m)$ and $N \in \text{Obj}(\mathcal{M}f)$ we define the space $J^A(M, N)$ by $J^A(M, N) = \{T_x^A f; f: M \rightarrow N\}$. For a local diffeomorphism $g: M_1 \rightarrow M_2$ and a smooth map $h: N_1 \rightarrow N_2$ we define the map $J^A(g, h): J^A(M_1, N_1) \rightarrow J^A(M_2, N_2)$ by

$$(1.7) \quad J^A(g, h)(T_x^A f) = T_{f(x)}^A h \circ T_x^A f \circ (T_x^A g)^{-1}.$$

In [19] we have proved a partial result valid for $m \geq k$, which states the identification of $J^A(M, N)$ with $J^r(M, N)$. In the conference paper [18] we have also sketched a proof concerning the cases of $m < k$, following the coordinate proof of the mentioned partial result from [17].

In the present paper, the main result will be proved and generalized in a new way, applying essentially Proposition 2.2 below, formulated and proved in a rigorous and explicit way, and Proposition 4.1, which states the rigidity of the so-called universally reparametrizable Weil algebras. In this proof we essentially reduce the computations with coordinates.

At the end of this section we observe that the elements $T_x^A f \in T_x^{A*}M$ form the nilpotent ideal $N_{B_{x,M}}$ of a Weil algebra $B_{x,M} = \mathbb{D}_m^r/J_{x,M}$ obtained from the algebraic operations defined on $T_x^{A*}M$ as follows. The vector space operations are defined by $(T_x^A f_1 + T_x^A f_2)(j^A \varphi_x) = T_x^A f_1(j^A \varphi_x) + T_x^A f_2(j^A \varphi_x)$ and $(c \cdot T_x^A f)(j^A \varphi_x) = c \cdot T_x^A f(j^A \varphi_x)$ for any $j^A \varphi_x \in T_x^A M$ and $c \in \mathbb{R}$. Analogously we define the multiplication on $T_x^{A*}M$ by

$$(1.8) \quad \begin{aligned} (T_x^A f_1 \cdot T_x^A f_2)(j^A \varphi_x) &:= \mu_A(T_x^A f_1(j^A \varphi_x), T_x^A f_2(j^A \varphi_x)) \\ &= T_{(0,0)}^A \mu \circ (T_x^A f_1(j^A \varphi_x), T_x^A f_2(j^A \varphi_x)), \end{aligned}$$

where μ denotes the standard multiplication on \mathbb{R} and $\mu_A = T^A\mu$ the multiplication on $A = T^A\mathbb{R}$. In other words, the product $T_x^A f_1 \cdot T_x^A f_2$ is defined by

$$(1.9) \quad T_x^A f_1 \cdot T_x^A f_2 = T_{(0,0)}^A \mu \circ (T_x^A f_1, T_x^A f_2) = T_x^A (\mu \circ (f_1, f_2)).$$

Clearly, every local diffeomorphism $g: M \rightarrow N$ defined near x determines the isomorphism $T_x^{A*}g: T_x^{A*}M \rightarrow T_{g(x)}^{A*}N$ determined by the map $T_x^{A*}g$, which is defined by the assignment $T_x^A f \mapsto T_x^A f \circ (T_x^A g)^{-1}$ for an arbitrary $T_x^A f \in T_x^{A*}M$. Therefore $T_x^{A*}M$ can be identified with the nilpotent ideal of $B_{x,M}$ and consequently with the nilpotent ideal of the Weil algebra $\mathbb{R} \oplus T_0^{A*}\mathbb{R}^m = B = \mathbb{D}_m^r/J$.

We present the already mentioned result $T^{r*}M \simeq P^r M[N_m^r, l]$ from [17] in a suitable form for direct application in the proof of the main result.

Lemma 1.1. *The multiplication in $T_0^{\mathbb{D}_k^r*}\mathbb{R}^m$ defined by (1.9) coincides with the restriction of the standard multiplication in \mathbb{D}_m^r to N_m^r .*

Proof. The elements $j_0^r f$ and $j_0^r g \in \mathbb{D}_m^r$ together with their product $j_0^r h = j_0^r f \times j_0^r g$, all of them considered in the polynomial form, satisfy in coordinates the identity $h_\alpha = f_\beta g_{\alpha-\beta} \alpha!/\beta!(\alpha-\beta)!$, applying the standard notation with multiindices. On the other hand, (1.8) yields

$$(1.10) \quad ((T_k^r)_0 f \cdot (T_k^r)_0 g)(j_0^r \varphi) = (T_k^r)_0 h(j_0^r \varphi) \\ = f_{l_1 \dots l_s} \varphi_{\mu_1}^{l_1} \dots \varphi_{\mu_s}^{l_s} \cdot g_{j_1 \dots j_t} \varphi_{\nu_1}^{j_1} \dots \varphi_{\nu_t}^{j_t} \tau^{\mu\nu} \text{ mod } \langle \tau_1, \dots, \tau_k \rangle^{r+1},$$

where $j_0^r \varphi \in (T_k^r)_0 \mathbb{R}^m$ is arbitrary and τ_1, \dots, τ_k denote the indeterminates of \mathbb{D}_k^r . We apply the standard derivative indices to the notation of the coordinates of $(T_k^r)_0 f$ and $(T_k^r)_0 g$ while in case of $j_0^r \varphi$ we apply the notation with multiindices. Put $j_0^r \varphi = (j_0^r \text{pr}_1)^m$ for the the first canonical projection $\text{pr}_1: \mathbb{R}^k \rightarrow \mathbb{R}$. In coordinates, we have $\varphi_j^l = \delta_j^l$ where $l = 1, \dots, m$. Applying the product $(T_k^r)_0 h$ from (1.10) to our $j_0^r \varphi$ and standard combinatorics, we obtain the coordinates of $(T_k^r)_0 h$ expressed in multiindices, which coincide with those of the coordinate identity in the very beginning of the proof. \square

Further, there is the projection homomorphism $p_B: \mathbb{R} \oplus J_0^r(\mathbb{R}^m, \mathbb{R})_0 \simeq \mathbb{D}_m^r \rightarrow B$ defined by

$$(1.11) \quad p_B((T_k^r)_0 f) = T_0^A f.$$

The verification of $p_B((T_k^r)_0 f \cdot (T_k^r)_0 g) = p_B((T_k^r)_0 f) \cdot p_B((T_k^r)_0 g) = T_0^A f \cdot T_0^A g$ is a direct consequence of (1.9). Indeed, $p_B((T_k^r)_0 f \cdot (T_k^r)_0 g) = p_B((T_k^r)_0 (f \cdot g)) = T_0^A (f \cdot g) = T_0^A f \cdot T_0^A g$.

Moreover, any surjective homomorphism $p_{\bar{A}, \hat{A}}: \bar{A} \rightarrow \hat{A}$ of Weil algebras determines the Weil algebra homomorphism $p_{\bar{B}, \hat{B}}: \bar{B} = T_0^{\bar{A}*} \mathbb{R}^m \rightarrow T_0^{\hat{A}*} \mathbb{R}^m = \hat{B}$ defined by

$$(1.12) \quad (p_{\bar{B}, \hat{B}}(T_0^{\bar{A}} f))(p_{\bar{A}, \hat{A}}(j^{\bar{A}} \varphi)) = p_{\bar{A}, \hat{A}}(T_0^{\bar{A}} f(j^{\bar{A}} \varphi))$$

for any $j^{\bar{A}} \varphi \in T_0^{\bar{A}} \mathbb{R}^m$, which can be easily and directly verified.

2. A -COVELOCITIES DETERMINED BY THEIR VALUES OVER $K = \max\{\text{width } A, \dim M\}$ ELEMENTS OF $\text{reg } T_x^A M$

In this section we prove that for any m -dimensional manifold M , every A -covelocality $T_x^A f \in T_x^{A*} M$ is determined by its values over arbitrary $\max\{\text{width } A, m\}$ regular and under the first jet projection linearly independent elements of $T_x^A M$. We precise the partial result from the conference paper [18], in which we sketched the proof of the fact that any A -covelocality is determined by its values over a linearly independent series of 1-jets of regular elements from $T_x^A M$. We keep the basic idea but we add an explicit formulation and rigorous construction of this determination applicable particularly in the proof of the main rigidity result. We also discuss properly the case of $m > k$ mentioned in that paper.

