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A CURVATURE IDENTITY ON A 6-DIMENSIONAL RIEMANNIAN MANIFOLD AND ITS APPLICATIONS

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Abstract. We derive a curvature identity that holds on any 6-dimensional Riemannian manifold, from the Chern-Gauss-Bonnet theorem for a 6-dimensional closed Riemannian manifold. Moreover, some applications of the curvature identity are given. We also define a generalization of harmonic manifolds to study the Lichnerowicz conjecture for a harmonic manifold "a harmonic manifold is locally symmetric" and provide another proof of the Lichnerowicz conjecture refined by Ledger for the 4-dimensional case under a slightly more general setting.

Keywords: Chern-Gauss-Bonnet theorem; curvature identity; locally harmonic manifold

MSC 2010: 53B20, 53C25

1. INTRODUCTION

Berger in [2] derived a curvature identity on 4-dimensional compact Riemannian manifolds from the Chern-Gauss-Bonnet theorem based on the well-known fact that the Euler number is a topological invariant. We demonstrated that the obtained curvature identity holds on any 4-dimensional Riemannian manifold which is not necessarily compact, see [11]. Further, Gilkey, Park and Sekigawa extended the result to the higher dimensional setting, the pseudo-Riemannian setting, manifolds with boundary setting and the Kähler setting, see [13], [14], [15], [16]. In this paper, we give a curvature identity explicitly which holds on any 6-dimensional Riemannian manifold using methods similar to those used in the 4-dimensional Chern-Gauss-

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Bonnet theorem, and also provide some applications of the obtained curvature identity. More precisely, we derive a symmetric 2-tensor valued curvature identity of degree 6 which holds on any 5-dimensional Riemannian manifold, from which a scalar-valued curvature identity can be derived ([13], Lemma 1.2 (3)). Furthermore, we derive a symmetric 2-tensor valued curvature identity of degree 6 on 4-dimensional Riemannian manifolds from the curvature identity on 5-dimensional Riemannian manifolds. Based on these obtained identities, we shall also discuss a question that arose in [6] related to the Lichnerowicz conjecture for a harmonic manifold "a harmonic manifold is locally symmetric". The original Lichnerowicz conjecture is the one for 4-dimensional harmonic manifolds which was proved by Walker ([30] and Corollary 1.2 in [6]). The Lichnerowicz conjecture was refined by Ledger since he showed that a locally symmetric manifold is harmonic if and only if it is locally isometric to a Euclidean space or a rank one symmetric space, see [20]. Concerning the Lichnerowicz conjecture, Szabó in [28] proved that the conjecture is true on the compact harmonic manifolds. For the non-compact case, Damek and Ricci in [3], [9] provided a counterexample demonstrating that the Lichnerowicz conjecture is not true for the case of dimensions greater than or equel to 7. As mentioned above, the Lichnerowicz conjecture is true for the 4-dimensional case. Further, Nikolayevsky in [23] showed that the Lichnerowicz conjecture refined by Ledger is also true for the 5-dimensional case. Presently, to the best of our knowledge, the Lichnerowicz conjecture is still open for the 6-dimensional case. In the present paper, we provide another proof of the Lichnerowicz conjecture, the refined version by Ledger for the 4-dimensional case and a brief review of the proof of the Lichnerowicz conjecture for the 5-dimensional case by Nikolayevsky under slightly general settings. For more detailed information concerning the Lichnerowicz conjecture, refer to [22], [23], [24], [18].

2. Preliminaries

In this section, we shall prepare several fundamental concepts, terminologies and notational conventions. In the present paper, we shall adopt notational conventions similar to those used in [13]. We denote by $\mathcal{I}_{m,n}$ the space of scalar invariant local formulas and by $\mathcal{I}^2_{m,n}$ the space of symmetric 2-tensor valued invariant local formulas, defined in the category of all Riemannian manifolds of dimension m and of degree n. We note that $\mathcal{I}_{m,n} = \{0\}$ and $\mathcal{I}^2_{m,n} = \{0\}$ if n is odd. We denote by r the restriction map $r\colon \mathcal{I}_{m,n} \to \mathcal{I}_{m-1,n}$ (or $r\colon \mathcal{I}^2_{m,n} \to \mathcal{I}^2_{m-1,n}$) given by restricting the summation to range from 1 to $m-1$.

Now, let $M = (M, g)$ be an m-dimensional Riemannian manifold and ∇ the Levi-Civita connection of g. We assume that the curvature tensor R is defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z \tag{2.1}$$

for $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M. We also denote the Ricci tensor and the scalar curvature of M by ϱ and τ, respectively. Let $\{e_i\} = \{e_1, e_2, \ldots, e_m\}$ be a local orthonormal frame and $\{e^i\}$ a dual frame field. Throughout the present paper, we assume that the components of the tensor fields are relative to a local orthonormal frame $\{e_i\}$ and also adopt the Einstein convention on sum over repeated indices unless otherwise specified. Further, we denote by $R_{abcd;i}, R_{abcd;ij}, \ldots$ the components of the covariant derivatives of the curvature tensor $R = (R_{abcd})$ with respect to the Levi-Civita connection ∇. The following theorems play fundamental roles in our forthcoming discussion.

Theorem 2.1 ([13]).

(1) *$r\colon \mathcal{I}_{m,n} \to \mathcal{I}_{m-1,n}$ is surjective.*
(2) *If n is even and if $m > n$, then $r\colon \mathcal{I}_{m,n} \to \mathcal{I}_{m-1,n}$ is bijective.*
(3) *Let m be even. Then $\ker\{r\colon \mathcal{I}_{m,m} \to \mathcal{I}_{m-1,m}\} = E_{m,m} \cdot \mathbb{R}$, where $E_{m,n} \in \mathcal{I}_{m,n}$ is the Pfaffian form defined by*

$$E_{m,n} := \sum_{i_1,\ldots,i_n,j_1,\ldots,j_n=1}^{m} R_{i_1 i_2 j_2 j_1} \ldots R_{i_{n-1} i_n j_n j_{n-1}} \times g(e^{i_1} \wedge \ldots \wedge e^{i_n}, e^{j_1} \wedge \ldots \wedge e^{j_n}). \tag{2.2}$$

Theorem 2.2 ([13]).

