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# BOUNDEDNESS OF PARA-PRODUCT OPERATORS ON SPACES OF HOMOGENEOUS TYPE

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*Abstract.* We obtain the boundedness of Calderón-Zygmund singular integral operators $T$ of non-convolution type on Hardy spaces $H^p(\mathcal{X})$ for $1/(1+\varepsilon) < p \leqslant 1$, where $\mathcal{X}$ is a space of homogeneous type in the sense of Coifman and Weiss (1971), and $\varepsilon$ is the regularity exponent of the kernel of the singular integral operator $T$. Our approach relies on the discrete Littlewood-Paley-Stein theory and discrete Calderón's identity. The crucial feature of our proof is to avoid atomic decomposition and molecular theory in contrast to what was used in the literature.

*Keywords*: boundedness; Calderón-Zygmund singular integral operator; para-product; spaces of homogeneous type

*MSC 2010*: 42B25, 42B30

## 1. Introduction and statements of results

In the 1970's, in order to extend the theory of Calderón-Zygmund singular integrals on $\mathbb{R}^n$ to a more general setting, R. Coifman and G. Weiss introduced spaces of homogeneous type which are equipped with a quasi-metric defined as follows.

For a set $\mathcal{X}$, we say that a function $\varrho\colon \mathcal{X}\times\mathcal{X} \to [0,\infty)$ is a quasi-metric on $\mathcal{X}$ if it satisfies that

(i) $\varrho(x,y) = 0$ if and only if $x = y$;
(ii) $\varrho(x,y) = \varrho(y,x)$ for all $x, y \in \mathcal{X}$;
(iii) there exists a constant $A \in [1,\infty)$ such that for all $x, y$ and $z \in \mathcal{X}$,

$$\varrho(x,y) \leqslant A[\varrho(x,z) + \varrho(z,y)].$$

Any quasi-metric $\varrho$ defines a topology, for which the balls $B(x,r) = \{y \in \mathcal{X}\colon \varrho(x,y) < r\}$ for all $x \in \mathcal{X}$ and all $r > 0$ form a basis.

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The following spaces of homogeneous type are variants of those introduced by Coifman and Weiss in [2].

**Definition 1.** Let $\theta \in (0,1]$. A space of homogeneous type, $(\mathcal{X}, \varrho, \mu)_\theta$, is a set $\mathcal{X}$ together with a quasi-metric $\varrho$ and a nonnegative measure $\mu$ on $\mathcal{X}$, and there exists a constant $C_0 > 0$ such that for all $0 < r < \operatorname{diam} \mathcal{X}$ and all $x, y, z \in \mathcal{X}$,

$$\mu(B(x,r)) \sim r \quad \text{and} \quad |\varrho(x,y) - \varrho(z,y)| \leqslant C_0 \varrho(x,z)^\theta [\varrho(x,y) + \varrho(z,y)]^{1-\theta}.$$

In the following, let $(\mathcal{X}, \varrho, \mu)_\theta$ be a space of homogeneous type as in Definition 1. The Hölder spaces on $\mathcal{X}$ are defined as follows.

**Definition 2.** Let $C_0^\eta(\mathcal{X})$, $\eta > 0$, be the space of all continuous functions on $\mathcal{X}$ with compact support and

$$\|f\|_{C^\eta} = \sup_{x,y \in \mathcal{X};\ x \neq y} \frac{|f(x) - f(y)|}{\varrho(x,y)^\eta} < \infty.$$

**Remark 1.** For $\eta \in (0, \theta]$, $C_0^\eta(\mathcal{X})$ is not empty. To see this, we can consider the function $g(x) = f(\varrho(x, x_0))$ with any fixed $x_0 \in \mathcal{X}$, where $f$ is a $C^1$ function defined on $\mathbb{R}$ with a compact support. It is easy to check that $g \in C_0^\eta(\mathcal{X})$ with $0 < \eta \leqslant \theta \leqslant 1$.

**Remark 2.** The dual space of $C^\beta(\mathbb{R})$ is not a functional space for $0 < \beta \leqslant 1$. However, it suffices to replace $C^\beta(\mathbb{R})$ by the closure $\mathring{C}^\beta(\mathbb{R})$ for the $C^\beta(\mathbb{R})$ norm of functions in $C^\gamma(\mathbb{R})$ where $\gamma > \beta$, and this closure does not depend on $\gamma$. Following this argument we define the function space $\mathring{C}_0^\eta(\mathcal{X})$ as the closure for the $C_0^\eta(\mathcal{X})$ norm of functions in $C_0^s(\mathcal{X})$ where $s > \eta$, and let $(\mathring{C}_0^\eta(\mathcal{X}))'$ be the dual space of $\mathring{C}_0^\eta(\mathcal{X})$. Here these two spaces do not depend on $s$. For more detail, see [11].

We now introduce the Calderón-Zygmund operator on $\mathcal{X}$. For convenience, in the following, we use $C$ to denote all constants only dependent on $\mathcal{X}$, which may vary from line to line.

**Definition 3** ([2]). A continuous function $K\colon \mathcal{X} \times \mathcal{X} \setminus \{(x,y)\colon x = y\} \to \mathbb{C}$ is said to be a Calderón-Zygmund singular integral kernel on $\mathcal{X}$ if there exist $\varepsilon \in (0, \theta]$ and constants $C > 0$ such that

$$|K(x,y)| \leqslant C \varrho(x,y)^{-1} \text{ for all } x \neq y;$$
$$|K(x,y) - K(x',y)| \leqslant C \varrho(x,x')^\varepsilon \varrho(x,y)^{-(1+\varepsilon)} \text{ for } \varrho(x,x') \leqslant \frac{1}{2A} \varrho(x,y);$$
$$|K(x,y) - K(x,y')| \leqslant C \varrho(y,y')^\varepsilon \varrho(x,y)^{-(1+\varepsilon)} \text{ for } \varrho(y,y') \leqslant \frac{1}{2A} \varrho(x,y).$$

The smallest such constant $C$ is denoted by $\|K\|_{CZ}$. And $\varepsilon$ is said to be the regularity exponent of the kernel $K$.

**Definition 4** ([2]). A continuous linear operator $T\colon \mathring{C}_0^\eta(\mathcal{X}) \to (\mathring{C}_0^\eta(\mathcal{X}))'$ for all $\eta \in (0,\theta]$ is said to be a Calderón-Zygmund singular integral operator on $\mathcal{X}$, if $T$ is associated with a Calderón-Zygmund kernel $K$ so that

$$\langle Tf, g\rangle = \iint K(x,y)f(y)g(x)\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)$$

for all $f$ and $g \in \mathring{C}_0^\eta(\mathcal{X})$ with disjoint supports.

**Remark 3** ([2]). Any Calderón-Zygmund singular integral operator which is bounded on $L^2(\mathcal{X})$ is also bounded on $L^p(\mathcal{X})$ for $1 < p < 1$; and is of weak type $(1,1)$.

We call an operator $T$ a Calderón-Zygmund operator if $T$ is a Calderón-Zygmund singular integral operator and is bounded on $L^2$.

From Remark 3 a question arises: Under what conditions a Calderón-Zygmund singular integral operator is bounded on $L^2$? This question was answered by the well-known T1 theorems of G. David and J. L. Journé, and G. David, J. L. Journé and S. Semmes in the standard case of $\mathbb{R}^n$ and in spaces of homogeneous type, respectively.