Without loss of generality, $A = \mathbb{D}_k^r / I$ will be supposed in the so-called normal form. The correctness of such assumption will be verified by the next lemma. The ideal I is said to be normal if $I = J \vee \langle \tau_{j_1}, \dots, \tau_{j_{k-l}} \rangle$ for another ideal J satisfying $J \subseteq \langle \tau_{i_1}, \dots, \tau_{i_l} \rangle^2$, where $\{j_1, \dots, j_{k-l}\} \cup \{i_1, \dots, i_l\} = \{1, \dots, k\}$. Further, A is said to be in the normal form if I is normal. Clearly, the definition of normality can be reduced to the condition $I \subseteq N_A^2$ in the case of $\text{width } A = k$. It follows from the comment after (1.1) that every Weil algebra is isomorphic to a normal one.

Lemma 2.1. *Let $t: A \rightarrow \hat{A}$ be an isomorphism of Weil algebras. Then any $T_x^A f \in T_x^{A*} M$ is determined by its values over elements $j^A \varphi_{1,x}, \dots, j^A \varphi_{h,x} \in T_x^A M$ if and only if $T_x^{\hat{A}} f$ is determined by its values over $\tilde{t}_M(j^A \varphi_{1,x}), \dots, \tilde{t}_M(j^A \varphi_{h,x})$.*

Proof. Fix $j^A \varphi_x \in T_x^A M$, put $j^A \eta_i = T_x^A f(j^A \varphi_{i,x})$ and $j^A \eta = T_x^A f(j^A \varphi_x)$. Then we represent the values of individual $T_x^A f$ over $j^A \varphi_x$ by the map $\omega: N_A^h \rightarrow N_A$ assigning some $j^A \eta$ to each h -tuple $(j^A \eta_1, \dots, j^A \eta_h)$. The same can be done with the space $T_x^{\hat{A}} M$, the elements $j^{\hat{A}} \hat{\varphi}_{i,x} = \tilde{t}_M(j^A \varphi_{i,x})$, $j^{\hat{A}} \hat{\varphi}_x = \tilde{t}_M(j^A \varphi_x)$, $j^{\hat{A}} \hat{\eta}_i = \tilde{t}_{\mathbb{R}}(j^A \eta_i)$ and $j^{\hat{A}} \hat{\eta} = \tilde{t}_{\mathbb{R}}(j^A \eta)$, obtaining the map $\hat{\omega}: N_{\hat{A}}^h \rightarrow N_{\hat{A}}$. To complete the proof it suffices to put $\hat{\omega} = \tilde{t}_{\mathbb{R}} \circ \omega \circ (\tilde{t}_M^{-1} \times \dots \times \tilde{t}_M^{-1})$, which follows from the commutativity of morphisms with natural transformations. \square

Let \tilde{B} be some Weil algebra of height r , $q < r$ and \tilde{B}_q denote the subordinate Weil algebra obtained by truncating \tilde{B} to the order q . By $\pi_{q,\tilde{B}}: \tilde{B} \rightarrow \tilde{B}_q$ we denote the projection homomorphism. Further, there is a homomorphism $\pi_{s,\tilde{B}}^q: \tilde{B}_q \rightarrow \tilde{B}_s$ determined by $\pi_{q,\tilde{B}}$ and $\pi_{s,\tilde{B}}$, which induces the natural transformation $(\tilde{\pi}_{s,\tilde{B}}^q)_M: T^{\tilde{B}_q}M \rightarrow T^{\tilde{B}_s}M$.

Consider the Weil algebra $B = \mathbb{R} \oplus N_B = \mathbb{D}_m^r/J$ with $N_B = T_0^{A*}\mathbb{R}^m$, see (1.11). Recall that for $A = \mathbb{D}_k^r$ the algebra B is identified with \mathbb{D}_m^r . Further, assume width $A = k$ in almost all of this section with the exception of Remark 2.2 and Corollary 2.1 at the very end.

In what follows, we need the cyclic permutations $\sigma_1, \dots, \sigma_m \in S_m$ considered in the form of the permutation matrices denoted by the same symbols. By S_m we denote the set of all permutations on $\{1, \dots, m\}$ and by $C_m \subseteq S_m$ the set of all cycles of length m . Further, for any permutation matrix σ , a real c and $l \in \{1, \dots, m\}$ let us define the matrix $\sigma(c, l)$ as

$$(2.1) \quad \sigma(c, l) = d(1, c^l, c^{2l}, \dots, c^{(m-1)l}) \cdot \sigma,$$

where $d(a_1, \dots, a_m)$ denotes in general the matrix with elements a_1, \dots, a_m on its diagonal and zeros on the other positions. Clearly, $\sigma(1, l) = \sigma$.

Lemma 2.2. *Let $c \notin \{0, 1, -1\}$. Then the matrices $\sigma(c, l)$ for $\sigma \in C_m$ and $l = 1, \dots, m$ form a linear basis of the Lie algebra $gl(m, \mathbb{R})$.*

Proof. The assertion is obtained from the assignment of an m^2 -dimensional row vector to every matrix $\sigma(c, l)$ read by rows and from the construction of the m^2 -th order matrix with rows formed by $\sigma_1(c, 1), \dots, \sigma_1(c, m), \dots, \sigma_m(c, 1), \dots, \sigma_m(c, m)$ considered in the vector form. By a suitable permutation of columns of such matrix we obtain a block-diagonal matrix. Expanding its determinant by its m -th order diagonal blocks we obtain its value in the form of the product of m Van der Monde determinants. This proves our claim. \square

In what follows, A is supposed to be normal. Let $B = \mathbb{D}_m^r/J = \mathbb{R} \oplus T_0^{A*}\mathbb{R}^m$ be the Weil algebra from (1.11). Further, let

$$(2.2) \quad \begin{aligned} T_0^A f &= p_B \circ (T_k^r)_0 f = p_B \left(\sum_{i=1}^r (T_k^r)_0 f^{(i)} \right) = \sum_{i=1}^r T_0^A f^{(i)} \\ &= p_B((T_k^r)_0 f^{(r)} + (T_k^r)_0 f^{(r-1)}) = T_0^A f^{(r)} + T_0^A f^{(r-1)} \end{aligned}$$

be some decomposition of $T_0^A f \in N_B$ such that $(T_k^r)_0 f^{(i)}$ corresponds to a homogeneous polynomial of order i and $(T_k^r)_0 f^{(i)}$ denotes the sum $\sum_{j=1}^i (T_k^r)_0 f^{(j)}$.

Let us denote by $\iota_k^r: \mathbb{D}_k^1 \rightarrow \mathbb{D}_k^r$ the canonical linear insertion of linear polynomials into the linear support of \mathbb{D}_k^r and by $\iota_k^{r,m}: (\mathbb{D}_k^1)^m \rightarrow (\mathbb{D}_k^r)^m$ the associated product linear map. Further, consider the canonical linear insertion $i_A: \mathbb{D}_k^1 \rightarrow A$ defined by $i_A = p_A \circ \iota_k^r$. Recalling our assumptions of width $A = k$ and the normality of A we deduce that this is a section with respect to $\pi_{1,A}: A \rightarrow A_1 \simeq \mathbb{D}_k^1$. Finally, consider the product map $i_A^m: (\mathbb{D}_k^1)^m \rightarrow A^m$ associated to $i_A: \mathbb{D}_k^1 \rightarrow A$. Clearly, this is a linear map again and since $k = \text{width } A$, the space $J_0^1(\mathbb{R}^k, \mathbb{R}^m)_0 \simeq i_A^m(J_0^1(\mathbb{R}^k, \mathbb{R}^m)_0)$ can be considered as a linear subspace of $T_0^A \mathbb{R}^m$. Clearly, $\iota_k^r = i_{\mathbb{D}_k^r}$ and $\iota_k^{r,m} = i_{\mathbb{D}_k^r}^m$.

