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# ON THE QUANTUM GROUPS AND SEMIGROUPS OF MAPS BETWEEN NONCOMMUTATIVE SPACES

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*Abstract.* We define algebraic families of (all) morphisms which are purely algebraic analogs of quantum families of (all) maps introduced by P. M. Sołtan. Also, algebraic families of (all) isomorphisms are introduced. By using these notions we construct two classes of Hopf-algebras which may be interpreted as the quantum group of all maps from a finite space to a quantum group, and the quantum group of all automorphisms of a finite noncommutative (NC) space. As special cases three classes of NC objects are introduced: quantum group of gauge transformations, Pontryagin dual of a quantum group, and Galois-Hopf-algebra of an algebra extension.

*Keywords*: Hopf-algebra; bialgebra; quantum group; noncommutative geometry

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## 1. Introduction

Our work is based on the extension to the noncommutative setting of the following picture. Let $X$ and $Y$ be compact Hausdorff spaces. If $S$ is a Hausdorff space then there is a canonical one-to-one correspondence between continuous maps $f\colon S\times X\to Y$ and continuous families $\overline{f}=\{f(s,\cdot)\}_{s\in S}$ of continuous maps from $X$ to $Y$ with parameter space $S$. Moreover, by the exponential law of topology, there is a canonical homeomorphism $Y^{S\times X}\simeq(Y^X)^S$ where the mapping spaces are endowed with compact-open topology. Indeed, the family of all continuous maps from $X$ to $Y$, that is the family $\overline{e}$ induced by the evaluation map $e\colon Y^X\times X\to Y$ defined by $e(a,x)=a(x)$, and the space $Y^X$ are completely characterized by the following universal property. For every Hausdorff space $S$ and any continuous map $f\colon S\times X\to Y$ there is a unique continuous map $\overline{f}$ that makes the following diagram

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commutative:

$$\begin{array}{ccc} Y^X \times X & \xrightarrow{e} & Y \\ {\scriptstyle \overline{f}\times \mathrm{id}_X}\uparrow & \nearrow_{f} & \\ S \times X & & \end{array}$$

In the context of NC geometry and operator algebra, Sołtan [15] and Woronowicz in [21] introduced the notion of the *quantum space of all maps* between two C*-algebraic NC spaces. To define this notion they used the C*-dual of the above universal property: For two quantum spaces $\mathfrak{Q}A$ and $\mathfrak{Q}B$, a quantum space $\mathfrak{Q}C$ together with a $*$-homomorphism $\phi\colon B \to A \otimes C$ is called the quantum family of all maps from $\mathfrak{Q}A$ to $\mathfrak{Q}B$ if for every $*$-homomorphism $\psi\colon B \to A \otimes D$, between C*-algebras, there is a unique $*$-homomorphism $\overline{\psi}\colon C \to D$ satisfying $(\mathrm{id}_A \otimes \overline{\psi})\phi = \psi$. Sołtan in [15] showed that such universal C*-algebra $C$ and $*$-homomorphism $\phi$ exist when $\mathfrak{Q}A$ is a finite quantum space, i.e. $A$ is a finite dimensional C*-algebra. He also showed ([15], [16]) that if $B = A$ then $C$ has a canonical C*-bialgebra structure (as in the classical case the space $X^X$ is a topological semigroup) and the bialgebra structure of some classes of quantum groups, e.g. quantum permutation groups ([20], [2]) and quantum isometry groups ([1]), are the same as that of $C$.

The aim of this paper is to study the purely algebraic version of the quantum space of all maps and its bialgebra structures. It seems that our algebraic formalism is more interesting and useful than the C*-formalism because the class of finite dimensional C*-algebras restricts only to finite direct sums of full matrix algebras over the complex field.

In Section 2 we define the *algebraic family of all morphisms* from an algebra $B$ to another algebra $A$ over a commutative ring $\mathbb{K}$. This will be a morphism $\mathfrak{m}(B, A)\colon B \to A \otimes \mathfrak{M}(B, A)$ between algebras which satisfies a universal property analogous to the above. As mentioned above, this is a purely algebraic reformulation of the notion of quantum family of all maps, see [15]. As an analogue of Theorem 3.3 of [15], we show that $\mathfrak{m}(B, A)$ exists if $A$ is a free finite rank $\mathbb{K}$-module. Also we consider some elementary properties of $\mathfrak{M}(\cdot, A)$ as a functor on the category of algebras.