To introduce the generalization of the $T1$ theorem to spaces of homogeneous type, we first need to define $T(1)$: The difficulty is that 1 is not a function in $\mathring{C}_0^\eta(\mathcal{X})$, hence $T(1)$ is not a distribution in $(\mathring{C}_0^\eta(\mathcal{X}))'$, but is a distribution modulo constant function. The definition is based on the following lemma (see [12]).

**Lemma 1.** *Let $S$ be a distribution in $(\mathring{C}_0^\eta(\mathcal{X}))'$. Suppose that there exists $R > 0$ such that the restriction of $S$ to the open set $\{x \in \mathcal{X}\colon\ \varrho(x,x_0) > R\}$, where $x_0$ is a fixed point in $\mathcal{X}$, is a continuous function such that $S(x) = O(\varrho(x,x_0))^{-1-\gamma}$ as $\varrho(x,x_0) \to \infty$. If $\gamma > 0$, then the integral*

$$\int_{\mathcal{X}} S(x)\,\mathrm{d}\mu(x) = \langle S, 1\rangle$$

*converges.*

We first write $1 = \varphi_1(x) + \varphi_2(x)$, where $\varphi_1 \in \mathring{C}_0^\eta(\mathcal{X})$ for some $\eta > 0$ and $\varphi_1(x) = 1$ for $\varrho(x,x_0) \leqslant R$. Then $\langle S, 1\rangle$ is defined by

$$\langle S,\varphi_1\rangle + \langle S,\varphi_2\rangle = \langle S,\varphi_1\rangle + \int_{\mathcal{X}} S(x)\varphi_2(x)\,\mathrm{d}\mu(x)$$

since the integral converges absolutely. It is easy to check that $\langle S, 1\rangle$ is independent of the decomposition.

Before defining $T1$, we define

$$\mathring{C}^{\eta}_{0,0}(\mathcal{X}) = \left\{ f \in \mathring{C}^{\eta}_{0}(\mathcal{X})\colon \int_{\mathcal{X}} f(x)\,\mathrm{d}\mu(x) = 0 \right\}.$$

If $f \in \mathring{C}^{\eta}_{0,0}(\mathcal{X})$, we define $\langle T1, f\rangle = \langle 1, T^{*}f\rangle$. Indeed, if the support of $f$ is contained in $\{x \in \mathcal{X}\colon \varrho(x, x_0) \leqslant R\}$, then

$$T^{*}(f)(x) = \int_{\mathcal{X}} [K(y, x) - K(x_0, x)] f(y)\,\mathrm{d}\mu(y) = O(\varrho(x, x_0)^{-1-\varepsilon})$$

for $\varrho(x, x_0) > R$ and $\varepsilon > 0$.

Now $T1$ is a continuous linear form on $\mathring{C}^{\eta}_{0,0}(\mathcal{X}) \subset \mathring{C}^{\eta}_{0}(\mathcal{X})$. We extend $T1$ to a distribution $S \in (\mathring{C}^{\eta}_{0}(\mathcal{X}))'$ as follows: let $\varphi \in \mathring{C}^{\eta}_{0}(\mathcal{X})$ be a function with $\int_{\mathcal{X}} \varphi(x)\,\mathrm{d}\mu(x) = 1$, then for all $f \in \mathring{C}^{\eta}_{0}(\mathcal{X})$, $f$ can be written uniquely as $f = \lambda\varphi + g$, where $\lambda = \int f(x)\,\mathrm{d}\mu(x)$ and $g \in \mathring{C}^{\eta}_{0,0}(\mathcal{X})$. Now we choose $S$ such that $\langle S, f\rangle = \lambda\langle S, \varphi\rangle + \langle T1, g\rangle$, then $T1 = S$ on $\mathring{C}^{\eta}_{0,0}(\mathcal{X})$, and is a distribution modulo the constant. $T^{*}1$ can be defined in a similar way.

For $\delta \in (0, \theta]$, $x_0 \in \mathcal{X}$ and $r > 0$, we define $A(\delta, x_0, r)$ to be the set of all $\varphi \in \mathring{C}^{\delta}_{0}(\mathcal{X})$ supported in $B(x_0, r)$ satisfying $\|\varphi\|_{\infty} < 1$ and $\|\varphi\|_{C^{\delta}} < r^{-\delta}$. To introduce $T1$ theorem on $\mathcal{X}$, we also need the following definition of weak boundedness.

**Definition 5.** An operator $T$ is weakly bounded if there exist $\delta \in (0, \theta]$ and $C < \infty$ such that for all $x_0 \in \mathcal{X}$, $r > 0$ and $\varphi$, $\psi \in A(\delta, x_0, r)$,

$$|\langle T\varphi, \psi\rangle| \leqslant C\mu(B(x_0, r)).$$

**Remark 4.** It is easy to see that weak boundedness is obviously implied by $L^2$ boundedness. And Calderón-Zygmund singular integral operator whose is antisymmetrical kernel, i.e., $K(x, y) = -K(y, x)$, has the weak boundedness property.

In 1985, using Coifman's idea on decomposition of the identity operator, G. David, J. L. Journé and S. Semmes developed the Littlewood-Paley analysis on spaces of homogeneous type and used it to give a proof of the following $T1$ theorem in this general setting.

**Theorem A** ([4]). *Let $T$ be a Calderón-Zygmund singular integral operator on $\mathcal{X}$. Then a necessary and sufficient condition for the extension of $T$ as a continuous linear operator on $L^2(\mathcal{X})$ is that the following conditions are all satisfied:* (a) $T1 \in \mathrm{BMO}$; (b) $T^{*}1 \in \mathrm{BMO}$; (c) *$T$ is weakly bounded. Here*

$$\mathrm{BMO}(\mathcal{X}) = \left\{ f \in L^1_{\mathrm{loc}}(\mathcal{X})\colon \sup_{r>0,\, x\in\mathcal{X}} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(y) - f_B|\,\mathrm{d}\mu(y) < \infty \right\},$$

*where $f_B = \mu(B(x, r))^{-1} \int_{B(x,r)} f(y)\,\mathrm{d}\mu(y)$.*

Deng and Han gave a new $T1$ theorem for the general spaces of homogeneous type as follows.

**Theorem B** ([5]). *Let $T$ be a Calderón-Zygmund singular integral operator on $\mathcal{X}$ with $T1 = T^*1 = 0$, and $T$ is weakly bounded. Then $T$ is bounded on $L^p$ for $1 < p < \infty$ and $H^p$ for $1/(1+\varepsilon) < p \leqslant 1$, where $\varepsilon$ is the regularity exponent of the kernel of the singular integral operator $T$.*

In the above theorem, the conditions $T1 = 0$ and $T^*1 = 0$ are sufficient conditions. A natural problem is when these conditions are also necessary. The following theorem answers this problem.

**Theorem 1** ([5]). *Let $T$ be a Calderón-Zygmund operator on $\mathcal{X}$, then $T$ is bounded on $H^p(\mathcal{X})$ for all $1/(1+\varepsilon) < p \leqslant 1$ if and only if $T^*1 = 0$.*

We remark here that the main tool used in the literature to prove Theorem 1 is the molecular theory of the Hardy space $H^p(\mathcal{X})$, see [3], [5].