Let us consider $j^A \eta \in T_0^A \mathbb{R} \simeq N_A$ in the form of $j^A \eta = T_0^A f(j^A \varphi)$ for some $j^A \varphi \in \text{reg } T_0^A \mathbb{R}^m$ and $T_0^A f \in T_0^{A*} \mathbb{R}^m$. Clearly, $j^A \eta$ can be decomposed (not necessarily uniquely) to

$$(2.3) \quad \begin{aligned} j^A \eta &= p_A(j_0^r \eta) = p_A \left(\sum_{i=1}^r j_0^r \eta_{(i)} \right) = \sum_{i=1}^r j^A \eta_{(i)} \\ &= p_A(j_0^r \eta_{(r)}) + p_A(j_0^r \eta^{(r-1)}) = j^A \eta_{(r)} + j^A \eta^{(r-1)}, \end{aligned}$$

where $j_0^r \eta_{(i)}$ coincides with a homogenous polynomial of order i and $j_0^r \eta^{(i)}$ with the sum $\sum_{j=1}^i j_0^r \eta_{(j)}$. It follows from the jet composition formula applied to any $(T_k^r)_0 f$ from (2.2) that $T_0^A f_{(r)}$ acts only on $j_0^1 \varphi \simeq i_A^m(j_0^1 \varphi)$ and that $T_0^A f_{(r)}$ affects only $j^A \eta_{(r)}$.

In the following investigations we will work with permutations, permutation matrices and their associated linear maps. Clearly, any permutation $\varrho \in S_m$ may be via its permutation matrix identified with the linear transformation s_ϱ on \mathbb{R}^m defined by $(x^1, \dots, x^m) \mapsto (x^{\sigma(1)}, \dots, x^{\sigma(m)})$ and consequently with an element of $\text{reg } J_0^1(\mathbb{R}^m, \mathbb{R}^m)_0$. Analogously we can do with the matrices $\sigma(c, l)$ determining the linear transformations on \mathbb{R}^m denoted by $s_{\sigma(c, l)}$.

The following modifications of σ and s_σ are applied in Proposition 2.1, namely in its second part concerning the case of $m > k$. Nevertheless, the principal part of Proposition 2.1 and its proof will correspond to the case of $m = k$, for which we can do only with σ and s_σ .

As for $m \geq k$ and $\sigma \in S_m$, denote by $\tilde{\sigma}: \{1, \dots, m\} \rightarrow \{0, 1, \dots, m\}$ the map assigning $\sigma(l)$ to l for $l \in \{1, \dots, k\}$ and 0 otherwise. Analogously to σ , the map $\tilde{\sigma}$ can be identified with the linear transformation $s_{\tilde{\sigma}}$ of \mathbb{R}^m defined by the assignment $(x^1, \dots, x^m) \mapsto s_\sigma(x^1, \dots, x^k, 0, \dots, 0)$. Finally, let us define the map $\bar{\sigma}: \{1, \dots, k\} \rightarrow \{1, \dots, m\}$ by the assignment $\bar{\sigma}(l) = \sigma(l)$ for $l = 1, \dots, k$ and the map $j: \mathbb{R}^k \rightarrow \mathbb{R}^m$ by

$$(2.4) \quad (x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0).$$

Clearly, $\bar{\sigma}$ can be identified with the linear map $s_{\bar{\sigma}}: \mathbb{R}^k \rightarrow \mathbb{R}^m$ defined by $s_{\bar{\sigma}} = s_{\bar{\sigma}} \circ j$ and consequently with the corresponding element of $\text{reg } J_0^1(\mathbb{R}^k, \mathbb{R}^m)_0$. In other words, we remove the last $m - k$ columns from the matrix of the linear transformation $s_{\bar{\sigma}}$. The same can be done with the matrices $\sigma(c, l)$, obtaining the matrices $\bar{\sigma}(c, l)$ and linear maps $\mathbb{R}^k \rightarrow \mathbb{R}^m$ denoted by $s_{\bar{\sigma}(c, l)}$ together with the corresponding elements of $J_0^1(\mathbb{R}^k, \mathbb{R}^m)_0$.

In order to be brief we simplify the notation from $s_{\bar{\sigma}}$ to $\bar{\sigma}$, from $s_{\bar{\sigma}}$ to $\tilde{\sigma}$ and from $s_{\sigma(c, l)}$ to $\sigma(c, l)$. As for matrices in general, we apply the same notation for them and the associated linear transformations of numerical spaces as well as for the corresponding 1-jets of zero-preserving maps.

Proposition 2.1. *Let A be in the normal form, $m \geq k = \text{width } A$ and $\sigma_1, \dots, \sigma_m$ be the cyclic permutations. Then every $T_0^A f_{(r)}$ contained in decomposition (2.2) is determined by its m images over $i_A^m(\bar{\sigma}_1), \dots, i_A^m(\bar{\sigma}_m)$.*

Proof. It follows from the deductions connected with (2.3) that for the purpose of our proof we can consider

$$(2.5) \quad T_0^A f_{(r)}(i_A^m(j_0^1 \varphi)) \simeq T_0^A f_{(r)}(j_0^1 \varphi)$$

instead of $T_0^A f_{(r)}(j^A \varphi)$, keeping in mind the normal form of I . By Lemma 2.2, a regular matrix $j_0^1 \varphi \in \text{reg } J_0^1(\mathbb{R}^m, \mathbb{R}^m)_0$ is identified with some linear combination of the matrices $\sigma(c, l)$ from (2.1), i.e., $j_0^1 \varphi = p^{\sigma, l} \sigma(c, l)$ for suitable coefficients $p^{\sigma, l} \in \mathbb{R}$, $\sigma \in C_m$ and $l = 1, \dots, m$.

In the first step, giving the principle idea of the proof, suppose $m = k$. In this case $\bar{\sigma}_i = \sigma_i$. Then for every $c \notin \{-1, 0, 1\}$ the value $T_0^A f_{(r)}(j_0^1 \varphi)$ can be expressed in the form of

$$(2.6) \quad T_0^A f_{(r)}(i_A^k(p^{\sigma, l} \sigma(c, l))), \quad \sigma \in C_k \text{ and } l = 1, \dots, k.$$

Fixing σ , the sum $p^{\sigma, l} \sigma(c, l)$ contained in (2.6) can be expressed in the form of $\sigma(1, l) \cdot D_{\sigma, c} = \sigma \cdot D_{\sigma, c}$ for some matrix $D_{\sigma, c}$ having all elements zeros except the diagonal ones. Clearly,

$$(2.7) \quad D_{\sigma, c} = \sigma^{-1} \cdot d(p^{\sigma, l} c^{0 \cdot l}, p^{\sigma, l} c^{1 \cdot l}, p^{\sigma, l} c^{2 \cdot l}, \dots, p^{\sigma, l} c^{(k-1) \cdot l}) \cdot \sigma,$$

where the components of the diagonal-like matrix in (2.7) are in the form of sums for $l = 1, \dots, k$. Clearly, for a dense subset of \mathbb{R}^k of all k -tuples $(p^{\sigma, 1}, \dots, p^{\sigma, k})$, the matrix $D_{\sigma, c} \simeq \iota_k^{r, k}(D_{\sigma, c}) \in G^A$. Applying (2.5) and recalling the notation (1.2) for

automorphisms of \mathbb{D}_k^r and the left action from (1.4), we deduce that for the dense subset of \mathbb{R}^k under discussion, every summand of (2.6) is of the form

$$(2.8) \quad T_0^A f_{(r)}(i_A^k(\sigma \cdot D_{\sigma,c})) = T_0^A f_{(r)}(l(t_{k,k}^{r,k}(D_{\sigma,c}^{-1}), i_A^k(\sigma))) \\ = t_{k,k}^{r,k}(D_{\sigma,c}^{-1}) \circ T_0^A f_{(r)}(i_A^k(\sigma)) = T_0^A f_{(r)}(i_A^k(\sigma)) \circ t_{k,k}^r(D_{\sigma,c}).$$

The standard density argument and the coincidence of the expressions in the second line of (2.8) prove our assertion for a fixed cycle $\sigma \in C_k$ in the sum (2.6). Since every partial summand from (2.6) corresponding to a fixed $\sigma \in C_k$ corresponds to a matrix of the same kind as σ in its nonzero positions, it follows from the well-known jet composition formula that $T_0^A f_{(r)}$ in (2.6) acts linearly and, consequently, (2.6) can be divided into k summands corresponding to the individual cycles σ . This completes the proof of the first step.