(1) *$r\colon \mathcal{I}^2_{m,n} \to \mathcal{I}^2_{m-1,n}$ is surjective.*
(2) *If n is even and if $m > n+1$, then $r\colon \mathcal{I}^2_{m,n} \to \mathcal{I}^2_{m-1,n}$ is bijective.*
(3) *If m is even, then $\ker\{r\colon \mathcal{I}^2_{m+1,m} \to \mathcal{I}^2_{m,m}\} = T^2_{m,m} \cdot \mathbb{R}$, where $T^2_{m,n} \in \mathcal{I}^2_{m,n}$ is the Pfaffian defined by*

$$T^2_{m,n} := \sum_{i_1,\ldots,i_{n+1},j_1,\ldots,j_{n+1}=1}^{m} R_{i_1 i_2 j_2 j_1} \ldots R_{i_{n-1} i_n j_n j_{n-1}} e^{i_{n+1}} \circ e^{j_{n+1}} \times g(e^{i_1} \wedge \ldots \wedge e^{i_{n+1}}, e^{j_1} \wedge \ldots \wedge e^{j_{n+1}}). \tag{2.3}$$

3. The universal curvature identity

Let $M = (M, g)$ be a 6-dimensional compact oriented Riemannian manifold. Then it is well-known that the Euler number of M is given by the following integral formula, namely, the Chern-Gauss-Bonnet theorem.

Theorem 3.1 ([25]).

$$\begin{aligned}(3.1)\quad \chi(M) &= \frac{1}{2^9\pi^3 3!}\int_M E_{6,6}\,\mathrm{d}v_g\\ &= \frac{1}{384\pi^3}\int_M\{\tau^3-12\tau|\varrho|^2+3\tau|R|^2+16\varrho_{ab}\varrho_{ac}\varrho_{bc}-24\varrho_{ab}\varrho_{cd}R_{acbd}\\ &\quad -24\varrho_{uv}R_{uabc}R_{vabc}+8R_{abcd}R_{aucv}R_{bvdu}-2R_{abcd}R_{abuv}R_{cduv}\}\,\mathrm{d}v_g.\end{aligned}$$

We here set

$$\hat{R}\equiv R_{abcd}R_{abuv}R_{cduv}$$

and

$$\mathring{R}\equiv R_{abcd}R_{aucv}R_{budv}.$$

We note that identity (3.1) is rearranged by our setting: the curvature of [25] has a negative sign difference to ours and the term $8R_{abcd}R_{aucv}R_{bvdu}$ has been changed by using the first Bianchi identity to $8\mathring{R}-2\hat{R}$.

Now, we regard the right hand side of (3.1) as a functional $\mathcal{F}$ on the space $\mathfrak{M}(M)$ of all Riemannian metrics on M. Let h be any symmetric $(0,2)$-tensor field in M and consider a one-parameter deformation of g by $g(t)=g+th$ for any $g\in\mathfrak{M}(M)$. Since the Euler number $\chi(M)$ is a topological invariant of M, $\mathcal{F}$ does not depend on the choice of Riemannian metrics on M, so we have

$$0=\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{F}(g(t))=0. \tag{3.2}$$

This holds for any symmetric $(0,2)$-tensor field h on M. Applying arguments similar to in [11], taking account of (3.1) and (3.2), we have the following equality as the corresponding Euler-Lagrange equation for the functional $\mathcal{F}$:

$$\begin{aligned}(3.3)\quad &\tfrac{1}{2}(\tau^3+3\tau|R|^2-12\tau|\varrho|^2+16\varrho_{ab}\varrho_{bc}\varrho_{ca}-24\varrho_{ab}\varrho_{cd}R_{acbd}\\ &-24\varrho_{uv}R_{abcu}R_{abcv}+8\mathring{R}-4\hat{R})g_{ij}-3\tau^2\varrho_{ij}-3|R|^2\varrho_{ij}\\ &+12|\varrho|^2\varrho_{ij}+12\tau\varrho_{ia}\varrho_{ja}+12\tau\varrho_{ab}R_{iabj}-6\tau R_{iabc}R_{jabc}\\ &-24\varrho_{ia}\varrho_{jb}\varrho_{ab}-24\varrho_{ac}\varrho_{bc}R_{iabj}+24\varrho_{aj}\varrho_{cd}R_{acid}\\ &+24\varrho_{ai}\varrho_{cd}R_{acjd}+24\varrho_{ab}R_{icdj}R_{acbd}+48\varrho_{cd}R_{iabc}R_{jabd}\\ &+6\varrho_{jd}R_{abci}R_{abcd}+6\varrho_{id}R_{abcj}R_{abcd}+12\check{R}_{ij}+12\hat{R}_{ij}-24\mathring{R}_{ij}=0,\end{aligned}$$

where $\check{R}_{ij}=R_{iuvj}R_{abcu}R_{abcv}$, $\hat{R}_{ij}=R_{ibac}R_{jbuv}R_{acuv}$ and $\mathring{R}_{ij}=R_{iabc}R_{jubv}R_{aucv}$. We here omit the detailed calculation. From (3.3) and Theorem 3.1, taking account of the results of ([10], Theorem 1.2) and ([11], Main theorem), we have

Theorem 3.2. *The curvature identity* (3.3) *holds on any* 6*-dimensional Riemannian manifold $M=(M,g)$ which is not necessarily compact and, further, it is universal in $\mathcal{I}^2_{6,6}$.*

Especially, we have

Corollary 3.3. *Let $M = (M, g)$ be a 6-dimensional Einstein manifold. Then the following identity holds on M:*

$$(3.4)\qquad (-\tau|R|^2 + 4\mathring{R} - 2\hat{R})g_{ij} + 12\check{R}_{ij} + 12\hat{R}_{ij} - 24\mathring{R}_{ij} + 4\tau R_{iabc}R_{jabc} = 0.$$

We note that the curvature identity (3.3) can also be obtained by making use of the equality $T^2_{6,6} = 0$ from Theorem 2.2 (3). However, we derived the same identity (3.3) without adopting this method in this paper. Further, we note that the curvature identity is universal in the same form for any 6-dimensional pseudo-Riemannian manifold, see [14].