In this paper, we will use a different approach to prove Theorem 1 without using atomic decomposition or molecular theory of $H^p(\mathcal{X})$. Moreover, we can get

**Theorem 2.** *If $T$ is a Calderón-Zygumnd operator on $\mathcal{X}$, then $T$ is bounded from $H^p(\mathcal{X})$ to $L^p(\mathcal{X})$ for all $1/(1+\varepsilon) < p \leqslant 1$.*

The main ideas are using almost estimates, the discrete Littlewood-Paley-Stein theory and discrete Calderón's identity together with the maximal and Littlewood-Paley characterizations of the Hardy spaces $H^p(\mathcal{X})$ to get the boundedness of the para-product which will be defined later (see Definition 8).

Our new approach includes the following steps.

*Step 1.* The discrete Calderón's identity, almost orthogonality estimates and the $H^p$ boundedness.

To recall the classical continous Calderón's identity, we begin with introducing the approximation to identity on the space of homogeneous type.

**Definition 6** ([10]). A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of linear operators is said to be an approximation to the identity of order $\varepsilon \in (0,\theta]$ on $\mathcal{X}$ if there exists $C > 0$ such that for all $k \in \mathbb{Z}$ and all $x$, $x'$, $y$ and $y' \in \mathcal{X}$, $S_k(x,y)$, the kernel of $S_k$, is a function from $\mathcal{X} \times \mathcal{X}$ into $\mathbb{C}$ satisfying

(1) $|S_k(x,y)| \leqslant C\dfrac{2^{-k\varepsilon}}{(2^{-k}+\varrho(x,y))^{1+\varepsilon}}$;

(2) $|S_k(x,y) - S_k(x',y)| \leqslant C\Big(\dfrac{\varrho(x,x')}{2^{-k}+\varrho(x,y)}\Big)^{\varepsilon}\dfrac{2^{-k\varepsilon}}{(2^{-k}+\varrho(x,y))^{1+\varepsilon}}$
for $\varrho(x,x') \leqslant (2A)^{-1}(2^{-k}+\varrho(x,y))$;

(3) $|S_k(x,y)-S_k(x,y')|\leqslant C\Big(\frac{\varrho(y,y')}{2^{-k}+\varrho(x,y)}\Big)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+\varrho(x,y))^{1+\varepsilon}}$
for $\varrho(y,y')\leqslant(2A)^{-1}(2^{-k}+\varrho(x,y))$;

(4) $|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|$
$\leqslant C\Big(\frac{\varrho(x,x')}{2^{-k}+\varrho(x,y)}\Big)^{\varepsilon}\Big(\frac{\varrho(y,y')}{2^{-k}+\varrho(x,y)}\Big)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+\varrho(x,y))^{1+\varepsilon}}$
for $\varrho(x,x')\leqslant(2A)^{-1}(2^{-k}+\varrho(x,y))$ and $\varrho(y,y')\leqslant(2A)^{-1}(2^{-k}+\varrho(x,y))$;

(5) $\int_{\mathcal{X}}S_k(x,y)\,\mathrm{d}\mu(y)=1$;

(6) $\int_{\mathcal{X}}S_k(x,y)\,\mathrm{d}\mu(x)=1$.

Next let us recall the definition of the space of test functions on spaces of homogeneous type.

**Definition 7** ([8]). Fix $0<\gamma,\ \beta<\theta$. A function $f$ defined on $\mathcal{X}$ is said to be a test function of type $(x_0,r,\beta,\gamma)$ with $x_0\in\mathcal{X}$ and $r>0$, if $f$ satisfies the following conditions:

(i) $|f(x)|\leqslant C\frac{r^{\gamma}}{(r+\varrho(x,x_0))^{1+\gamma}}$;

(ii) $|f(x)-f(y)|\leqslant C\Big(\frac{\varrho(x,y)}{r+\varrho(x,x_0)}\Big)^{\beta}\frac{r^{\gamma}}{(r+\varrho(x,x_0))^{1+\gamma}}$
for $\varrho(x,y)\leqslant(2A)^{-1}[r+\varrho(x,x_0)]$;

(iii) $\int_{\mathcal{X}}f(x)\,\mathrm{d}\mu(x)=0$.

If $f$ is a test function of type $(x_0,r,\beta,\gamma)$, we write $f\in\mathcal{G}(x_0,r,\beta,\gamma)$, and the norm of $f$ in $\mathcal{G}(x_0,r,\beta,\gamma)$ is defined by

$$\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)}=\inf\{C\colon\ \text{(i) and (ii) hold}\}.$$

Now fix $x_0\in\mathcal{X}$ and let $\mathcal{G}(\beta,\gamma)=\mathcal{G}(x_0,1,\beta,\gamma)$. It is easy to see that

$$\mathcal{G}(x_1,r,\beta,\gamma)=\mathcal{G}(\beta,\gamma)$$

with an equivalent norm for all $x_1\in\mathcal{X}$ and $r>0$. Furthermore, it is easy to check that $\mathcal{G}(\beta,\gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta,\gamma)$. Also, let the dual space $(\mathcal{G}(\beta,\gamma))'$ consist of all linear functionals $\mathcal{L}$ from $\mathcal{G}(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists $C\geqslant 0$ such that for all $f\in\mathcal{G}(\beta,\gamma)$,

$$|\mathcal{L}(f)|\leqslant C\|f\|_{\mathcal{G}(\beta,\gamma)}.$$

We denote by $\langle h,f\rangle$ the natural pairing of elements $h\in(\mathcal{G}(\beta,\gamma))'$ and $f\in\mathcal{G}(\beta,\gamma)$. Clearly, for all $h\in(\mathcal{G}(\beta,\gamma))'$, $\langle h,f\rangle$ is well defined for all $f\in\mathcal{G}(x_0,r,\beta,\gamma)$ with $x_0\in\mathcal{X}$ and $r>0$.

It is well-known that even when $\mathcal{X} = \mathbb{R}^n$, $\mathcal{G}(\beta_1, \gamma)$ is not dense in $\mathcal{G}(\beta_2, \gamma)$ if $\beta_1 > \beta_2$, which will cause us some inconvenience. To overcome this defect, in what follows, for a given $\varepsilon \in (0, \theta]$, we let $\mathring{\mathcal{G}}(\beta, \gamma)$ be the completion of the space $\mathcal{G}(\varepsilon, \varepsilon)$ in $\mathcal{G}(\beta, \gamma)$ when $0 < \beta,\ \gamma < \varepsilon$.

We also need the following construction given by Christ in [1], which provides an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type. A similar construction was independently given by Sawyer and Wheeden in [14].