In the case of $m > k$, let us extend A to $\tilde{A} = \mathbb{D}_m^r / (I \vee \langle \tau_{k+1}, \dots, \tau_m \rangle \langle \tau_1, \dots, \tau_m \rangle)$, which is of width m . Further, A can be identified with

$$(2.9) \quad \bar{A} = \mathbb{D}_m^r / (I \vee \langle \tau_{k+1}, \dots, \tau_m \rangle),$$

adding formally $m - k$ indeterminates. Clearly, there is the projection homomorphism $\bar{p}: \tilde{A} \rightarrow \bar{A} \simeq A$. Consider $j_0^1 \varphi \in J_0^1(\mathbb{R}^k, \mathbb{R}^m)_0$, which can be also considered as an element of $J_0^1(\mathbb{R}^m, \mathbb{R}^m)_0$. First we observe that for any $c \notin \{-1, 0, 1\}$ the equality $i_A^m(j_0^1 \varphi) = i_A^m(p^{\sigma,l} \sigma(c, l))$ is valid if and only if so is the equality $i_{\bar{A}}^m(j_0^1 \varphi) = i_{\bar{A}}^m(p^{\sigma,l} \bar{\sigma}(c, l))$ in which all $\sigma \in C_m$ are considered. One can verify easily that $i_{\bar{A}}^m(j_0^1 \varphi)$ equals to the sum $\bar{\sigma} \cdot D_{\bar{\sigma},c}$ for $\sigma \in C_m$ and the matrices $D_{\bar{\sigma},c}$ obtained from $D_{\sigma,c}$ as their first diagonal blocks of order k . Further, $T_0^A f_{(r)}(i_A^m(j_0^1 \varphi)) = T_0^A f_{(r)}(i_{\bar{A}}^m(p^{\sigma,l} \bar{\sigma}(c, l))) = \bar{p} \circ T_0^{\tilde{A}} f_{(r)}(i_{\tilde{A}}^m(p^{\sigma,l} \sigma(c, l)))$.

Applying the first step of the present proof to \tilde{A} we obtain the above expression in the form of the sum $\bar{p} \circ T_0^{\tilde{A}} f_{(r)}(l(t_{m,m}^{r,m}(D_{\sigma,c}^{-1}), i_{\tilde{A}}^m(\sigma))) = \bar{p} \circ (t_{k,k}^{r,m}(D_{\sigma,c}^{-1}) \circ T_0^{\tilde{A}} f_{(r)}(i_{\tilde{A}}^m(\sigma))) = \bar{p} \circ T_0^{\tilde{A}} f_{(r)}(i_{\tilde{A}}^m(\sigma)) \circ t_{k,k}^r(D_{\sigma,c})$ (for $\sigma \in C_m$) on the dense subset of all m^2 -tuples consisting of the coefficients $p^{\sigma,l}$. It is easy to see that this expression equals to $T_0^A f_{(r)}(i_A^m(\bar{\sigma})) \circ t_{k,k}^r(D_{\bar{\sigma},c})$. Thus we have proved that $T_0^A f_{(r)}(i_A^m(j_0^1 \varphi))$ is really determined by the values $T_0^A f_{(r)}(i_A^m(\bar{\sigma}))$ and the coefficients $D_{\bar{\sigma},c}$ constructed from $D_{\sigma,c}$. This completes the proof. \square

Proposition 2.2. *Let $m = \dim M$, $A = \mathbb{D}_k^r / I$, $k = \text{width } A$ and $K = \max\{k, m\}$. Then every $T_x^A f \in T_x^{A*} M$ is determined by its values over arbitrary, in the first jet projection linearly independent regular elements $j^A \varphi_{1,x}, \dots, j^A \varphi_{K,x} \in T_x^A M$.*

Proof. First assume $m \geq k$. We prove that for every $T_0^A f \in T_0^{A*} \mathbb{R}^m$ and $j^A \varphi \in T_0^A \mathbb{R}^m$, the value $T_0^A f(j^A \varphi) = j^A \eta$ is determined by m values $T_0^A f(i_A^m(\bar{\sigma}_i))$. For $s =$

1, \dots, r-1 consider the chain of Weil algebra homomorphisms $\pi_{s, A_{s+1}} : A_{s+1} \rightarrow A_s$ of truncated Weil algebras associated to $A = A_r$, recalling the notation introduced immediately after the proof of Lemma 2.1. Further, consider the corresponding chain of Weil algebra homomorphisms $\pi_{s, B_{s+1}} : B_{s+1} \rightarrow B_s$ setting $B_s = T_0^{A_s^*} \mathbb{R}^m$, see (1.12). By Proposition 2.1, for any $s = 1, \dots, r$ the map $T_0^{A_s} f_{(s)}$ is determined by its m values over $i_A^m(\bar{\sigma}_i)$ and consequently over $\pi_{s, A} \circ i_A^m(\bar{\sigma}_i)$.

In the first step observe that $T_0^A f$ is determined by its m values over $i_A^m(\bar{\sigma}_i)$ up to $T_0^{A_1} f_{(1)} = T_0^{A_1} f^{(1)}$, which is actually determined by its values over $\pi_{1, A} \circ i_A^m(\bar{\sigma}_i) = \bar{\sigma}_i$. We step by step particularize the determination of $T_0^A f$ by its values over $i_A^m(\bar{\sigma}_i)$ by means of $T_0^{A_s} f = T_0^{A_s} f^{(s)}$, increasing s up to r .

More exactly, let us deduce the $(s+1)$ -st step from the assumption of $T_0^{A_s} f$ being determined by its values over $i_A^m(\bar{\sigma}_i)$. Consider $T_0^A f$ as an element of $T_0^{A_s} f = T_0^{A_s} f^{(s)} \in B_s$ and some of its decomposition (2.2) together with the corresponding decomposition (2.3) of $j^A \eta$. We observe from the well-known jet composition formula that $T_0^A f_{(s)}$ affects not only $j^A \eta_{(s)}$ but also $j^A \eta^{(t)}$ for any $s \leq t \leq r$. However, the remainder of $j^A \eta$ in the form of $j^A \varrho_{s+1} = T_0^A f(j^A \varphi) - T_0^A f^{(s)}(j^A \varphi)$ is affected only by $T_0^A f_{(t)}$ for $t \geq s+1$. Then $j^A \varrho_{s+1}$ is determined exactly by the values of $T_0^A f_{(s+1)}$ over $i_A^m(\bar{\sigma}_i)$ and consequently, $j_{s+1}^A \eta^{(s+1)}$ is determined by the values of $T^{A_{s+1}} f^{(s+1)} = T_0^{A_{s+1}} f$ over $i_A^m(\bar{\sigma}_i)$.

Replacing \mathbb{R}^m by M , 0 by x and applying $T_x^A f(j^A \varphi_x) = T_0^A (f \circ u_x^{-1}) \circ j^A (u_x \circ \varphi_x)$ for any coordinate map $u_x : M_x \rightarrow \mathbb{R}_0^m$ completes the proof for $m \geq k$. For $m < k$ the assertion is obtained immediately if we identify every $f : M \rightarrow \mathbb{R}$ with the map $\hat{f} : M \times \mathbb{R}^{k-m} \rightarrow \mathbb{R}$ defined by $\hat{f}(x, y) = f(x)$. \square

Remark 2.1. An element $j^A \varphi_x \in T_x^A M$ is said to be regular if near $0 \in \mathbb{R}^k$ and for $k \leq m$ the map φ_x is an immersion while for $k \geq m$ it is a submersion.

