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DIOPHANTINE EQUATIONS INVOLVING FACTORIALS

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Abstract. We study the Diophantine equations $(k!)^n - k^n = (n!)^k - n^k$ and $(k!)^n + k^n = (n!)^k + n^k$, where k and n are positive integers. We show that the first one holds if and only if $k = n$ or $(k, n) = (1, 2), (2, 1)$ and that the second one holds if and only if $k = n$.

Keywords: Diophantine equation; factorial

MSC 2010: 11D61

1. Introduction

The theory of Diophantine equations with its long and rich history has attracted the attention of numerous mathematicians, who published many books and research papers on this beautiful part of number theory. An introduction to Diophantine equations with interesting historical information can be found, for example, in the monographs [1] and [2].

In this note, we are concerned with two closely related Diophantine equations involving factorials. Our work has been motivated by an interesting result published by Carnal [3] in 2012. He offered a short and elegant treatment of

$$k^n \cdot n! = n^k \cdot k!, \quad k, n \in \mathbb{N}. \tag{1.1}$$

He proved that the relation (1.1) holds if and only if $k = n$ or $(k, n) = (1, 2), (2, 1)$. Here, we study the following two additive counterparts of (1.1):

$$(k!)^n - k^n = (n!)^k - n^k \quad \text{and} \quad (k!)^n + k^n = (n!)^k + n^k.$$

We present all positive integer solutions (k, n) of these equations.

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2. Our results

Theorem 2.1. *Let n and k be positive integers. The equation*

$$(k!)^n - k^n = (n!)^k - n^k \tag{2.1}$$

holds if and only if

$$k = n \quad \text{or} \quad (k, n) = (1, 2), (2, 1).$$

Proof. Obviously, if $k = n$ or $(k, n) = (1, 2), (2, 1)$ then (2.1) is valid. Next, we show that if (2.1) holds with $k < n$, then we obtain $(k, n) = (1, 2)$. We distinguish three cases.

Case 1. $k = 1$. Then, (2.1) reduces to

$$0 = n! - n = n((n-1)! - 1).$$

Since $n > 1$, we find $n = 2$.

Case 2. $k = 2$. Equation (2.1) yields

$$0 = (n!)^2 - n^2 = n(n! + n)((n-1)! - 1),$$

which has no solution $n > 2$.

Case 3. $k \geqslant 3$. The sequences $\{(n!)^{1/n}\}_{n=1}^{\infty}$ and $\{-n^{1/n}\}_{n=3}^{\infty}$ are strictly increasing, so that we obtain for $n > k \geqslant 3$:

$$(n!)^k > (k!)^n \quad \text{and} \quad -n^k > -k^n.$$

Adding up the above inequalities gives

$$(n!)^k - n^k > (k!)^n - k^n.$$

The proof is complete. □

Theorem 2.2. *Let n and k be positive integers. The equation*

$$(k!)^n + k^n = (n!)^k + n^k \tag{2.2}$$

holds if and only if $n = k$.

P r o o f. We show that (2.2) implies $k = n$. By symmetry, we may assume that $n \geqslant k$. We consider again three cases.

Case 1. $k = 1$. Then, (2.2) reads $2 = n! + n$. This gives $n = 1$.

Case 2. $k = 2$. We have

$$2^{n+1} = n^2(((n-1)!)^2 + 1).$$

It follows that $n = 2^a$ for some $a \geqslant 1$. Then we obtain

$$2^{n+1-2a} = ((n-1)!)^2 + 1.$$

If $n \geqslant 3$, then $((n-1)!)^2$ is even so the right-hand side above is odd and greater than 1 whereas the left-hand side is either 1 or even. Therefore, $n = 2$.

Case 3. $k \geqslant 3$. Since

$$k^k \mid (k!)^n, \quad k^k \mid k^n \quad \text{and} \quad k^k \mid (n!)^k,$$

we conclude from (2.2) that $k^k \,|\, n^k$. This gives $k \,|\, n$. Let $n = bk$ for some $b \geqslant 1$. We assume that $b \geqslant 2$. Since $\{n^{1/n}\}_{n=3}^{\infty}$ is strictly decreasing, we find $k^n - n^k > 0$. Thus,

$$k^n - n^k = (n!)^k - (k!)^{bk} > 0, \quad \text{therefore} \quad n! - (k!)^b \geqslant 1.$$

It follows that

$$k^{bk} = k^n > k^n - n^k = (n! - (k!)^b) \sum_{j=0}^{k-1} (n!)^j \, (k!)^{b(k-j-1)} > (k!)^{b(k-1)}.$$

This gives

$$k^k > (k!)^{k-1}, \quad \text{so} \quad k^{k/(k-1)} > k!.$$

But, this is false, since

$$k^{k/(k-1)} \leqslant k^{3/2} = k\sqrt{k} < k(k-1) \leqslant k!.$$

Thus, $b = 1$ and $k = n$. □

R e m a r k. For more Diophantine equations involving factorials we refer to the research articles [4], [5], [6] and the references therein.

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