4. Derived curvature identities on 4- and 5-dimensional Riemannian manifolds

In this section, we shall provide further curvature identities on 4- and 5-dimensional Riemannian manifolds derived from the curvature identity (3.3) on 6-dimensional Riemannian manifolds.

Now, let $M = (M, g)$ be a 5-dimensional Riemannian manifold and $\overline{M} = (M \times \mathbb{R}, g \oplus 1)$ the Riemannian product of $M = (M, g)$ and a real line $\mathbb{R}$. Then, applying Theorem 3.1 to the Riemannian manifold $\overline{M} = (M \times \mathbb{R}, g \oplus 1)$, we see that the curvature identity

$$(4.1)\qquad \begin{aligned} &\tau^3 - 12\tau|\varrho|^2 + 3\tau|R|^2 + 16\varrho_{ab}\varrho_{bc}\varrho_{ca} \\ &\quad - 24\varrho_{ab}\varrho_{cd}R_{acbd} - 24\varrho_{uv}R_{abcu}R_{abcv} + 8\mathring{R} - 4\hat{R} = 0 \end{aligned}$$

holds on M and further, it is universal in $\mathcal{I}_{5,6}$ ([13], Lemma 1.2 (3)).

Now, taking account of Theorem 2.2 (1), we see that (3.3) holds on M in the same form by restricting the range of the indices from 1 to 5. Therefore, from (3.3) and (4.1), we have

Theorem 4.1. *Let $M = (M, g)$ be a 5-dimensional Riemannian manifold. Then, in addition to (4.1), the identity*

$$(4.2)\qquad \begin{aligned} &\tau^2\varrho_{ij} + |R|^2\varrho_{ij} - 4|\varrho|^2\varrho_{ij} - 4\tau\varrho_{ia}\varrho_{ja} - 4\tau\varrho_{ab}R_{iabj} + 2\tau R_{iabc}R_{jabc} \\ &\quad + 8\varrho_{ia}\varrho_{jb}\varrho_{ab} + 8\varrho_{ac}\varrho_{bc}R_{iabj} - 8\varrho_{aj}\varrho_{cd}R_{acid} - 8\varrho_{ai}\varrho_{cd}R_{acjd} \\ &\quad - 8\varrho_{ab}R_{icdj}R_{acbd} - 16\varrho_{cd}R_{iabc}R_{jabd} - 2\varrho_{jd}R_{abci}R_{abcd} \\ &\quad - 2\varrho_{id}R_{abcj}R_{abcd} - 4\check{R}_{ij} - 4\hat{R}_{ij} + 8\mathring{R}_{ij} = 0 \end{aligned}$$

holds on M.

Remark 1. Transvecting (4.2) with g_{ij}, we may also obtain (4.1).

From Theorem 4.1, we have

Corollary 4.2. *Let $M = (M, g)$ be a 5-dimensional Einstein manifold. Then we have*

$$\Big(\frac{\tau^3}{25} + \frac{\tau}{5}|R|^2\Big)g_{ij} - 2\tau R_{iabc}R_{jabc} - 4\check{R}_{ij} - 4\hat{R}_{ij} + 8\mathring{R}_{ij} = 0. \tag{4.3}$$

From (4.2), taking account of Theorem 2.2 (1) and Equation (1.2) in [12], we have

Corollary 4.3. *Let $M = (M, g)$ be a 4-dimensional Riemannian manifold. Then the identity (4.2) holds in the same form by restricting the range of the indices from 1 to 4 and further, it is universal in $\mathcal{I}^2_{4,6}$. Especially, if M is Einstein, the identity reduces to the identity*

$$\Big(\frac{\tau^3}{8} - \frac{3}{4}\tau|R|^2\Big)g_{ij} - 4\hat{R}_{ij} + 8\mathring{R}_{ij} = 0.$$

Here, a 6-dimensional, 5-dimensional and 4-dimensional Riemannian manifold satisfying the curvature identities in Corollaries 3.3, 4.2 and 4.3 will be called a 6-dimensional, 5-dimensional and 4-dimensional weakly Einstein manifold of degree 6, respectively. Based on our current work, the definition of a 4-dimensional weakly Einstein manifold introduced in our papers [11], [12] may be made more precise and the definition becomes that of a 4-dimensional weakly Einstein manifold of degree 4. We note that Arias-Marco and Kowalski recently obtained a classification theorem for 4-dimensional homogeneous weakly Einstein manifolds, see [1].