Remark 2.2. In Proposition 2.2, the assumption of width $A = k$ may be omitted. To verify this assertion, assume $m \geq h \geq k = \text{width } A$ and consider $\hat{A} = \mathbb{D}_h^r / (I \vee \langle \tau_{k+1}, \dots, \tau_h \rangle)$. We remark that for $h = m$ the algebra \hat{A} coincides with \bar{A} from (2.9). Let $j : \mathbb{R}^k \rightarrow \mathbb{R}^h$ be the map defined in (2.4), substituting h for m . Clearly, the map $t : A \rightarrow \hat{A}$ defined by the assignment $j^A \eta \mapsto j^{\hat{A}} \hat{\eta}$ for any $j^{\hat{A}} \hat{\eta}$ satisfying $\eta = \hat{\eta} \circ j$ is an isomorphism. Applying Lemma 2.1 we immediately deduce our claim formulated as follows.

Corollary 2.1. Let $A = \mathbb{D}_k^r / I$ be a Weil algebra, $\dim M = m$ and $m \geq k$. Then every $T_x^A f \in T_x^{A^*} M$ is determined by its values over m arbitrary linearly independent regular elements $j_0^1 \varphi_{1,x}, \dots, j_0^1 \varphi_{m,x} \in J_0^1(\mathbb{R}^k, M)_x$.

In the very end of the present section we give a lemma applied in the proof of the main rigidity result below.

Lemma 2.3. Let $\tilde{t}_M: T^{A_1}M \rightarrow T^{A_2}M$ be the natural equivalence induced by an isomorphism $t: A_1 \rightarrow A_2$ and $\tilde{t}_{x,M}$ be its restriction to $T_x^{A_1}M$. Then for any $x \in M$ there is an isomorphism $t_{x,M}^*: T_x^{A_1^*}M \rightarrow T_x^{A_2^*}M$ defined by one of the following two mutually equivalent conditions:

- (1) $t_{x,M}^*(T_x^{A_1}f)(\tilde{t}_M(j^{A_1}\varphi_x)) = t \circ T_x^{A_1}f(j^{A_1}\varphi_x)$ for any $j^{A_1}\varphi_x \in T_x^{A_1}M$,
- (2) $t_{x,M}^*(T_x^{A_1}f) = T_x^{A_2}f = t \circ T_x^{A_1}f \circ \tilde{t}_{x,M}^{-1}$.

Proof. First we recall that any natural transformation of Weil functors covers the identity on base manifolds (see [8]). The equivalence of the conditions (1), (2) follows from the commutativity of natural transformations with morphisms and the surjectivity of $t_{x,M}: T_x^{A_1}M \rightarrow T_x^{A_2}M$. Further, every $t_{x,M}^*: T_x^{A_1^*}M \rightarrow T_x^{A_2^*}M$ is an isomorphism of Weil algebras. In order to verify this we check that $t_{x,M}^*$ is a homomorphism, i.e., $(T_x^{A_2}f_1 \cdot T_x^{A_2}f_2)(j^{A_2}\varphi_x) = t \circ (T_x^{A_1}f_1 \cdot T_x^{A_1}f_2)(j^{A_1}\varphi_x)$ for any $j^{A_1}\varphi_x \in T_x^{A_1}M$ and $j^{A_2}\varphi_x = t_{x,M}(j^{A_1}\varphi_x)$. Applying (1.8) and the commutativity of natural transformations with morphisms we immediately verify that $(T_x^{A_2}f_1 \cdot T_x^{A_2}f_2)(j^{A_2}\varphi_x) = T_{(0,0)}^{A_2}\mu \circ (T_x^{A_2}f_1(j^{A_2}\varphi_x), T_x^{A_2}f_2(j^{A_2}\varphi_x)) = T_{(0,0)}^{A_2}\mu \circ (T_x^{A_2}f_1 \circ \tilde{t}_{x,M}(j^{A_1}\varphi_x), T_x^{A_2}f_2 \circ \tilde{t}_{x,M}(j^{A_1}\varphi_x)) = T_{(0,0)}^{A_2}\mu \circ \tilde{t}_{\mathbb{R}^2}(T_x^{A_1}f_1(j^{A_1}\varphi_x), T_x^{A_1}f_2(j^{A_1}\varphi_x)) = t \circ T_{(0,0)}^{A_1}\mu(T_x^{A_1}f_1(j^{A_1}\varphi_x), T_x^{A_1}f_2(j^{A_1}\varphi_x)) = t \circ (T_x^{A_1}f_1 \cdot T_x^{A_1}f_2)(j^{A_1}\varphi_x)$. Since $t_{x,M}$ is obviously invertible, it is really an isomorphism. \square

3. THE GENERAL RIGIDITY RESULT

In the present section we prove the general rigidity result $T^{A^*}M \simeq T^{r^*}M \simeq P^rM[N_m^r, l]$. Besides Proposition 2.2 and its corollary we essentially apply the so-called rigidity of universally reparametrizable Weil algebras, formulated and proved in Proposition 4.1 below. The proof for the lower-dimensional cases of M is obtained by means of an essential application of Proposition 2.2 as well.

In the next Proposition 3.1 we are searching for a Weil algebra whose nilpotent ideal coincides with $T_0^{A^*}\mathbb{R}^m$. In Subsection 1.2 before Lemma 1.1 and in Section 2 before (2.2) it was denoted by B and its defining ideal by J . Nevertheless, in the proof of Proposition 3.1 we use those symbols in a different meaning for convenience. Unlike the notation introduced after Lemma 2.1, the indices by symbols like B_i do not indicate subordinate Weil algebras but only their indexing.

In the whole section we assume height $A = r$. We will also work with the subalgebras $\mathbb{D}_{(i_1, \dots, i_l)}^r \subseteq \mathbb{D}_k^r$ consisting of polynomials in indeterminates $\tau_{i_1}, \dots, \tau_{i_l}$ only. For the next proof, we recall the linear map $\iota_k^r: \mathbb{D}_k^1 \rightarrow \mathbb{D}_k^r$, its associated product map $\iota_k^{r,m}: (\mathbb{D}_k^1)^m \rightarrow (\mathbb{D}_k^r)^m$, the linear map $i_A: \mathbb{D}_k^1 \rightarrow A$ defined by $i_A = p_A \circ \iota_k^r$ and its associated product map $i_A^m: (\mathbb{D}_k^1)^m \rightarrow A^m$, all defined between (2.2) and (2.3).

Proposition 3.1. For $m \geq \text{width } A$ and height $A = r$ the algebraic structure of $T_0^{A*} \mathbb{R}^m$ coincides with the nilpotent ideal N_m^r of the jet algebra \mathbb{D}_m^r .

Proof. It follows from Lemma 2.3 applied to A and \bar{A} defined by (2.9) that without loss of generality A can be assumed in the form of \mathbb{D}_m^r/I . Consider $T_0^A f \in T_0^{A*} \mathbb{R}^m$ in the form of the sum

$$(3.1) \quad \frac{1}{\alpha!} a_\alpha T_0^A \text{pr}_{\mathbb{R}^m}^\alpha, \quad 1 \leq |\alpha| \leq r,$$

where $\text{pr}_{\mathbb{R}^m}^\alpha: \mathbb{R}^m \rightarrow \mathbb{R}$ denotes a monomial constructed from the projections $\text{pr}_{\mathbb{R}^m}^1, \dots, \text{pr}_{\mathbb{R}^m}^m: \mathbb{R}^m \rightarrow \mathbb{R}$ corresponding to a multiindex α . We prove that all a_α are necessary for determining $T_0^A f$.

(i) In the first step consider $T_0^A f|_{\text{Orb}(\text{Aut } A, j^A \text{id}_{\mathbb{R}^m})}$, which can be identified with $j^A f = T_0^A f(j^A \text{id}_{\mathbb{R}^m})$ (see (1.5)). Put $T_0^A f(j^A \text{id}_{\mathbb{R}^m}) = j^A(a_\alpha \tau^\alpha / \alpha!) = [a_\alpha \tau^\alpha / \alpha!]_I$. This identification determines the coefficients a_α up to the decomposition classes $[(a_\alpha)_{|\alpha| \leq r}]_I$ corresponding to the I -classes $[(a_\alpha \tau^\alpha / \alpha!)]_I$. Notice that in the case of monomial A all a_α are uniquely determined except those corresponding to $\tau^\alpha \in I$ which may be for present arbitrary.