**Lemma 2.** *For every integer $k \in \mathbb{Z}_+$, there exists a collection of open subsets $\{Q_\tau^k \subset \mathcal{X} \colon \tau \in I_k\}$, where $I_k$ denotes some index set depending on $k$, and $c_1, c_2 > 0$, are such that*

(i) $\mu(\{X \setminus \bigcup Q_\tau^k\}) = 0$;

(ii) *if $l \geqslant k$, then for all $\tau' \in I_l$ and $\tau \in I_k$ either $Q_{\tau'}^l \subset Q_\tau^k$ or $Q_{\tau'}^l \cap Q_\tau^k = \emptyset$;*

(iii) *if $l < k$, for each $\tau \in I_k$, there is a unique $\tau' \in I_l$ such that $Q_\tau^k \subset Q_{\tau'}^l$, $\operatorname{diam}(Q_\tau^k) \leqslant c_1 2^{-k}$, and each $Q_\tau^k$ contains some ball $B(z_\tau^k, c_2 2^{-k})$.*

In the following, we say that a cube $Q \subset \mathcal{X}$ is a dyadic cube in $\mathcal{X}$ if $Q = Q_\tau^k$ for some $k \in \mathbb{Z}_+$ and $\tau \in I_k$, and denote it by $\operatorname{diam} Q \sim 2^{-k}$. Denote by $Q_\tau^{k,\nu}$, $\nu = 1, 2, \ldots, N(k, \tau)$, the set of all cubes $Q_{\tau'}^{k+j} \subset Q_\tau^k$ where $j$ is a fixed large positive integer, and denote by $y_\tau^{k,\nu}$ a point in $Q_\tau^{k,\nu}$.

We now recall the discrete Calderón reproducing formulae on spaces of homogeneous type in [9].

**Lemma 3.** *Let $\varepsilon \in (0, \theta]$ for $k \in \mathbb{Z}$, let $\{S_k\}_{k \in \mathbb{Z}}$ be an approximation to the identity of order $\varepsilon$, $D_k = S_k - S_{k-1}$, let $\{Q_\tau^{k,\nu} \colon \tau \in I_k,\ \nu = 1, \ldots, N(k, \tau)\}$ be the dyadic cubes of $\mathcal{X}$ defined in Lemma 2 with $j \in \mathbb{N}$ large enough. Then there are two families of linear operators $\{\widetilde{D}_k\}_{k \in \mathbb{Z}}$, $\{\overline{D}_k\}_{k \in \mathbb{Z}}$ on $\mathcal{X}$ such that for all $f \in \mathcal{G}(\beta, \gamma)$ with $\beta, \gamma \in (0, \varepsilon)$ and any point any $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$,*

$$
\begin{aligned}
f(x) &= \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \widetilde{D}_k(x, y_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu}) \\
&= \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(x, y_\tau^{k,\nu}) \overline{D}_k(f)(y_\tau^{k,\nu}),
\end{aligned}
\tag{1}
$$

*where the series converge in the norm of both the space $\mathcal{G}(\beta', \gamma')$ with $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$ and the space $L^p(X)$ with $p \in (1, \infty)$.*

By an argument of duality, Han in [9] also established the following discrete Calderón reproducing formulae on spaces of distributions, $(\mathring{\mathcal{G}}(\beta, \gamma))'$ with $\beta, \gamma \in (0, \varepsilon)$.

**Lemma 4.** *With all the notation as in Lemma* 3*, for all* $f \in (\mathring{\mathcal{G}}(\beta,\gamma))'$ *with* $\beta,\gamma \in (0,\varepsilon)$, (1) *holds in* $(\mathring{\mathcal{G}}(\beta',\gamma'))'$ *with* $\beta < \beta' < \varepsilon$ *and* $\gamma < \gamma' < \varepsilon$.

Applying the above lemma, it was proved in [5] that $H^p(\mathcal{X})$ can be characterized by discrete Littlewood-Paley square functions

**Proposition 1.** *Let* $\theta' \in (0,\theta)$*, let* $D_k$ *and* $Q_\tau^{k,\nu}$ *be the same as in Lemma* 3*. Then for* $1/(1+\theta') < p \leqslant 1$, $f \in H^p(\mathcal{X})$ *if and only if* $f \in (\mathring{\mathcal{G}}(\beta,\gamma))'$ *with* $\beta,\gamma \in (0,\theta')$ *and*

$$\|f\|_{H^p} \sim \left\| \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |D_k(f)|^2 \chi_{Q_\tau^{k,\nu}}(\cdot) \right\}^{1/2} \right\|_p < \infty.$$

**Remark 5.** $H^p(\mathcal{X})$ also can be characterized by classical continuous Littlewood-Paley square functions, i.e.,

$$\|f\|_{H^p(\mathcal{X})} \sim \left\| \left\{ \sum_{k=-\infty}^{\infty} |D_k(f)(\cdot)|^2 \right\}^{1/2} \right\|_p.$$

These two kinds of definition of $H^p(\mathcal{X})$ are both independent of the choice of the approximation to identity, see [5] for the proof.

Proposition 1 and the almost orthogonality estimates provide a direct proof of the following $H^p(\mathcal{X})$ boundedness.

**Theorem 3.** *If* $T$ *is a Calderón-Zygmund operator with regularity exponent* $\varepsilon > 0$ *and* $T1 = T^*1 = 0$*, then* $T$ *is bounded on* $H^p(\mathcal{X})$ *for* $1/(1+\varepsilon) < p \leqslant 1$.

The proof of this theorem is elementary. The basic idea is to apply the orthogonality estimates stated as follows.

**Lemma 5.** *Let* $D_k$ *be the same as in Lemma* 3*. If* $T$ *satisfies the conditions in Theorem* 3*, then*

$$|D_k T(D_l)(x,y)| \leqslant C 2^{-|k-l|\varepsilon'} \frac{2^{-(k\wedge l)\varepsilon'}}{(2^{-(k\wedge l)} + \varrho(x,y))^{1+\varepsilon'}},$$

*where* $\varepsilon' \in (0,\varepsilon)$*, and the constant depends only on* $\varepsilon'$ *and* $D_k$.

**Remark 6.** We remark that the conditions $T1 = T^*1 = 0$ are crucial in deriving Lemma 5. The classical orthogonality estimates are

$$|D_k T(D_l)(x,y)| \leqslant C 2^{-|k-l|L} \frac{2^{-(k\wedge l)M}}{(2^{-(k\wedge l)} + \varrho(x,y))^{1+M}},$$

for any $L, M$ and the constant $C$ depends only on $L, M$ and $D_k$. See [4], [10], [5] for details of its proof.

We also need the following lemma, which can be found in [7], pages 147–148, for $\mathbb{R}^n$ and [5], page 93, for spaces of homogeneous type.