In Section 3 we define *algebraic families of all isomorphisms.* The idea of the definition comes from the following trivial fact: A family $\overline{f}$ associated with the continuous map $f\colon S \times X \to Y$ is a family of homeomorphisms from $X$ onto $Y$ if and only if there is another family $\overline{g}$ associated with a continuous map $g\colon S \times Y \to X$ such that $g(s, f(s, x)) = x$ and $f(s, g(s, y)) = y$ for every $x \in X$, $y \in Y$ and $s \in S$. It follows that the family of all homeomorphisms must be universal with respect to this property. Then by dualizing this universal property we find the desired notion of the algebraic family of all isomorphisms $\mathfrak{i}(B, A)\colon B \to A \otimes \mathfrak{I}(B, A)$ from $B$ onto $A$. We will show that $\mathfrak{i}(B, A)$ exists if both $A$ and $B$ are free finite rank $\mathbb{K}$-modules.

In Section 4 we consider the existence of duals of the following classical objects:

(1) The space of all bundle morphisms between two fibre bundles over a classical space.
(2) The space of all continuous maps between two classical spaces which are identity over a common subspace.
(3) The space of all continuous group homomorphisms between two topological groups.
(4) The space of all $G$-maps between two $G$-spaces for a topological group $G$.

If $G$ is a compact group then $G^X$ is a topological group for every compact space $X$. Section 5 is devoted to the noncommutative analog of this fact. More precisely, we will show that if $B$ is a Hopf-algebra and $A$ is commutative and finite-rank then $\mathfrak{M}(B, A)$ has a canonical Hopf-algebra structure. In the C*-case this construction has been considered in [17] and [13].

In Sections 6 and 7, by using the results of the preceding sections, we construct two classes of Hopf-algebras which may be interpreted as the quantum group of gauge transformations over trivial quantum vector bundles in the sense of [4], and as the Pontryagin dual of finite-rank Hopf-algebras.

If $X$ is a compact Hausdorff space then the space of all homeomorphisms of $X$ is a topological group. In Section 8 we show this is also the case in the dual context. Indeed, we will show that for any finite-rank Hopf-algebra $A$ the algebra $\mathfrak{I}(A, A)$ has a canonical Hopf-algebra structure. Finally, we consider a notion dual to the notion of Galois group of algebra extensions.

We announce that there is a C*-algebraic concept of the *family of quantum invertible maps* due to Podleś, see [12], [14,] which is different from our notion of the algebraic family of isomorphisms and because of its topological nature it seems that its purely algebraic formulation is not interesting. Also, after preparation of this paper we received a recent paper [14] of Skalski and Sołtan where they consider C*-algebraic quantum group structures on Podleś' families of quantum invertible maps. We delegate the study of relations between the results of the present paper and those of [14] to a future work.

**Notation and preliminaries.** The identity map on a set $X$ is denoted by $i_X$. Throughout, all rings have unit and ring homomorphisms preserve units. $\mathbb{K}$ denotes a fixed nonzero commutative ring. Further $\otimes$ denotes the tensor product over $\mathbb{K}$. For any permutation $\sigma$ of $\{1, \ldots, n\}$, $F_\sigma$ denotes the flip homomorphism $x_1 \otimes \ldots \otimes x_n \mapsto x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}$ on the tensor product $M_1 \otimes \ldots \otimes M_n$ of $\mathbb{K}$-modules; when there is no risk of confusion we will write $F$ instead of $F_\sigma$. An algebra is a ring together with a structural ring homomorphism $\mathbb{K} \to A$. A morphism between algebras is a ring homomorphism which commutes with structural homomorphisms. $\mathbb{K}[x_i]_{i\in I}$ and $\mathbb{K}^c[x_i]_{i\in I}$ denote, respectively, the polynomial algebra in non-commuting and commuting variables $\{x_i\}_{i\in I}$. Let $B$ be an algebra. If $S\subseteq B$ generates $B$ as an algebra, we write $I_S$ for the kernel of the canonical morphism $\mathbb{K}[x_s]_{s\in S}\to B$ defined by $x_s\mapsto s$. Alg denotes the category of algebras and $\mathrm{Alg}(B,A)$ is the set of morphisms from $B$ to $A$. The elements of $\mathrm{Alg}(B,\mathbb{K})$ are called characters of $B$. The full subcategory of commutative algebras is denoted by $\mathrm{Alg}^c$. The multiplication of $A$ is denoted by $\mu_A\colon A\otimes A\to A$. Note that if $A$ is commutative then $\mu_A$ is a morphism. By a finite-rank algebra we mean an algebra which is a free $\mathbb{K}$-module of finite rank. Throughout, *finite-rank* is abbreviated to FR. A comultiplication on an algebra $B$ is a morphism $\Delta\colon B\to B\otimes B$ which is coassociative: $(\Delta\otimes i_B)\Delta=(i_B\otimes\Delta)\Delta$. So $\Delta$ is cocommutative if $\Delta=F\Delta$. A character $\varepsilon$ on $B$ is a counit if $(i_B\otimes\varepsilon)\Delta=(\varepsilon\otimes i_B)\Delta=i_B$. So a morphism $S\colon B\to B^{\mathrm{op}}$ is an antipode if $\mu_B(S\otimes i_B)\Delta=\mu_B(i_B\otimes S)\Delta=\varepsilon 1$. Then $(B,\Delta,\varepsilon)$ and $(B,\Delta,\varepsilon,S)$ are called the bialgebra and the Hopf-algebra, respectively. By a (right) $B$-comodule we mean an algebra $V$ together with a right coaction of $B$, i.e. a morphism $\varrho\colon V\to V\otimes B$ satisfying $(\varrho\otimes i_B)\varrho=(i_V\otimes\Delta)\varrho$ and $(i_V\otimes\varepsilon)\varrho=i_V$.