(ii) Given $j^A \varphi \in N_A$ we deduce that $\text{Orb}(\text{Aut } A, j^A \varphi) = N_A$ implies $A = \mathbb{D}_m^r$. It is easy to see that our claim is equivalent to $G^A = G_m^r$. This follows from the definition of G^A and G_A , the latter being identified with the stabilizing subgroup $\text{St}(A) \subseteq G_m^r$ of all elements from A and from the identification $G^A/G_A \simeq \text{Aut } A$ (see (1.3)). Thus every $j_0^r g \in G_m^r$ determines an A -admissible A -velocity. Applying Proposition 4.1 from Section 4 with m substituted for k yields immediately $A = \mathbb{D}_m^r$.

(iii) In the case of $A \neq \mathbb{D}_m^r$, select $j^A \psi \notin \text{Orb}(\text{Aut } A, j^A \text{id}_{\mathbb{R}^m})$. Further, consider some $j_0^r \psi_0 \in j^A \psi$. Put $J = I \circ (j_0^r \psi_0)^{-1}$ and $B = \mathbb{D}_m^r/J$. Clearly, $(j_0^r \psi_0)^{-1}$ determines an A -admissible B -velocity and consequently a natural equivalence $\tilde{t}_{j_0^r \psi_0}: T^A \rightarrow T^B$.

Then the commutativity of natural transformations with morphisms yields $T_0^A f(j^A \psi) = T_0^B f(j^B \text{id}_{\mathbb{R}^m}) \circ j_0^r \psi_0$ and consequently $j^B f = T_0^B f(j^B \text{id}_{\mathbb{R}^m}) = T_0^A f(j^A \psi) \circ (j_0^r \psi_0)^{-1}$. For $j_0^r \psi_0$ we select $\iota_m^{r,m}(\sigma)$ for some cycle $\sigma \in C_m$. Then the coefficients a_α of $T_0^A f$ are determined up to the decomposition J -classes $[(a_\alpha)_{|\alpha| \leq r}]_J$ corresponding to $[(a_\alpha \tau^\alpha / \alpha!)]_J = j^B(a_\alpha \tau^\alpha / \alpha!)$ and consequently to the decomposition $I \cap J$ -classes $[(a_\alpha)_{|\alpha| \leq r}]_{I \cap J}$ corresponding to $j^C(a_\alpha \tau^\alpha / \alpha!)$, if we denote by C the Weil algebra $\mathbb{D}_m^r/I \cap J$.

(iv) Conversely, consider $T_0^B f$. We identify $T_0^B f|_{\text{Orb}(\text{Aut } B, j^B \text{id}_{\mathbb{R}^m})}$ with $j^B f = T_0^B f(j^B \text{id}_{\mathbb{R}^m})$ analogously as in (i). This enables us to determine the coefficients a_α of $T_0^B f$ up to the decomposition classes $[(a_\alpha)_{|\alpha| \leq r}]_J$ corresponding to the J -classes $[(a_\alpha \tau^\alpha / \alpha!)]_J \simeq j^B(a_\alpha \tau^\alpha / \alpha!)$. Further, consider $j_0^r \psi_0$ from (iii) and its in-

verse $j_0^r \psi_0^{-1}$. This is an $\iota_m^{r,m}$ -image of some cycle again, which follows from the definition of $\iota_k^{r,m}$ presented after (2.2) and the coordinate expression of the multiplication in G_m^r . Clearly, $j_0^r \psi_0$ determines a B -admissible A -velocity and hence a natural equivalence $\tilde{t}_{j_0^r \psi_0^{-1}}: T^B \rightarrow T^A$. Then the commutativity of natural transformations with morphisms yields $T_0^B f(j^B \psi^{-1}) = T_0^A f(j^A \text{id}_{\mathbb{R}^m}) \circ j_0^r \psi_0^{-1}$ and consequently $j^A f = T_0^A f(j^A \text{id}_{\mathbb{R}^m}) = T_0^B f(j^B \psi) \circ (j_0^r \psi_0)$.

Analogously as in (iii) we obtain that the coefficients of $T_0^B f$ are determined by its evaluation over $j^B \text{id}_{\mathbb{R}^m}$ and $j^B \psi_0^{-1}$ up to the decomposition $I \cap J$ -classes $[(a_\alpha)_{|\alpha| \leq r}]_{I \cap J}$ corresponding to $j^C(a_\alpha \tau^\alpha / \alpha!)$ for $C = \mathbb{D}_m^r / I \cap J$ again.

(v) Consider the elements $j_0^r \psi_i = \iota_m^{r,m}(\sigma_i)$ for all cycles $\sigma_i \in C_m$. Then $j_0^r \text{id}_{\mathbb{R}^m}$ coincides with $j_0^r \psi_1$ in this notation, while $j_0^r \psi_0$ from (iii) corresponds to the minimal $i \in \{1, \dots, m\}$ and $\iota_m^{r,m}(\sigma_i)$ for which $\tilde{p}_{A, \mathbb{R}^m} \circ \iota_m^{r,m}(\sigma_i) = i_A^m(\sigma_i) \notin \text{Orb}(\text{Aut } A, j^A \text{id}_{\mathbb{R}^m})$. For each $j_0^r \psi_i$ we put $J_i = I \circ (j_0^r \psi_i)^{-1}$ and $B_i = \mathbb{D}_m^r / J_i$. We step by step particularize the coefficients a_α of $T_0^A f$ from (3.1). We do so with the coefficients of $T_0^{B_i} f$ and thus extend the step (iv), applying the fact of $\{\iota_m^{r,m}(\sigma) : \sigma \in C_m\}$ being closed with respect to inverses in G_m^r . We conclude that the coefficients of the individual elements $T_0^{B_i} f$, $i = 1, \dots, m$, are determined up to the $J_1 \cap \dots \cap J_m$ -classes $[(a_\alpha)_{|\alpha| \leq r}]_{J_1 \cap \dots \cap J_m}$ corresponding to $j^C(a_\alpha \tau^\alpha / \alpha!)$ for $C = \mathbb{D}_m^r / J_1 \cap \dots \cap J_m$.

(vi) Applying Proposition 2.2 and Corollary 2.1 we conclude after m steps that every element of the form (3.1) with the set of the coefficients $(a_\alpha)_{|\alpha| \leq r}$ contained in a $J_1 \cap \dots \cap J_m$ -class $[(a_\alpha)_{|\alpha| \leq r}]_{J_1 \cap \dots \cap J_m} \simeq j^C(a_\alpha \tau^\alpha / \alpha!)$ determines each of the maps $T_0^{B_i} f$. Moreover, each of $T_0^{B_i} f$ coincides with $T_0^C f$ for $C = \mathbb{D}_m^r / J_1 \cap \dots \cap J_m$. This follows from the fact that the maps $T_0^{B_i} f$ mutually coincide and that they respect J_i -classes. Further, suppose $J = J_1 \cap \dots \cap J_m \neq \{j_0^r 0_{\mathbb{R}^m}\}$. By (ii) there is $j_0^r \chi \in G_m^r$ such that $J \circ (j_0^r \chi)^{-1} \not\subseteq J$. This enforces another limiting condition on the coefficients a_α from (3.1). Nevertheless, this is a contradiction with Proposition 2.2 and Corollary 2.1.

(vii) In the final step we are to verify the algebraic isomorphism $T_0^{A^*} \mathbb{R}^m \cong N_m^r$. Its existence is obtained directly from $T_0^{A^*} \mathbb{R}^m = T_0^{\mathbb{D}_m^r} \mathbb{R}^m$ and Lemma 1.1. This completes the proof. \square

Remark 3.1. The assumption of height $A = r$ has been applied in (ii) by means of Proposition 4.1 from Section 4. This assumption is necessary for its validity and it is explicitly formulated there.