**Lemma 6.** *Let $k$, $\eta \in \mathbb{Z}_+$ with $\eta \leqslant k$. If for any dyadic cube $Q_\tau^{k,\nu} \subset \mathcal{X}$,*

$$|f_{Q_\tau^{k,\nu}}(x)| \leqslant (1 + 2^\eta \varrho(x, y_\tau^{k,\nu}))^{-1-\varepsilon},$$

*where $x \in \mathcal{X}$, $y_\tau^{k,\nu}$ is any point in $Q_\tau^{k,\nu}$ and $\varepsilon > 0$, then*

$$\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_{Q_\tau^{k,\nu}}||f_{Q_\tau^{k,\nu}}(x)| \leqslant C 2^{(k-\eta)} \left[ M\left( \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_{Q_\tau^{k,\nu}}| \chi_{Q_\tau^{k,\nu}} \right)^r (x) \right]^{1/r},$$

*where $r > 1/(1+\varepsilon)$, $C$ is independent of $x$, $k$ and $\eta$, $\lambda_{Q_\tau^{k,\nu}}$ is any constant only depending on $Q_\tau^{k,\nu}$. Here and in the sequel, $M$ is the Hardy-Littlewood maximal operator on $\mathcal{X}$, which is defined by*

$$M(f)(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| \, \mathrm{d}\mu(y).$$

We now return to the proof of Theorem 3. By Proposition 1, we only need to show that for $1/(1+\varepsilon) < p \leqslant 1$, $f \in L^2(\mathcal{X}) \cap H^p(\mathcal{X})$, we have

$$\left\| \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |D_k(Tf)|^2 \chi_{Q_\tau^{k,\nu}}(\cdot) \right\}^{1/2} \right\|_p < \|f\|_{H^p(\mathcal{X})}.$$

Note that $T$ is bounded on $L^2(\mathcal{X})$. Therefore, by Lemma 4, we can rewrite $D_k(Tf)$ as

$$\begin{aligned} D_k(Tf) &= D_k\left( T \sum_{k'=-\infty}^{\infty} \sum_{\tau \in I_{k'}} \sum_{\nu=1}^{N(k',\tau)} \mu(Q_\tau^{k',\nu}) D_{k'}(\cdot, y_\tau^{k',\nu}) \overline{D}_{k'}(f)(y_\tau^{k',\nu}) \right) \\ &= \sum_{k'=-\infty}^{\infty} \sum_{\tau \in I_{k'}} \sum_{\nu=1}^{N(k',\tau)} \mu(Q_\tau^{k',\nu}) D_k T D_{k'}(\cdot, y_\tau^{k',\nu}) \overline{D}_{k'}(f)(y_\tau^{k',\nu}). \end{aligned}$$

Using the orthogonality estimates yields

$$|D_k(Tf)| \leqslant C \sum_{k'=-\infty}^{\infty} \sum_{\tau \in I_{k'}} \sum_{\nu=1}^{N(k',\tau)} 2^{-|k-k'|^{\varepsilon'}} \frac{2^{-(k \wedge k')\varepsilon'} \mu(Q_\tau^{k',\nu})}{(2^{-(k\wedge k')} + \varrho(\cdot, y))^{1+\varepsilon'}} \overline{D}_{k'}(f)(y_\tau^{k',\nu}),$$

where $\varepsilon' \in (0, \varepsilon)$.

Then by applying Lemma 6, we have

$$\sum_{k=-\infty}^{\infty} |D_k(Tf)|^2$$
$$\leqslant C \sum_{k=-\infty}^{\infty} \Big[ \sum_{k'=-\infty}^{\infty} 2^{-|k-k'|\varepsilon'} \Big\{ M\Big( \sum_{\tau\in I_{k'}} \sum_{\nu=1}^{N(k',\tau)} \overline{D}_{k'}(f)(y_\tau^{k',\nu}) \chi_{Q_\tau^{k',\nu}} \Big)^r \Big\}^{1/r} \Big]^2.$$

Finally, by the Fefferman-Stein vector valued maximal function inequality in [6] on $L^2(\mathcal{X})$, we obtain

$$\Big\| \Big\{ \sum_{k=-\infty}^{\infty} |D_k(Tf)|^2 \Big\}^{1/2} \Big\|_p$$
$$\leqslant C \Big\| \Big\{ \sum_{k=-\infty}^{\infty} \Big[ \sum_{k'=-\infty}^{\infty} 2^{-|k-k'|\varepsilon'} \Big\{ M\Big( \sum_{\tau\in I_{k'}} \sum_{\nu=1}^{N(k',\tau)} \overline{D}_{k'}(f)(y_\tau^{k',\nu}) \chi_{Q_\tau^{k',\nu}}(\cdot) \Big)^r \Big\}^{1/r} \Big]^2 \Big\}^{1/2} \Big\|_p$$
$$\leqslant C \Big\| \Big( \sum_{k'=-\infty}^{\infty} \sum_{\tau\in I_{k'}} \sum_{\nu=1}^{N(k',\tau)} |\overline{D}_{k'}(f)(y_\tau^{k',\nu})|^2 \chi_{Q_\tau^{k',\nu}}(\cdot) \Big)^{1/2} \Big\|_p \leqslant C\|f\|_{H^p}.$$

Since $L^2(\mathcal{X}) \cap H^p(\mathcal{X})$ is dense in $H^p(\mathcal{X})$, the above estimates give the proof of Theorem 3.

*Step 2.* A new discrete Calderón's identity for $\mathrm{BMO}(\mathcal{X})$.

**Proposition 2.** *Let $\theta' \in (0,\theta)$, $1/(1+\theta') < p \leqslant 1$. Then for any $f \in L^2(\mathcal{X}) \cap H^p(\mathcal{X})$, there exists some $\widetilde{f} \in L^2(\mathcal{X}) \cap H^p(\mathcal{X})$ with $\|f\|_2 \sim \|\widetilde{f}\|_2$ and $\|f\|_{H^p} \sim \|\widetilde{f}\|_{H^p}$ and*

$$f = \sum_{k=-\infty}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(\cdot, y_\tau^{k,\nu}) \overline{D}_k(\widetilde{f})(y_\tau^{k,\nu}) \tag{2}$$

*where $Q_\tau^{k,\nu}, y_\tau^{k,\nu}, D_k$ are the same as in Lemma 3, and the series converges in $L^2(\mathcal{X}) \cap H^p(\mathcal{X})$.*

Proof. We begin with the classical Calderón's identity on $L^2(\mathcal{X})$:

$$f = \sum_{k=-\infty}^{\infty} D_k \overline{D}_k(f).$$

Using Coifman's idea of decomposition of identity yields

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} \int_{\mathcal{X}} D_k \overline{D}_k(x,y)(f)(y)\,\mathrm{d}\mu(y) \\ &= \sum_{k=-\infty}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \int_{Q_\tau^{k,\nu}} D_k \overline{D}_k(x,y)(f)(y)\,\mathrm{d}\mu(y) \\ &= \sum_{k=-\infty}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(x, y_\tau^{k,\nu}) \overline{D}_k(f)(y_\tau^{k,\nu}) + R(f)(x). \end{aligned}$$

It was proved by Deng and Han in [5] that $R$ is a Calderón-Zygmund operator on $\mathcal{X}$. Note that $R(1) = R^*(1) = 0$, hence by Theorem 3, $R$ is bounded on $H^p(\mathcal{X})$. Moreover, there exists $\delta > 0$ such that $\|R(f)\|_2 \leqslant C2^{-N\delta}\|f\|_2$ and $\|R(f)\|_{H^p} \leqslant C2^{-N\delta}\|f\|_{H^p}$.