## 2. Algebraic family of all morphisms

In this section we consider the notions of the *algebraic family of morphisms* and the *algebraic family of all morphisms* from an algebra $B$ to another algebra $A$. We shall show that the latter exists when $A$ is a FR algebra. Also we consider some functorial properties of these notions and some simple explicit examples and computations.

**Definition 2.1.** Let $A$, $B$, $B'$, $C$ and $C'$ be algebras.

(a) A morphism $\psi\colon B\to A\otimes C$ is called an algebraic family of morphisms from $B$ to $A$ with parameter-algebra $C$. When there are no ambiguity about $A$, $B$ and $C$, we call $\psi$ just 'family'.

(b) A family with parameter-algebra 0 is called the empty family of morphisms. If $f_1,\dots,f_n$ is a finite collection of morphisms from $B$ to $A$ then $\psi\colon B\to A\otimes\mathbb{K}^n$, defined by $\psi(b)=\sum_{i=1}^{n}f_i(b)\otimes e_i$ where $e_1,\dots,e_n$ denote the standard basis, is called the trivial family.

(c) $\mathcal{M}(B,A)$ denotes the class of all families of morphisms from $B$ to $A$; $\mathcal{M}(B,A)$ can be viewed as a category such that objects are families and morphisms from $\psi\colon B\to A\otimes C$ to $\psi'\colon B\to A\otimes C'$ are algebra morphisms $\varphi\colon C\to C'$ satisfying $\psi'=(i_A\otimes\varphi)\psi$.

(d) Let $\psi\colon B\to A\otimes C$ and $\psi'\colon B'\to B\otimes C'$ be families of morphisms from $B$ to $A$ and from $B'$ to $B$, respectively. Then the family $(\psi\otimes i_{C'})\psi'$ of morphisms from $B'$ to $A$ (with parameter-algebra $C\otimes C'$) is denoted by $\psi\circ\psi'$ and called the composition of $\psi$ and $\psi'$. (Note that the composition of families can be viewed as a functor from $\mathcal{M}(B,A) \times \mathcal{M}(B',B)$ to $\mathcal{M}(B',A)$.)

(e) Let $\mathcal{C}$ be a subclass of $\mathcal{M}(B,A)$. Then $\psi \in \mathcal{C}$ is called a $\mathcal{C}$-universal family if it is an initial object of $\mathcal{C}$ when $\mathcal{C}$ is viewed as a full subcategory of $\mathcal{M}(B,A)$. (Note that if a $\mathcal{C}$-universal family exists then it is unique up to isomorphism.)

(f) Let $\mathcal{C}$ be a subclass of $\mathcal{M}(B,A)$. Then $\mathcal{C}^{\mathrm{c}}$ denotes the subclass of $\mathcal{C}$ containing families with commutative parameter-algebras.

**Lemma 2.2.** *Let $\mathcal{C} \subseteq \mathcal{M}(B,A)$. Suppose that $\phi\colon B \to A \otimes C$ is a $\mathcal{C}$-universal family. Consider the morphism $\psi = (i_A \otimes \varphi)\phi\colon B \to A \otimes C/[C,C]$, where $[C,C]$ denotes the commutator ideal of $C$ and $\varphi\colon C \to C/[C,C]$ is the canonical projection. Then $\psi$ is $\mathcal{C}^{\mathrm{c}}$-universal if it belongs to $\mathcal{C}$.*

Proof. Straightforward. □

**Proposition 2.3.** *Let $A$ and $B$ be algebras such that $A$ is* FR. *Then there exists an $\mathcal{M}(B,A)$-universal family, say $\phi\colon B \to A \otimes Z$. This family has the following additional properties.*

(a) *The set $\{(\alpha \otimes i_Z)\phi(b)\}$, where $b \in B$ and $\alpha$ runs over all module homomorphisms from $A$ to $\mathbb{K}$, generates $Z$ as an algebra.*
(b) *If $B$ is finitely generated or presented algebra then $Z$ is finitely generated or presented algebra, respectively.*
(c) *For any algebra $C$ there is a canonical one-to-one correspondence between morphisms from $B$ to $A \otimes C$ and morphisms from $Z$ to $C$. In particular there is a one-to-one correspondence between morphisms from $B$ to $A$ and characters on $Z$.*
(d) *$Z$ represents the covariant functor* $\mathrm{Alg}(B, A \otimes \cdot)$ *from* Alg *to the category of sets.*

The morphism $\phi$ is called the *algebraic family of all morphisms from $B$ to $A$*. We will also use the symbols $\mathfrak{m}(B,A)$ and $\mathfrak{M}(B,A)$ for $\phi$ and $Z$, respectively.