5. A generalization of harmonic manifolds

An m-dimensional Riemannian manifold $M = (M, g)$ is called a *locally harmonic* manifold (briefly, harmonic manifold) if for every point $p \in M$, the volume density function $\theta_p(q) = \sqrt{\det(g_{ij})(q)}$ is a radial function in a normal neighborhood $U_p = U_p(x^1, \dots, x^m)$ centered at p, where $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$, namely, there exists a positive real number $\varepsilon(p)$ and a smooth function $\Theta_p\colon [0, \varepsilon(p)) \to M$ such that $\theta_p(q) = \Theta_p(d(p, q))$ for $q \in U_p$ where $d(p, q)$ is the distance from p to q. We note that there are several equivalent definitions for harmonic manifolds, see [4]. A locally Euclidean space and a locally rank one symmetric space are harmonic manifolds. Concerning the converse, there is a well-known conjecture known as the Lichnerowicz conjecture that every harmonic manifold is locally isometric to a Euclidean space or a rank one symmetric space. Copson and Ruse in [8], Lichnerowicz in [21] and Ledger in [19] have shown that each harmonic manifold must satisfy an infinite sequence $\{H_n\}_{n=1,2,\ldots}$ of conditions on the curvature tensor and its covariant derivatives. The first three of these conditions are given as follows, see [4], [33]:

$$
\begin{aligned}
&H_1\colon\ R_{aija} = \Lambda_1 g_{ij},\\
&H_2\colon\ \mathfrak{S}(R_{aijb}R_{bkla}) = \Lambda_2 \mathfrak{S}(g_{ij}g_{kl}),\\
&H_3\colon\ \mathfrak{S}(32R_{aijb}R_{bklc}R_{cuva} - 9R_{aijb;k}R_{buva;l}) = \Lambda_3 \mathfrak{S}(g_{ij}g_{kl}g_{uv}),
\end{aligned}
$$

where each Λ_n, $n = 1, 2, 3$, is a constant and $\mathfrak{S}$ denotes the summation taken over all permutations of the free indices appearing inside the parentheses. From the condition H_1, it follows immediately that a harmonic manifold is Einstein and hence real analytic as a Riemannian manifold. We may note that the conditions H_1, H_2, H_3 are equivalent to the following conditions H_1', H_2', H_3', respectively, see [6]:

$$
\begin{aligned}
&H_1'\colon\ R_{axxa} = \Lambda_1 |x|^2,\\
&H_2'\colon\ R_{axxb}R_{bxxa} = \Lambda_2 |x|^4,\\
&H_3'\colon\ 32R_{axxb}R_{bxxc}R_{cxxa} - 9R_{axxb;x}R_{bxxa;x} = \Lambda_3 |x|^6,
\end{aligned}
$$

for any $x = \xi^i e_i \in T_pM$ at $p \in M$, where $R_{axxb} = R_{aijb}\xi^i\xi^j$ and $R_{axxb;c} = R_{aijb;k}\xi^i\xi^j\xi^k$.

Remark 1. The condition H_3 in [6] is incorrect ([4], page 162) and should be changed to the above H_3'.

In [6], Carpenter, Gray and Willmore raised the following question:

Question A. Does there exist a Riemannian manifold $M = (M, g)$ which satisfies some but not all of the conditions $\{H_n\}_{n=1,2,\ldots}$?

Concerning Question A, they discussed the case where $M = (M, g)$ is a non-flat locally symmetric space satisfying the condition H_1 and some other condition H_k and obtained some partial answers to the question ([6], Theorem 1.1). Taking account of these observations, it seems worthwhile to consider Question A under a more general setting.

Now, we shall define a generalization of harmonic manifolds.

Definition 5.1. A Riemannian manifold $M = (M, g)$ satisfying the conditions $\{H_n\}_{n=1,\ldots,k}$ is called an *asymptotic harmonic manifold up to order k*.

By the above definition, it follows immediately that an asymptotic harmonic manifold up to order k is an asymptotic harmonic manifold up to order for any l, $1 \leqslant l < k$.

Further, we may check that a locally symmetric asymptotic harmonic manifold up to order k is k-stein, see [6], [19], [20].

Let $M = (M, g)$ be an m-dimensional asymptotic harmonic manifold up to order 2. Then we have

$$\varrho_{ij} = \Lambda_1 g_{ij} \quad \text{and hence,} \quad \Lambda_1 = \frac{\tau}{m}, \tag{5.1}$$

and

$$\begin{aligned} &R_{aijb}R_{aklb} + R_{aijb}R_{alkb} + R_{aikb}R_{ajlb} \\ &+R_{aikb}R_{aljb} + R_{ailb}R_{akjb} + R_{ailb}R_{ajkb} = 2\Lambda_2(g_{ij}g_{kl} + g_{ik}g_{jl} + g_{jk}g_{il}). \end{aligned} \tag{5.2}$$

Transvecting (5.2) with g_{kl} and taking account of (5.1), we have

$$2\Big(\frac{\tau}{m}\Big)^2 g_{ij} + 3R_{iabc}R_{jabc} = 2(m+2)\Lambda_2 g_{ij}. \tag{5.3}$$

From (5.3), we have

$$\Lambda_2 = \frac{1}{2m(m+2)}\Big(\frac{2\tau^2}{m} + 3|R|^2\Big). \tag{5.4}$$

Thus, from (5.4), it follows immediately that $|R|^2$ is constant on M. Further, from (5.3) and (5.4), we have

$$R_{iabc}R_{jabc} = \frac{1}{3}\Big\{\frac{1}{m}\Big(\frac{2\tau^2}{m} + 3|R|^2\Big) - \frac{2\tau^2}{m^2}\Big\}g_{ij} = \frac{1}{m}|R|^2 g_{ij}, \tag{5.5}$$

and hence, M is a super-Einstein manifold with constant $|R|^2$ (see [5], [17]).

Remark 2. By definition, an m ($\geqslant 3$) -dimensional asymptotic harmonic manifold up to order 2 is a 2-stein manifold with constant $|R|^2$, see [6]. It is known that for each 2-stein manifold of dimension m ($\neq 4$), $|R|^2$ is constant. An explicit example of a 4-dimensional 2-stein manifold with non-constant $|R|^2$ has been provided in [7]. It is also known that every 2-stein manifold is super-Einstein. We may reconfirm this fact by the above equality (5.5).