See [4], [5], [10] for details of the proofs. Now for any $f \in L^2(\mathcal{X}) \cap H^p(\mathcal{X})$, we set $\widetilde{f} = \sum\limits_{n=0}^{\infty} R^n(f)$. This implies

$$f(x) = \sum_{k=-\infty}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(x, y_\tau^{k,\nu}) \overline{D}_k(\widetilde{f})(y_\tau^{k,\nu}).$$

We remark that $R$ is also bounded on $\mathrm{BMO}(\mathcal{X})$ with the inequality $\|R(f)\|_{\mathrm{BMO}} \leqslant C2^{-N\delta}\|f\|_{\mathrm{BMO}}$. For any $f \in L^2(\mathcal{X}) \cap H^1(\mathcal{X})$, the same proof implies

$$\widetilde{f}(x) = \sum_{n=0}^{\infty} R^n \Bigg( \sum_{k=-\infty}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(\cdot, y_\tau^{k,\nu}) \overline{D}_k(\widetilde{f})(y_\tau^{k,\nu}) \Bigg)(x),$$

where the series converges in $H^1(\mathcal{X})$. Therefore, for any $h \in \mathrm{BMO}(\mathcal{X})$,

$$\begin{aligned} \langle \widetilde{f}, h\rangle &= \Bigg\langle \sum_{n=0}^{\infty} R^n \sum_{k=-\infty}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(\cdot, y_\tau^{k,\nu}) \overline{D}_k(\widetilde{f})(y_\tau^{k,\nu}), h \Bigg\rangle \\ &= \Bigg\langle \sum_{k=-\infty}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(\cdot, y_\tau^{k,\nu}) \overline{D}_k(\widetilde{f})(y_\tau^{k,\nu}), \widetilde{h} \Bigg\rangle \\ &= \Bigg\langle \widetilde{f}, \sum_{k=-\infty}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(\cdot, y_\tau^{k,\nu}) \overline{D}_k(\widetilde{h})(y_\tau^{k,\nu}) \Bigg\rangle \end{aligned}$$

where $\widetilde{h} = \sum\limits_{n=0}^{\infty} R^n(h) \in \mathrm{BMO}(\mathcal{X})$ with $\|h\|_{\mathrm{BMO}} \sim \|\widetilde{h}\|_{\mathrm{BMO}}$.

We now obtain the discrete Calderón's identity for BMO functions: for any $h \in \mathrm{BMO}(\mathcal{X})$, there exists $\widetilde{h} \in \mathrm{BMO}(\mathcal{X})$ such that

$$h = \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(\cdot, y_\tau^{k,\nu}) \overline{D}_k(\widetilde{h})(y_\tau^{k,\nu})$$

where the series converges in $(H^1, \mathrm{BMO})$ sense. □

*Step 3.* The discrete para-product operators.

We now introduce the discrete para-product operators.

**Definition 8.** Let $\varepsilon \in (0, \theta]$ for $k \in \mathbb{Z}$, let $\{S_k\}_{k \in \mathbb{Z}}$ be an approximation to the identity of order $\varepsilon$, $D_k = S_k - S_{k-1}$, $\overline{D}_k$, $Q_\tau^{k,\nu}$ and let $y_\tau^{k,\nu}$ be the same as in Lemma 3. For the convenience, let

$$\Lambda = \{\lambda = (k, \tau, \nu)\colon\ k \in \mathbb{Z},\ \tau \in I_k,\ \nu = 1, \dots, N(k,\tau)\},$$

and $Q_\lambda$, $y_\lambda$ are used to denote the associated $Q_\tau^{k,\nu}$ and $y_\tau^{k,\nu}$.

Then the discrete para-product $\pi_b$ for $b \in \mathrm{BMO}(\mathcal{X})$ is defined by

$$\pi_b(f)(x) = \sum_{\lambda \in \Lambda} \mu(Q_\lambda) D_k(x, y_\lambda) \overline{D}_k(\widetilde{b})(y_\lambda) S_k(f)(y_\lambda)$$

and

$$\pi_b^*(f)(x) = \sum_{\lambda \in \Lambda} \mu(Q_\lambda) S_k(x, y_\lambda) \overline{D}_k(\widetilde{b})(y_\lambda) D_k(f)(y_\lambda),$$

where $\widetilde{h}$ is the same as in Proposition 2.

Note that for $b \in \mathrm{BMO}(\mathcal{X})$, $\pi_b$ is a Calderón-Zygmund operator on $\mathcal{X}$. Then for $b \in \mathrm{BMO}(\mathcal{X})$ and $g \in \mathring{C}_{0,0}^{\eta}(\mathcal{X})$, by Proposition 2 we have

$$\begin{aligned}\langle T_b(1), g\rangle &= \left\langle \sum_{\lambda \in \Lambda} \mu(Q_\lambda) D_k(x, y_\lambda) \overline{D}_k(\widetilde{b})(y_\lambda) S_k(1)(y_\lambda), g \right\rangle \\ &= \left\langle \sum_{\lambda \in \Lambda} \mu(Q_\lambda) D_k(x, y_\lambda) \overline{D}_k(\widetilde{b})(y_\lambda)(y_\lambda), g \right\rangle = \langle b, g\rangle.\end{aligned}$$

Therefore, we have $\pi_b(1) = b$. Similarly, we can get $\pi_b^*(1) = 0$. Then using an idea of the proof of the $T1$ theorem given by David and Journé, one can decompose a Calderón-Zygmund singular integral operator $T$ into

$$T = \widetilde{T} + \pi_{T1} + \pi_{T^*1}^*,$$

where $\widetilde{T}$ is a Calderón-Zygmund singular integral operator. Moreover, note that

$$\langle T1, g\rangle = \langle \widetilde{T}_1, g\rangle + \langle \pi_{T1}(1), g\rangle + \langle \pi^*_{T^*1}(1), g\rangle = \langle \widetilde{T}1, g\rangle + \langle T1, g\rangle,$$

so we have $\widetilde{T}1 = 0$. And similarly we get $\widetilde{T}^*1 = 0$. Moreover, if $T$ is bounded on $L^2(\mathcal{X})$, then $T1$ and $T^*1$ are bounded on $\mathrm{BMO}(\mathcal{X})$ and $\widetilde{T}$, $\pi_{T1}$ and $\pi^*_{T^*1}$ are all bounded on $L^2(\mathcal{X})$.

Note that Theorem B implies that $\widetilde{T}$ is bounded on $H^p(\mathcal{X})$, since $L^2$ boundedness of $\widetilde{T}$ implies its weak boundedness. Therefore, to prove Theorem 1 we only need to show that $\pi_b$ is bounded on $H^p(\mathcal{X})$, to prove Theorem 2 we only need to show that $\pi^*_b$ and $\pi_b$ are bounded from $H^p(\mathcal{X})$ to $L^p(\mathcal{X})$ for all $1/(1+\varepsilon) < p \leqslant 1$ and $b \in \mathrm{BMO}(\mathcal{X})$.

**Lemma 7.** *Let $b \in \mathrm{BMO}(\mathcal{X})$. Then $\pi_b$ is bounded on $H^p(\mathcal{X})$, $\pi^*_b$ and $\pi_b$ are bounded from $H^p(\mathcal{X})$ to $L^p(\mathcal{X})$ for all $1/(1+\varepsilon) < p \leqslant 1$.*

P r o o f. We first show that $\pi_b$ is bounded on $H^p(\mathcal{X})$ for all $1/(1+\varepsilon) < p \leqslant 1$ and $b \in \mathrm{BMO}(\mathcal{X})$.