Let us return to the notation $B = \mathbb{R} \oplus T_0^{A^*} \mathbb{R}^m = \mathbb{D}_m^r / J$. Clearly, $B = \mathbb{D}_m^r$ for any $m \geq k$. In what follows, consider the subalgebras $A_{(i_1, \dots, i_m)} = p_A(\mathbb{D}_{(i_1, \dots, i_m)}^r)$. We prove the coincidence of B with \mathbb{D}_m^r for $m < \text{width } A$.

Proposition 3.2. *For $m < k = \text{width } A$ and height $A = r$, the algebraic structure of $T_0^{A^*} \mathbb{R}^m$ coincides with the nilpotent ideal N_m^r of \mathbb{D}_m^r .*

Proof. We prove that every $T_0^A f \in T_0^{A^*} \mathbb{R}^m$, obviously identified with $T_0^A \widehat{f}$: $T_0^A \mathbb{R}^m \times T_0^A \mathbb{R}^{k-m} \rightarrow T_0^A \mathbb{R}$ free of the second set of arguments, coincides with $(a_\alpha/\alpha!) T_0^A \text{pr}_{\mathbb{R}^m}^\alpha$ from (3.1), satisfying $a_\alpha = 0$ whenever α contains an index from the set $\{m+1, \dots, k\}$. We make the most of the fact that $T_0^A \widehat{f}$ is determined by its values over $i_A^k(\sigma_1), \dots, i_A^k(\sigma_k)$ for the cycles $\sigma_i \in C_k$ and analogously we do with the restrictions of $T_0^A f$ to $A_{(\sigma(1), \dots, \sigma(m))}^m$, cycling $\sigma(1), \dots, \sigma(m)$. Clearly, the restrictions of $T_0^A f$ to $A_{(\sigma(1), \dots, \sigma(m))}^m$ under discussion are identified with the restrictions of $T_0^A \widehat{f}$ to $A_{(\sigma(1), \dots, \sigma(m))}^m \times A_{(\sigma(m+1), \dots, \sigma(k))}^{k-m}$.

For every $\sigma \in C_k$, define the ideal $J_\sigma \subseteq \mathbb{D}_m^r$ by

$$(3.2) \quad J_\sigma(t_1, \dots, t_m) = (I \circ \sigma \vee \langle \tau_{\sigma(m+1)}, \dots, \tau_{\sigma(k)} \rangle) \cap \mathbb{D}_m^r(\tau_{\sigma(1)}, \dots, \tau_{\sigma(m)}),$$

where t_1, \dots, t_m denote the indeterminates $\tau_{\sigma(1)}, \dots, \tau_{\sigma(m)}$ and $\langle \tau_{\sigma(m+1)}, \dots, \tau_{\sigma(k)} \rangle$ denotes the ideal in \mathbb{D}_k^r generated by $\tau_{\sigma(m+1)}, \dots, \tau_{\sigma(k)}$. Briefly speaking, we have substituted $t_1 = \tau_{\sigma(1)}, \dots, t_k = \tau_{\sigma(k)}$ for τ_1, \dots, τ_k in I and joining $I \circ \sigma$ obtained in this way with $\langle \tau_{\sigma(m+1)}, \dots, \tau_{\sigma(k)} \rangle$ we receive an ideal of width m which can be considered only in t_1, \dots, t_m after the obvious renaming of the indeterminates presented above. Finally, we put $B_\sigma = \mathbb{D}_m^r/J_\sigma$.

The elements $T_0^A f$ restricted to $A_{(\sigma(1), \dots, \sigma(m))}^m$ can be identified with the elements $T_0^{B_\sigma} f$ and so can be the corresponding restrictions of T^{A^*} -maps with T^{B_σ} -maps. By the partial rigidity result obtained in Proposition 3.1, the algebra of such elements coincides with the nilpotent ideal $N_m^{q_\sigma} \subseteq \mathbb{D}_m^{q_\sigma}$, putting $q_\sigma = \text{height } A_{(\sigma(1), \dots, \sigma(m))}$. Nevertheless, for the restrictions of $T_0^A \widehat{f}$ to $A_{(\sigma(1), \dots, \sigma(m))}^m \times A_{(\sigma(m+1), \dots, \sigma(k))}^{k-m}$ the coefficients a_α corresponding to α satisfying $\alpha \cap \{m+1, \dots, k\} \neq \emptyset$ may be put to zero.

This step can be done with every $\sigma \in C_k$. Clearly, the coefficients a_α corresponding to $\alpha \cap \{m+1, \dots, k\} \neq \emptyset$ may be put to zero in all of those cases. Since every $T_0^A f$ under discussion, identified with $T_0^A \widehat{f}$ free of the second argument, is by Proposition 2.2 determined by its values over $i_A^k(\sigma_1), \dots, i_A^k(\sigma_k)$, the coefficients a_α containing any index from $\{m+1, \dots, k\}$ are really zeros and our claim is proved. \square

We state the general rigidity result in the following theorem.

Theorem 3.1. *Let $A = \mathbb{D}_k^r/I$, height $A = r$ and N_m^r be the nilpotent ideal of \mathbb{D}_m^r . Further, let $l: G_m^r \times N_m^r \rightarrow N_m^r$ be the left action defined by*

$$(3.3) \quad l(j_0^r g, j_0^r \alpha) = j_0^r(\alpha \circ g^{-1}).$$

Then the system of spaces $T^{A*}M$ and T^{A*} -maps from (1.6) over m -dimensional manifolds and local diffeomorphisms forms the natural bundle $P^r M[N_m^r, l]$ identified with the r -th order covelocities bundle $T^{r*}M$.

P r o o f. In the context of the general theory of natural bundles (see [8]) consider the formula

$$(3.4) \quad l(j_0^r g, T_0^A f) = T_0^A(f \circ g^{-1}).$$

Further, any $T_x^A f$ is identified with $\{j_0^r \alpha_x, T_x^A f \circ (T_0^A \alpha_x)\} = \{j_0^r \alpha_x, T_0^A(f \circ \alpha_x)\}$ and conversely $\{j_0^r \alpha_x, T_0^A f\}$ is identified with $T_x^A(f \circ \alpha_x^{-1})$. The fact that $\{j_0^r \alpha_x \circ j_0^r g, T_0^A(f \circ g)\}$ and $\{j_0^r \alpha_x, T_0^A f\}$ determine the same element of $T_x^{A*}M$ implies the identification $T_x^{A*}M \simeq P_x^r M[T_0^{A*}\mathbb{R}^m, l]$ with l defined by (3.4).

As for morphisms, it is easy to verify that the maps $T^{A*}g$ correspond to the morphisms of associated bundles which are of the form $\{P^r g, \text{id}_{T_0^{A*}\mathbb{R}^m}\}$.

Let us denote by $\omega: T_0^{A*}\mathbb{R}^m \rightarrow N_m^r$ the algebraic identification introduced by (3.1) and verified by the proofs of Proposition 3.1 and Proposition 3.2. Clearly, ω is of the form $T_0^A f \cong (1/\alpha!)a_\alpha T_0^A \text{pr}_{\mathbb{R}^m}^\alpha \cong (1/\alpha!)a_\alpha \tau^\alpha$, $1 \leq |\alpha| \leq r$.

Checking the equivariancy of ω with respect to (3.3) and (3.4), which reads $\omega \circ l(j_0^r g, T_0^A f) = l(j_0^r g, \omega(T_0^A f))$, completes the proof. \square

Corollary 3.1. For $A = \mathbb{D}_k^r/I$ the system of spaces $J^A(M, N)$ and their $J^A(g, h)$ -maps (see (1.7)) is identified with the jet functor J^r .