From (5.2), taking account of (5.4), we may show

Proposition 5.1. *Let $M = (M, g)$ be an m ($\geqslant 3$) -dimensional non-flat asymptotic harmonic manifold up to order 2. Then, M is irreducible.*

The following identity always holds. We shall use it to derive the Lichnerowicz formula:

$$
\begin{aligned}
(5.6)\quad (R_{iabc}R_{jabc})_{;kk} &= 2B_{ij} + 8\mathring{R}_{ij} + 2\hat{R}_{ij} + 4\varrho_{cd}R_{iabc}R_{jabd} + 2\varrho_{ic;ab}R_{jabc} \\
&\quad + 2\varrho_{jc;ab}R_{iabc} + 2\varrho_{ab;ic}R_{jabc} + 2\varrho_{ab;jc}R_{iabc}.
\end{aligned}
$$

Especially, if the Riemannian manifold $M = (M, g)$ is Einstein with constant $|R|^2$, from (5.6), we have easily

$$
(5.7)\qquad |\nabla R|^2 = -4\mathring{R} - \hat{R} - \frac{2\tau}{m}|R|^2.
$$

In the sequel, we assume that every Riemannian manifold $M = (M, g)$ is an m ($\geqslant 4$) -dimensional asymptotic harmonic manifold up to order 3 unless otherwise specified. Then, from the condition H_3, taking account of (5.5), we have

$$
\begin{aligned}
(5.8)\quad & g_{kl}g_{uv}\mathfrak{S}(R_{aijb;k}R_{buva;l}) = 48(A_{ij} + 2B_{ij}), \\
(5.9)\quad & g_{kl}g_{uv}\mathfrak{S}(R_{aijb}R_{bklc}R_{cuva}) = 48\Big(\frac{\tau^3}{m^3}g_{ij} + \frac{3}{2}\check{R}_{ij} - \frac{7}{2}\hat{R} + \mathring{R}_{ij} + \frac{3}{m}\tau R_{iabc}R_{jabc}\Big) \\
& \qquad = 48\Big\{-\frac{7}{2}\hat{R}_{ij} + \mathring{R}_{ij} + \Big(\frac{\tau^3}{m^3} + \frac{9\tau}{2m^2}|R|^2\Big)g_{ij}\Big\}, \\
(5.10)\quad & g_{kl}g_{uv}\mathfrak{S}(g_{ij}g_{kl}g_{uv}) = 48(m+2)(m+4)g_{ij},
\end{aligned}
$$

where $A_{ij} = R_{abcd;i}R_{abcd;j}$ and $B_{ij} = R_{ibcd;a}R_{jbcd;a}$. Thus, from H_3 and (5.8)–(5.10), we have

$$
\begin{aligned}
(5.11)\quad 32\Big\{-\frac{7}{2}\hat{R}_{ij} + \mathring{R}_{ij} + \Big(\frac{\tau^3}{m^3} + \frac{9\tau}{2m^2}|R|^2\Big)g_{ij}\Big\} - 9(A_{ij} + 2B_{ij}) \\
= (m+2)(m+4)\Lambda_3 g_{ij}.
\end{aligned}
$$

Multiplying (5.11) by m, we have the equation

$$
\begin{aligned}
(5.12)\quad 32\Big\{-\frac{7m}{2}\hat{R}_{ij} + m\mathring{R}_{ij} + \Big(\frac{\tau^3}{m^2} + \frac{9\tau}{2m}|R|^2\Big)g_{ij}\Big\} - 9m(A_{ij} + 2B_{ij}) \\
= m(m+2)(m+4)\Lambda_3 g_{ij}.
\end{aligned}
$$

Transvecting (5.11) with g_{ij}, we further have

$$
(5.13)\quad 32\Big(-\frac{7}{2}\hat{R} + \mathring{R} + \frac{\tau^3}{m^2} + \frac{9\tau}{2m}|R|^2\Big) - 27|\nabla R|^2 = m(m+2)(m+4)\Lambda_3.
$$

Thus, from (5.12) and (5.13), we have

$$
\begin{aligned}
9m(A_{ij} + 2B_{ij}) - 27|\nabla R|^2 g_{ij} &= 32\Big\{-\frac{7m}{2}\hat{R}_{ij} + m\mathring{R}_{ij} + \Big(\frac{\tau^3}{m^2} + \frac{9\tau}{2m}|R|^2\Big)g_{ij}\Big\} \\
&\quad + 27|\nabla R|^2 g_{ij} - 32\Big(-\frac{7}{2}\hat{R} + \mathring{R} + \frac{\tau^3}{m^2} + \frac{9\tau}{2m}|R|^2\Big)g_{ij},
\end{aligned}
$$

and hence,

$$9m(A_{ij}+2B_{ij})-27|\nabla R|^2 g_{ij}=32(m\mathring{R}_{ij}-\mathring{R}g_{ij})-112(m\hat{R}_{ij}-\hat{R}g_{ij}). \tag{5.14}$$

From (5.6), taking account of (5.5), we have

$$B_{ij}=-4\mathring{R}_{ij}-\hat{R}_{ij}-\frac{2\tau}{m^2}|R|^2 g_{ij}. \tag{5.15}$$

Thus, from (5.1), (5.4), (5.7) and (5.13), we have

Proposition 5.2. *Let $M=(M,g)$ be an m-dimensional asymptotic harmonic manifold up to order 3. Then M is a 2-stein manifold with constant $|R|^2$, and further, $|\nabla R|^2+\hat{R}+4\mathring{R}$, $27|\nabla R|^2+112\hat{R}-32\mathring{R}$ are constant and hence, $17\hat{R}-28\mathring{R}$ is constant on M.*

Remark 3. Proposition 6.68 in [4] should be corrected as above.