By the Littlewood-Paley characterization of $H^p(\mathcal{X})$ in Proposition 1, we only need to prove that

$$\Big\|\Big\{\sum_{\lambda=(k,\tau,\nu)\in\Lambda} |D_k(\pi_b(f))(y_\lambda)|^2\chi_{Q_\lambda}(\cdot)\Big\}^{1/2}\Big\|_p^p \leqslant C_p\|f\|^p_{H^p}.$$

Using the almost orthogonality, we get

$$\begin{aligned}
&\Big\|\Big\{\sum_{\lambda\in\Lambda} |D_k(\pi_b(f))(y_\lambda)|^2\chi_{Q_\lambda}(\cdot)\Big\}^{1/2}\Big\|_p^p \\
&= \Big\|\Big\{\sum_{\lambda\in\Lambda}\Big|D_k\Big(\sum_{\lambda'\in\Lambda'} \mu(Q_{\lambda'})D_{k'}(\cdot, y_{\lambda'})\overline{D}_{k'}(\widetilde{b})(y_{\lambda'})S_{k'}(f)(y_{\lambda'})\Big)(y_\lambda)\Big|^2\chi_{Q_\lambda}(\cdot)\Big\}^{1/2}\Big\|_p^p \\
&\leqslant C\Big\|\Big\{\sum_{\lambda'\in\Lambda'} |\overline{D}_{k'}(\widetilde{b})(y_{\lambda'})S_{k'}(f)(y_{\lambda'})|^2\chi_{Q_{\lambda'}}(\cdot)\Big\}^{1/2}\Big\|_p^p.
\end{aligned}$$

Set

$$\Omega_l = \Big\{x \in \mathcal{X}\colon \sup_k |S_k(f)(x)|^2 > 2^l\Big\}$$

and

$$B_l = \Big\{Q' \text{ is a dyadic cube in } \mathcal{X}\colon\ \mu(Q'\cap\Omega_l) > \frac{1}{2}\mu(Q') \text{ and } \mu(Q'\cap\Omega_{l+1}) \leqslant \frac{1}{2}\mu(Q')\Big\}.$$

Then by Remark 5, i.e., the maximal characterization of the Hardy space given in [5], we get

$$\sum_l 2^{lp}\mu(\Omega_l) \leqslant \|f\|_{H^p}^p.$$

Then

$$\begin{aligned}\sum_{\lambda'\in\Lambda'} &|\overline{D}_{k'}(\widetilde{b})(y_{\lambda'})S_{k'}(f)(y_{\lambda'})|^2\chi_{Q_\lambda}(\cdot)\\ &=\sum_{k'}\sum_l\sum_{\widetilde{Q}\in B_l}\sum_{Q'\subset\widetilde{Q},Q'\in B_l}|\overline{D}_{k'}(\widetilde{b})(y_{Q'})S_{k'}(f)(y_{Q'})|^2\chi_{Q'}(\cdot),\end{aligned}$$

where $\widetilde{Q}$ are maximal dyadic cubes in $B_l$ and $y_{Q'}$ is any point in $Q'$. This leads to the estimate

$$\begin{aligned}&\Big\|\sum_{\lambda'\in\Lambda'}\{|\overline{D}_{k'}(\widetilde{b})(y_{\lambda'})S_{k'}(f)(y_{\lambda'})|^2\chi_{Q_\lambda}(\cdot)\}^{1/2}\Big\|_p^p\\ &\quad\leqslant\sum_l\sum_{\widetilde{Q}\in B_l}\Big\|\sum_{Q'\subset\widetilde{Q},Q'\in B_l}\sum_{k'}\{|\overline{D}_{k'}(\widetilde{b})(y_{Q'})S_{k'}(f)(y_{Q'})|^2\chi_{Q'}(\cdot)\}^{1/2}\Big\|_p^p,\end{aligned}$$

where the inequality $(a+b)^p \leqslant a^p + b^p$ for $0 < p \leqslant 1$ is used. Using the Hölder inequality to control the $L^p$ norm by the $L^2$ norm for functions with compact support, we get

$$\begin{aligned}&\Big\|\sum_{Q'\subset\widetilde{Q},Q'\in B_l}\sum_{k'}\{|\overline{D}_{k'}(\widetilde{b})(y_{Q'})S_{k'}(f)(y_{Q'})|^2\chi_{Q'}(\cdot)\}^{1/2}\Big\|_p^p\\ &\quad\leqslant C\mu(\widetilde{Q})^{1-p/2}\Big(\sum_{Q'\subset\widetilde{Q},Q'\in B_l}\sum_{k'}\mu(Q')|\overline{D}_{k'}(\widetilde{b})(y_{Q'})|^2|S_{k'}(f)(y_{Q'})|^2\Big)^{p/2}.\end{aligned}$$

This yields

$$\begin{aligned}&\sum_l\sum_{\widetilde{Q}\in B_l}\Big\|\sum_{Q'\subset\widetilde{Q},Q'\in B_l}\sum_{k'}\{|\overline{D}_{k'}(\widetilde{b})(y_{Q'})|^2|S_{k'}(f)(y_{Q'})|^2\chi_{Q'}(\cdot)\}^{1/2}\Big\|_p^p\\ &\leqslant\sum_l\sum_{\widetilde{Q}\in B_l}C\mu(\widetilde{Q})^{1-p/2}\Big(\sum_{Q'\subset\widetilde{Q},Q'\in B_l}\sum_{k'}\mu(Q')|\overline{D}_{k'}(\widetilde{b})(y_{Q'})|^2|S_{k'}(f)(y_{Q'})|^2\Big)^{p/2}\\ &\leqslant\sum_l\Big(\sum_{\widetilde{Q}\in B_l}C\mu(\widetilde{Q})\Big)^{1-p/2}\Big(\sum_{Q'\subset\widetilde{Q},Q'\in B_l}\sum_{k'}\mu(Q')|\overline{D}_{k'}(\widetilde{b})(y_{Q'})|^2|S_{k'}(f)(y_{Q'})|^2\Big)^{p/2}.\end{aligned}$$

Note that if $Q' \in B_l$, then

$$Q' \subset \widetilde{\Omega}_l = \Big\{x \in \mathcal{X}\colon\ M_{\chi_{\Omega_l}}(x) > \frac{1}{2}\Big\}$$

and since $y_{Q'}$ is any fixed point in $Q' \in B_l$, where $\mu(Q' \cap \Omega_{l+1}) \leqslant \mu(Q')/2$ so we can take $y_{Q'} \in \Omega_{l+1}$, then $|S_{k'}(f)(y_{Q'})| \leqslant 2^{l+1}$. Therefore,

$$\begin{aligned}
\Big(\sum_{Q' \subset \widetilde{Q}, Q' \in B_l} \sum_{k'} \mu(Q')|\overline{D}_{k'}(\widetilde{b})(y_{Q'})|^2 |S_{k'}(f)(y_{Q'})|^2\Big)^{p/2}& \\
\leqslant C 2^{lp} \Big(\sum_{Q' \subset \widetilde{Q}, Q' \in B_l} \sum_{k'} \mu(Q')|\overline{D}_{k'}(\widetilde{b})(y_{Q'})|^2\Big)^{p/2}& \\
\leqslant C 2^{lp} \Big(\sum_{\widetilde{Q} \in B_l} \mu(\widetilde{Q})\Big)^{p/2} \leqslant C 2^{lp} \mu(\widetilde{\Omega}_l)^{p/2} \leqslant C 2^{lp} \mu(\Omega_l)^{p/2}&
\end{aligned}$$

where we have used the fact that $b \in \mathrm{BMO}$ and a result concerning the Carleson measure ([5], page 118, Theorem 4.13)

$$\sum_{Q' \subset \widetilde{Q}} \sum_{k'} \mu(Q')|\overline{D}_{k'}(\widetilde{b})(y_{Q'})|^2 \leqslant C\mu(\widetilde{Q}).$$