P r o o f. For any $M \in \text{Obj}(\mathcal{M}f_m)$ consider the bundle functor $G_M: \mathcal{M}f \rightarrow \mathcal{F}\mathcal{M}$ defined by $G_M N = J^r(M, N)$ on objects and by $G_M h = J^r(\text{id}_M, h)$ on morphisms. From the point of view of the objects, it follows from Theorem 3.1 that $J^r(M, N) = G_M N = P^r N[(T^{r*}M)^n, l_M] = P^r N[(T^{A*}M)^n, l_M]$, where $l_M: G_n^r \times (T^{r*}M)^n \rightarrow (T^{r*}M)^n$ is defined by $l_M(j_0^r g, T_x^r f) = T_0^r g \circ T_x^r f = T_0^A g \circ T_x^A f$. Therefore $J^r(M, N) = J^A(M, N)$. For morphisms, we have $J^A(g, h)(T_x^A f) = T_{g(x)}^A(h \circ f \circ g^{-1}) = J_{g(x)}^r(h \circ f \circ g^{-1}) = J^r(g, h)(j_x^r f)$. By g^{-1} we denote the inverse to g considered near x . \square

4. THE RIGIDITY OF UNIVERSALLY REPARAMETRIZABLE WEIL ALGEBRAS

Let us consider a Weil algebra invariant to all reparametrizations. A Weil algebra of this property will be said to be universally reparametrizable. The following proposition states the rigidity of such Weil algebras.

Proposition 4.1. *Let $A = \mathbb{D}_k^r/I$ be a Weil algebra considered in the polynomial form $\mathbb{R}[\tau_1, \dots, \tau_k]/I$. Further, let any local diffeomorphism $f: \mathbb{R}_0^k \rightarrow \mathbb{R}_0^k$ determine an A -admissible A -velocity. Then the existence of a nonzero $j_0^r \eta \in I$ implies $\langle \tau_1, \dots, \tau_k \rangle^r \subseteq I$. In particular, $A = \mathbb{D}_k^r$ whenever height $A = r$.*

Proof. First we observe that the second assertion is a direct consequence of the first one. Indeed, $0 \neq j_0^r \eta \in I$ implies $\langle \tau_1, \dots, \tau_k \rangle^r \subseteq I$. Nevertheless, this is a contradiction with height $A = r$.

As for the first assertion, we prove by induction with respect to l that the existence of a nonzero $j_0^r \eta \in I \cap \mathbb{D}_{(i_1, \dots, i_l)}$ implies $\langle \tau_{i_1}, \dots, \tau_{i_l} \rangle^r \subseteq I$ for any l -elementary subset $\{i_1, \dots, i_l\}$ of $\{1, \dots, k\}$. Clearly, the polynomial $j_0^r \eta$ may be assumed to be from $\langle \tau_{i_1}, \dots, \tau_{i_l} \rangle^r$. First we observe that for any permutation σ of $\{1, \dots, k\}$ and any polynomial p under discussion it holds that $p(\tau_{\sigma(1)}, \dots, \tau_{\sigma(k)}) \in I$ whenever $p(\tau_1, \dots, \tau_k) \in I$ independently of p being a polynomial of exactly k or less indeterminates. A symmetry of this kind follows immediately from the universal reparametrization property of A .

The first step corresponding to $l = 1$ is trivial. As for the induction step, suppose the assertion being satisfied for $l \leq k$. Further, assume $i_1 = 1, \dots, i_l = l$ and $j_0^r \eta$ in the form of a polynomial $p \in I$ in indeterminates $\tau_1, \dots, \tau_{l+1}$. The proof will be divided into two steps.

(a) Let p be a monomial. The cases of $r = 1$ and $k = 1$ are trivial. One observes easily that it suffices to discuss l only up to $K = \min\{k, r\}$. In the induction step we can suppose p without loss of generality in the form of $\tau_{l+1}^h \tau_l^{s-h} q_{r-s}(\tau_1, \dots, \tau_{l-1})$ for some $h \leq 1$ and $h < s$, where $s \in \{2, \dots, r - l + 1\}$ and $q_{r-s}(\tau_1, \dots, \tau_{l-1})$ is a monomial of order $r - s$ in the indicated indeterminates exactly. Consider $s + 1$ reparametrizations of the form

$$(4.1) \quad \tau_{l+1} \mapsto c^j \tau_{l+1} + \tau_l, \quad \tau_l \mapsto \tau_l, \quad \dots, \quad \tau_1 \mapsto \tau_1, \quad j = 0, 1, \dots, s.$$

By the universal reparametrization property we have the following system of $s + 1$ elements from I whose h -th element, $h = 0, \dots, s$, is of the form

$$(4.2) \quad (c^h \tau_{l+1} + \tau_l)^s q_{r-s}(\tau_1, \dots, \tau_{l-1}) = q_{r-s}(\tau_1, \dots, \tau_{l-1}) \sum_{j=0}^s \binom{s}{j} c^{hj} \tau_{l+1}^j \tau_l^{s-j}.$$

We show that for $j = 0, 1, \dots, s$ each monomial $\tau_{l+1}^j \tau_l^{s-j} q_{r-s}(\tau_1, \dots, \tau_{l-1})$ is an element of I provided p is. In other words, we show that each monomial $\tau_{l+1}^j \tau_l^{s-j}$ can be expressed in the form a linear combination of $(\tau_{l+1} + \tau_l)^s, \dots, (c^s \tau_{l+1} + \tau_l)^s$. Consider the matrix of the coefficients of the monomials $q_{r-s}(\tau_1, \dots, \tau_{l-1}) \cdot \tau_{l+1}^j \tau_l^{s-j}$

from the right-hand sides of the system (4.2). Its determinant equals to

$$(4.3) \quad \binom{s}{0} \binom{s}{1} \dots \binom{s}{s} \det M(s, c),$$

where the matrix $M(s, c) = (m(s, c))_{i,j}$ is defined by $(m(s, c))_{i,j} = c^{(i-1)(j-1)}$.

We particularly obtain that $\tau_l^s \cdot q_{r-s}(\tau_1, \dots, \tau_{l-1}) \in I$ and conclude that $p \in I$ implies that every monomial in the indeterminates τ_1, \dots, τ_l only is an element of I . The same holds for all monomials in arbitrary l indeterminates.

Now we are to deduce that if any monomial of order r in τ_1, \dots, τ_l is an element of I , then so is any monomial in $\tau_1, \dots, \tau_{l+1}$. However, one can deduce that applying the set of $s + 1$ reparametrizations

$$(4.4) \quad \tau_{l+1} \mapsto \tau_{l+1}, \tau_l \mapsto c^j \tau_{l+1} + \tau_l, \dots, \tau_1 \mapsto \tau_1, \quad j = 0, 1, \dots, s$$

and the deductions connected with (4.2) applied in the converse direction, i.e., from $j = 0$ to other cases of j .

(b) Without loss of generality, p may be assumed to be a sum of t monomials of order r , otherwise p could be multiplied by some monomial truncating the monomials of the maximal degree in the sum and consequently reducing its length. Iterating this step we would obtain either a sum of monomials of the same degree or one monomial, which would lead to the step (a). Consider p in the form of $q \cdot p_1 + p_2$ for a monomial q in $\tau_1, \dots, \tau_{l+1}$ exactly, a sum p_1 of t_1 monomials in the same indeterminates in which at least one summand does not contain τ_{l+1} and a residual sum of monomials in the indeterminates τ_1, \dots, τ_l only.

Let us denote by s the maximal degree of τ_{l+1} in p . For $c \neq 0$, take the reparametrization $\tau_{l+1} \mapsto c\tau_{l+1}, \tau_l \mapsto \tau_l, \dots, \tau_1 \mapsto \tau_1$ and the polynomial \bar{p} obtained from p by such reparametrization. Clearly, the polynomial $\bar{p} - c^s p \in I$ and there is a nonzero $c \in \mathbb{R}$ such that $\bar{p} - c^s p \neq j_0^r 0$. Clearly, its length is shorter than that of p . Iterating this step we achieve either a polynomial in τ_1, \dots, τ_l only or a monomial in the indeterminates $\tau_1, \dots, \tau_{l+1}$, both of them being elements of I . The first case coincides with the already proved step in the induction proof, while the second one leads to the case (a). \square

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Author’s address: Jiří Tomáš, Department of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Technická 2, Brno, Czech Republic, e-mail: Tomas@fme.vutbr.cz.