Here, we set

$$\begin{aligned}
\alpha_{ij}&=A_{ij}-\frac{1}{m}|\nabla R|^2 g_{ij},\\
\beta_{ij}&=B_{ij}-\frac{1}{m}|\nabla R|^2 g_{ij},\\
\hat{\gamma}_{ij}&=\hat{R}_{ij}-\frac{1}{m}\hat{R}g_{ij},\\
\mathring{\gamma}_{ij}&=\mathring{R}_{ij}-\frac{1}{m}\mathring{R}g_{ij}.
\end{aligned} \tag{5.16}$$

Then, from (5.14) and (5.16), we have

$$9(\alpha_{ij}+2\beta_{ij})=32\mathring{\gamma}_{ij}-112\hat{\gamma}_{ij}, \tag{5.17}$$

and hence, from (5.15)–(5.16), we have

$$\begin{aligned}
B_{ij}=&-4\mathring{R}_{ij}-\hat{R}_{ij}-\frac{2\tau}{m^2}|R|^2 g_{ij}\\
=&-4\mathring{\gamma}_{ij}-\hat{\gamma}_{ij}-\frac{1}{m}\Big(4\mathring{R}+\hat{R}+\frac{2\tau}{m}|R|^2\Big)g_{ij}\\
=&-4\mathring{\gamma}_{ij}-\hat{\gamma}_{ij}+\frac{1}{m}|\nabla R|^2 g_{ij}.
\end{aligned}$$

Thus we obtain

$$\beta_{ij}=-4\mathring{\gamma}_{ij}-\hat{\gamma}_{ij}. \tag{5.18}$$

Hence, we have

Proposition 5.3. *Let $M = (M, g)$ be an m-dimensional asymptotic harmonic manifold up to order 3. Then the following equalities hold:*

$$9\alpha_{ij} = 104\mathring{\gamma}_{ij} - 94\hat{\gamma}_{ij}, \quad \beta_{ij} = -4\mathring{\gamma}_{ij} - \hat{\gamma}_{ij}.$$

5.1. 4-dimensional asymptotic harmonic manifolds. Let $M = (M, g)$ be a 4-dimensional asymptotic harmonic manifold up to order 3. Then, since M is a 2-stein manifold (with constant $|R|^2$) for each point $p \in M$, we may choose a Singer-Thorpe basis $\{e_i\} = \{e_1, e_2, e_3, e_4\}$ such that

$$\begin{gathered} R_{1212} = R_{3434} = a, \quad R_{1313} = R_{2424} = b, \quad R_{1414} = R_{2323} = c, \\ R_{1234} = \alpha, \quad R_{1342} = \beta, \quad R_{1423} = \gamma \end{gathered} \tag{5.19}$$

satisfying $\alpha + \beta + \gamma = 0$ and $\alpha = a + \tau/12$, $\beta = b + \tau/12$, $\gamma = c + \tau/12$ (or $-\alpha = a + \tau/12$, $-\beta = b + \tau/12$, $-\gamma = c + \tau/12$), see [27].

Without loss of generality, it suffices to consider the case,

$$\alpha = a + \frac{\tau}{12}, \quad \beta = b + \frac{\tau}{12}, \quad \gamma = c + \frac{\tau}{12}. \tag{5.20}$$

Then, by straightforward calculation, we obtain

$$\begin{gathered} \tau = -4(a + b + c), \quad |R|^2 = \frac{5}{6}\tau^2 - 32(ab + bc + ca), \\ \hat{R}_{ij} = \frac{1}{4}\hat{R}g_{ij} \quad (\hat{\gamma}_{ij} = 0), \quad \hat{R} = 192abc + 32\tau(ab + bc + ca) - \frac{7}{12}\tau^3, \\ \mathring{R}_{ij} = \frac{1}{4}\mathring{R}g_{ij} \quad (\mathring{\gamma}_{ij} = 0), \quad \mathring{R} = 96abc + 4\tau(ab + bc + ca) - \frac{\tau^3}{24}. \end{gathered} \tag{5.21}$$

Further, from (5.1), (5.4), (5.13), and (5.21), we have

$$\Lambda_1 = \frac{\tau}{4}, \tag{5.22}$$

$$\Lambda_2 = \frac{1}{48}\Big(\frac{1}{2}\tau^2 + 3|R|^2\Big), \tag{5.23}$$

$$192\Lambda_3 = -27|\nabla R|^2 + 32\Big(-\frac{7}{2}\hat{R} + \mathring{R} + \frac{\tau^3}{16} + \frac{9}{8}\tau|R|^2\Big). \tag{5.24}$$

From (5.22) and (5.23), taking account of Corollary 4.3, we see that $\hat{R} - 2\mathring{R}$ is constant. Thus, from Proposition 5.2, it follows that both $\hat{R}$ and $\mathring{R}$ are constant, and hence $|\nabla R|^2$ is also constant. Thus, a, b, and c are the real roots of the equation

$$t^3 + \frac{\tau}{4}t^2 + \frac{1}{32}\Big(\frac{5}{6}\tau^2 - |R|^2\Big)t - 192\Big(\hat{R} - \tau|R|^2 - \frac{1}{4}\tau^3\Big) = 0 \tag{5.25}$$

at each point $p \in M$ and hence, a, b and c can be expressed in terms of constant-valued functions τ, $|R|^2$, $\hat{R}$ and $\mathring{R}$ at each point of M, respectively. Therefore, M is a 4-dimensional 2-stein curvature harmonic manifold, and hence M is a locally symmetric manifold by virtue of ([27], page 281). Further, taking account of the result [26] and Proposition 5.1, we may show

Theorem 5.4. *A 4-dimensional asymptotic harmonic manifold up to order 3 is locally flat or locally isometric to a rank one symmetric space.*

Thus, from Theorem 5.4, the refinement of Walker's result follows immediately [30].