Substituting all these estimates into the above inequality we get

$$\begin{aligned}
\Big\|\sum_{\lambda' \in \Lambda'} \{|\overline{D}_{k'}(\widetilde{b})(y_{\lambda'}) S_{k'}(f)(y_{\lambda'})|^2 \chi_{Q_\lambda}(\cdot)\}^{1/2}\Big\|_p^p& \\
\leqslant \sum_l \Big(\sum_{\widetilde{Q} \in B_l} C\mu(\widetilde{Q})\Big)^{1-p/2} 2^{lp} \mu(\Omega_l)^{p/2}& \\
\leqslant C \sum_l 2^{lp} \mu(\widetilde{\Omega})^{1-p/2} \mu(\Omega_l)^{p/2} \leqslant C \sum_l 2^{lp} \mu(\Omega_l) \leqslant C\|f\|_{H^p}^p.&
\end{aligned}$$

This shows that $\pi_b$ is bounded on $H^p(\mathcal{X})$.

We now prove that $\pi_b^*$ is bounded from $H^p(\mathcal{X})$ to $L^p(\mathcal{X})$. A similar result for $\pi_b$ can be obtained by the same method. We first note that $\pi_b^*$ is bounded on $L^2$, thus

$$\|\pi_b^* f\|_p^p \leqslant \sum_l \sum_{\widetilde{Q} \in B_l} \Big\|\sum_{Q \subset \widetilde{Q}, Q \in B_l} \mu(Q) S_k(\cdot, y_Q) \overline{D}_k(b)(y_Q) D_k(f)(y_Q)\Big\|_p^p.$$

Set

$$\Omega_l = \Big\{x \in \mathcal{X}\colon\ \Big\{\sum_k \sum_Q |D_k(f)(y_Q)|^2 \chi_Q(x)\Big\}^{1/2} > 2^l\Big\}$$

and

$$B_l = \Big\{Q \text{ is a dyadic cube in } \mathcal{X}\colon\ \mu(Q\cap\Omega_l) > \frac{1}{2}\mu(Q) \text{ and } \mu(Q\cap\Omega_{l+1}) \leqslant \frac{1}{2}\mu(Q)\Big\},$$

using the Hölder inequality yields

$$\begin{aligned}
&\sum_l \sum_{\widetilde{Q}\in B_l} \Big\| \sum_{Q\subset\widetilde{Q}, Q\in B_l} \mu(Q)S_k(\cdot,y_Q)\overline{D}_k(b)(y_Q)D_k(f)(y_Q)\Big\|_p^p \\
&\leqslant C\mu(\widetilde{Q})^{1-p/2}\bigg(\Big\|\sum_l \sum_{Q\subset\widetilde{Q}, Q\in B_l} \mu(Q)S_k(\cdot,y_Q)\overline{D}_k(b)(y_Q)D_k(f)(y_Q)\Big\|_2^2\bigg)^{p/2}.
\end{aligned}$$

Therefore,we have

$$\begin{aligned}
&\sum_l \sum_{\widetilde{Q}\in B_l} \Big\| \sum_{Q\subset\widetilde{Q}, Q\in B_l} \mu(Q)S_k(\cdot,y_Q)\overline{D}_k(b)(y_Q)D_k(f)(y_Q)\Big\|_p^p \\
&\leqslant C\sum_l \sum_{\widetilde{Q}\in B_l} \mu(\widetilde{Q})^{1-p/2}\bigg(\Big\| \sum_{Q\subset\widetilde{Q}, Q\in B_l} \mu(Q)S_k(\cdot,y_Q)\overline{D}_k(b)(y_Q)D_k(f)(y_Q)\Big\|_2^2\bigg)^{p/2} \\
&\leqslant C\sum_l \bigg(\sum_{\widetilde{Q}\in B_l} \mu(\widetilde{Q})\bigg)^{1-p/2}\bigg(\Big\| \sum_{Q\subset\widetilde{Q}, Q\in B_l} \mu(Q)S_k(\cdot,y_Q)\overline{D}_k(b)(y_Q)D_k(f)(y_Q)\Big\|_2^2\bigg)^{p/2}.
\end{aligned}$$

We claim that

$$\Big\| \sum_{Q\subset\widetilde{Q}, Q\in B_l} \mu(Q)S_k(\cdot,y_Q)\overline{D}_k(b)(y_Q)D_k(f)(y_Q)\Big\|_2^2 \leqslant C2^{2l}\mu(\widetilde{\Omega}_l),$$

which implies

$$\|\pi_b^* f\|_p^p \leqslant C2^{2l}\mu(\Omega_l) \leqslant C\Big\|\sum_k\sum_Q |D_k(f)(y_Q)|^2\chi_Q(\cdot)\}^{1/2}\Big\|_p^p \leqslant C\|f\|_{H^p}^p.$$

To show the claim, we use the duality argument to get

$$\begin{aligned}
&\Big\|\sum_{Q\in B_l} \mu(Q)S_k(\cdot,y_Q)\overline{D}_k(b)(y_Q)D_k(f)(y_Q)\Big\|_2 \\
&\qquad\times \sup_{\|h\|_2\leqslant 1}\Big|\Big\langle \sum_{Q\in B_l} \mu(Q)S_k(\cdot,y_Q)\overline{D}_k(b)(y_Q)D_k(f)(y_Q), h\Big\rangle\Big|
\end{aligned}$$

$$\leqslant \sup_{\|h\|_2\leqslant 1} \sum_{Q\in B_l} \mu(Q)S_k(h)(y_Q)\overline{D}_k(b)(y_Q)D_k(f)(y_Q)$$

$$\leqslant \sup_{\|h\|_2\leqslant 1} \Bigl(\sum_{Q\in B_l} |\mu(Q)S_k(h)(y_Q)|^2|\overline{D}_k(b)(y_Q)|^2\Bigr)^{1/2}\Bigl(\sum_{Q\in B_l} \mu(Q)|D_k(f)(y_Q)|^2\Bigr)^{1/2}$$

$$\leqslant C\Bigl(\sum_{Q\in B_l} \mu(Q)|D_k(f)(y_Q)|^2\Bigr)^{1/2},$$

where the last inequality follows from the fact that $b \in \mathrm{BMO}(\mathcal{X})$ and from the Carleson measure estimate. To complete the proof of the claim, we have

$$C2^{2l}\mu(\widetilde{\Omega}) \geqslant \int_{\widetilde{\Omega}_l\setminus\Omega_{l+1}} \Bigl\{\sum_k\sum_Q |D_k(f)(y_Q)|^2\chi_Q(x)\Bigr\}\,\mathrm{d}\mu(x)$$
$$\geqslant \frac{1}{2}\sum_{Q\in B_l} \mu(Q)|D_k(f)(y_Q)|^2.$$

This finishes the proof of Lemma 7. □

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