5.2. 5-dimensional asymptotic harmonic manifolds. First, let $M = (M, g)$ be a 5-dimensional asymptotic harmonic manifold up to order 2. Then M is a 2-stein manifold with constant $|R|^2$. From Corollary 4.2 and from (5.5) with $m = 5$, we see that M satisfies the equality

$$2\mathring{R}_{ij} - \hat{R}_{ij} = \frac{\tau}{100}(9|R|^2 - \tau^2)g_{ij}. \tag{5.26}$$

Hence, transvecting (5.26) with g_{ij}, we have

$$2\mathring{R} - \hat{R} = \frac{\tau}{20}(9|R|^2 - \tau^2). \tag{5.27}$$

Next, let $M = (M, g)$ be a 5-dimensional asymptotic harmonic manifold up to order 3. Then, from Proposition 5.2 and (5.27), we see that $\hat{R}$, $\mathring{R}$ and $|\nabla R|^2$ are constant on M ([30], Proposition 3.1). From (5.13) with $m = 5$, in addition to the equalities (5.26) and (5.27), we have the equality

$$315\Lambda_3 = -27|\nabla R|^2 - 112\hat{R} + 32\mathring{R} + \frac{32}{25}\tau^3 + \frac{144\tau}{5}|R|^2. \tag{5.28}$$

Thus, from (5.27) and (5.28), we have

$$27|\nabla R|^2 + 96\hat{R} - \frac{12}{25}\tau^3 - 36\tau|R|^2 = -315\Lambda_3. \tag{5.29}$$

Now, we recall the following result of Nikolayevsky ([23], Proposition 1).

Proposition 5.5. *A 5-dimensional 2-stein manifold $M = (M, g)$ is either of constant sectional curvature or locally homothetic to the symmetric space* SU(3)/SO(3) *or to its noncompact dual* SL(3)/SO(3).

In this section, we give a brief review on Proposition 5.5 under a slightly more general setting from the view point of Question A. We note that the following result ([23], Proposition 4) plays an essential role in the proof of Proposition 5.5.

Proposition 5.6. *Let $M = (M, g)$ be a 5-dimensional 2-stein manifold. Then, at each point $p \in M$, there exists an orthonormal basis $\{e_i\}$ such that*

$$\begin{gathered} R_{1212} = R_{1313} = R_{2323} = R_{2424} = R_{3434} = \mu - \nu, \quad R_{1414} = \mu - 4\nu, \\ R_{1515} = R_{4545} = \mu, \quad R_{2525} = R_{3535} = \mu - 3\nu, \\ R_{1234} = \nu, \quad R_{1235} = \sqrt{3}\,\nu, \quad R_{1324} = -\nu, \quad R_{1325} = \sqrt{3}\,\nu, \\ R_{1423} = -2\nu, \quad R_{2425} = \sqrt{3}\,\nu, \quad R_{3435} = -\sqrt{3}\,\nu \end{gathered}$$

and all the other components of R vanish.

From Proposition 5.6, by direct calculation, we have

$$\tau = -20\mu + 30\nu, \tag{5.30}$$

$$R_{iabc}R_{jabc} = (8\mu^2 - 24\mu\nu + 60\nu^2)\delta_{ij}, \tag{5.31}$$

and hence,

$$|R|^2 = 40\mu^2 - 120\mu\nu + 300\nu^2. \tag{5.32}$$

Further we obtain

$$\check{R}_{ij} = (-32\mu^3 + 144\mu^2\nu - 384\mu\nu^2 + 360\nu^3)\delta_{ij}, \tag{5.33}$$

$$\hat{R}_{ij} = (16\mu^3 - 72\mu^2\nu + 360\mu\nu^2 - 600\nu^3)\delta_{ij}, \tag{5.34}$$

and hence, $\hat{R} = 80\mu^3 - 360\mu^2\nu + 1800\mu\nu^2 - 3000\nu^3$,

$$\mathring{R}_{ij} = (12\mu^3 - 54\mu^2\nu + 18\mu\nu^2 - 30\nu^3)\delta_{ij} \tag{5.35}$$

and hence, $\mathring{R} = 60\mu^3 - 270\mu^2\nu + 90\mu\nu^2 - 150\nu^3$. Thus, from (5.30) and (5.32) we see that μ and ν are represented in terms of the constant valued functions τ and $|R|^2$ at each point $p \in M$, and hence, μ and ν are constant on M. Therefore, M is curvature homogeneous. From (5.7) with $m = 5$, taking account of (5.30)–(5.35), we have

$$|\nabla R|^2 = 1680\mu\nu^2. \tag{5.36}$$

Thus, from (5.36), it follows that M is locally symmetric if and only if $\mu = 0$ or $\nu = 0$. Here, if $\nu = 0$, then, from Proposition 5.6, it follows that M is a space of constant sectional curvature $-\mu$. Now, we assume that $\nu \neq 0$. Then, by applying the second Bianchi identity to the curvature form obtained by making use of Proposition 5.6, we may check that M is locally symmetric (and hence, $\mu = 0$), and further that M is locally homothetic to the symmetric space SU(3)/SO(3) or to its noncompact dual SL(3)/SO(3) ([23], pages 32–34). Thus, we have Proposition 5.5.

We now show that any 5-dimensional Riemannian manifold $M = (M, g)$ which is locally homothetic to the symmetric space SU(3)/SO(3) (or SL(3)/SO(3)) with a fixed canonical Riemannian metric is never an asymptotic harmonic manifold up to order 3. In order to do this, without loss of generality, it suffices to establish it in the case where the Riemannian manifold M is locally homothetic to the symmetric space SL(3)/SO(3) equipped with the metric given by ([23], page 34). Now, we assume that M is an asymptotic harmonic manifold up to order 3. Then, we may easily check that $\nu < 0$ for M. Since $\nabla R = 0$ and $\mu = 0$ hold on M, from (5.29), taking account of (5.30)–(5.35), we have

$$\Lambda_3 = 1984\nu^3. \tag{5.37}$$

On the other hand, choosing an orthonormal basis $\{e_i\} = \{e_1 = x, e_2, e_3, e_4, e_5\}$ of the tangent space T_pM at any point $p \in M$ satisfying the condition in Proposition 5.6 and calculating the equality in the condition H_3' by making use of the orthonormal basis $\{e_i\}$, we have also

$$\Lambda_3 = 2012\nu^3. \tag{5.38}$$

Thus, from (5.37) and (5.38), it follows that $\nu = 0$. But this is a contradiction. Summing up the above arguments, we have finally

Theorem 5.7. *Let $M = (M, g)$ be a 5-dimensional asymptotic harmonic manifold up to order 3. Then M is a space of constant sectional curvature.*

From Theorem 5.7, we have immediately ([23], Theorem 1)

Corollary 5.8. *A 5-dimensional harmonic manifold is a space of constant sectional curvature.*

Corollary 5.8 gives an affirmative answer to the Lichnerowicz conjecture (refined version by Ledger) for the 5-dimensional case.

Remark 4. The result that the symmetric space SU(3)/SO(3) (or SL(3)/SO(3)) is not asymptotic harmonic manifold up to order 3 can be also obtained by taking account of the fact that SU(3)/SO(3) (or SU(3)/SL(3)) is not 3-stein ([6], page 58). We here give another explicit proof for the same result by making use of the curvature identities on 5-dimensional Riemannian manifolds derived from the universal curvature identity on 6-dimensional Riemannian manifolds.

6. Concluding remarks

Based on the discussions in the previous sections, while grappling with the Lichnerowicz conjecture for the 6-dimensional case, it seems effective to find an orthonormal basis at each point of a 6-dimensional 2-stein manifold such as the Singer-Thorpe basis for the 4-dimensional case and the Nikolayevsky basis for the 5-dimensional case. As an approach to the Lichnerowicz conjecture for the 6-dimensional case, it also seems worthwhile to provide the universal curvature identity on the 8-dimensional Riemannian manifold through a method similar to the 4- and 5-dimensional cases and further the curvature identities on the 6- and 7-dimensional Riemannian manifolds derived from the universal curvature identities obtained.

Lastly, we shall explain a reason why we introduced the notion of asymptotic harmonic manifolds. As mentioned at the beginning of Section 5, there are several equivalent definitions for harmonic manifolds. One of them is the one expressed in terms of the characteristic function $f = f(\Omega)$, where $\Omega = 1/2s^2$, $s = d(p,q)$ for $q \in U_p$ ($U_p = U_p(x^1, x^2, \dots, x^m)$ denoting a sufficiently small normal coordinate neighborhood centered at each point $p \in M$). The characteristic function plays an important role in the geometry of harmonic manifolds. We refer to [4], [21], [29] for more details on the characteristic functions. Due to these observations concerning Question A, it is natural to discuss the relationships between the constants $\{H_n\}_{n=1,2,\dots}$ and $\{f^{(n)}(0)\}_{n=1,2,\dots}$. Here, we denote by "$\,'\,$" the derivative with respect to the variable Ω. Now, let $M = (M,g)$ be an m-dimensional harmonic manifold with the characteristic function $f = f(\Omega)$. Then it is known that between the constants $\{\Lambda_1, \Lambda_2, \Lambda_3\}$ and $\{f'(0), f''(0), f'''(0)\}$, the following relations hold [21], [29]:

$$\Lambda_1 = -\frac{3}{2}f'(0), \quad \Lambda_2 = -\frac{45}{8}f''(0), \quad \Lambda_3 = -315f'''(0). \tag{6.1}$$

Lichnerowicz in [21] has proved the following.

Theorem 6.1. *In any m-dimensional harmonic manifold $M = (M,g)$, the characteristic function $f = f(\Omega)$ satisfies the inequality*

$$f'(0)^2 + \frac{5}{2}(m-1)f''(0) \leqslant 0. \tag{6.2}$$

The equality sign is valid if and only if M is of constant sectional curvature.

From (5.1) and (5.4), taking account of (6.1) and (6.2), we can see that the above Theorem 6.1 is generalized as follows:

Theorem 6.2. *Let $M = (M,g)$ be an m-dimensional asymptotic harmonic manifold up to order 2. Then M satisfies the inequality*

$$\Lambda_1^2 - (m-1)\Lambda_2 \leqslant 0. \tag{6.3}$$

The equality sign is valid if and only if M is of constant sectional curvature $\tau/(m(m-1))$.

Tachibana in [29] has proved the following.

Theorem 6.3. *Any $2n$-dimensional harmonic Kähler manifold $M = (M, J, g)$ satisfies the inequality*

$$f'(0)^2 + \frac{5(n+1)^2}{n+7} f''(0) \leqslant 0 \tag{6.4}$$

and the equality sign is valid if and only if M is of constant holomorphic sectional curvature.

From (5.1) and (5.4), taking account of (6.1) and (6.2), we can see that the above Theorem 6.3 is generalized as follows:

Theorem 6.4. *Let $M = (M, J, g)$ be a $2n$-dimensional asymptotic harmonic Kähler manifold up to order 2. Then M satisfies the inequality*

$$\Lambda_1^2 - \frac{2(n+1)^2}{n+7} \Lambda_2 \leqslant 0 \tag{6.5}$$

and the equality sign is valid if and only if M is of constant holomorphic sectional curvature $\tau/(n(n+1))$.

Similarly, from (5.1), (5.4) and (5.13), taking account of (6.1), we can see that the corresponding generalizations for the results ([31], Theorem 5.2, and [32], Theorem 5.5) are obtained.

Taking account of the discussions in the present paper and in [6] concerning Question A we obtain that if the dimension is 4 then the least integer of the series is not greater than 3 and if the dimension is 5 then the least integer of the series is 3. Based on the arguments developed the following question naturally arises:

Question B. For any integer m ($m \geqslant 6$), does there exist the least integer $K(m)$ such that any m-dimensional asymptotic harmonic manifold up to order k ($k \geqslant K(m)$) is necessarily harmonic?

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