Proof. Suppose that $S \subseteq B$ generates $B$ as an algebra and $\{a_i\colon i = 1,\dots,n\}$ is a basis for $A$ as a free module. Let $Z$ be the universal algebra generated by $z_{si}$, $s \in S$, such that for every polynomial $p \in I_S$ the equation $p\Big(\Big\{\sum_{i=1}^{n} a_i \otimes z_{si}\Big\}_{s\in S}\Big) = 0$ (in $A \otimes Z$) is satisfied. Then the assignment $s \mapsto \sum_{i=1}^{n} a_i \otimes z_{si}$ defines a morphism $\phi$. Let $\psi\colon B \to A \otimes C$ be a morphism and suppose that $c_{s,i} \in C$ is such that $\psi(s) = \sum_{i=1}^{n} a_i \otimes c_{si}$. Thus the equation $p\Big(\Big\{\sum_{i=1}^{n} a_i \otimes c_{si}\Big\}_{s\in S}\Big) = 0$ is satisfied for every $p \in I_S$. Then the universality of $Z$ shows that the assignment $z_{si} \mapsto c_{si}$ defines a morphism $\overline{\psi}$

and this morphism is the only one that satisfies $\psi = (i_A \otimes \overline{\psi})\phi$. This shows that $\phi$ is a universal family in $\mathcal{M}(B, A)$. We set that (a) and (b) immediately follow from the construction of $Z$, and it is clear that the assignment $\psi \mapsto \overline{\psi}$ defines the desired correspondence in (c). Also (d) trivially follows from (c). □

**Remark 2.4.** Let $A$ and $B$ be algebras such that $A$ is FR. It follows from Proposition 2.3 and Lemma 2.2 that there exists a family $\mathfrak{m}^{\mathrm{c}}\colon B \to A \otimes \mathfrak{M}^{\mathrm{c}}(B, A)$ which is $\mathcal{M}^{\mathrm{c}}(B, A)$-universal. It also follows that $\mathfrak{M}^{\mathrm{c}}(B, A)$ is canonically isomorphic to the quotient of $\mathfrak{M}(B, A)$ by its commutator ideal, and analogs of conditions (a)–(d) of Proposition 2.3 are satisfied for $\mathfrak{M}^{\mathrm{c}}(B, A)$.

The universality of families of all morphisms easily implies the following proposition.

**Proposition 2.5.** *Let $A$ and $B$ be algebras such that $A$ is* FR *and commutative. Then $\mathfrak{M}^{\mathrm{c}}(B, A)$ and $\mathfrak{M}^{\mathrm{c}}(B/[B, B], A)$ are isomorphic.*

P r o o f. Straightforward. □

**Remark 2.6.** Let $A$ and $B$ be algebras such that $A$ is FR.

(a) It follows easily from the universal property that $\mathfrak{M}(B, \mathbb{K}) = B$, $\mathfrak{M}^{\mathrm{c}}(B, \mathbb{K}) = B/[B, B]$, and $\mathfrak{M}(\mathbb{K}, A) = \mathfrak{M}^{\mathrm{c}}(\mathbb{K}, A) = \mathbb{K}$.

(b) From the construction of the universal parameter-algebra in the proof of Proposition 2.3 we see that $\mathfrak{M}(B, \mathbb{K}^n)$ is the free product $B * \ldots * B$ of $n$ copies of $B$. (Here free product means coproduct in Alg.) Similarly, we have $\mathfrak{M}^{\mathrm{c}}(B, \mathbb{K}^n) = \bigotimes_{i=1}^{n} B/[B, B]$ by Remark 2.4. It follows that $\mathfrak{M}(\mathbb{K}[x], \mathbb{K}^n) = \mathbb{K}[x_1, \ldots, x_n]$ and $\mathfrak{M}^{\mathrm{c}}(\mathbb{K}[x], \mathbb{K}^n) = \mathbb{K}^{\mathrm{c}}[x_1, \ldots, x_n]$. Also if $\mathbb{K}$ is an algebraically closed field and $B$ is the algebra of polynomial functions on an affine variety $V$ then $\mathfrak{M}^{\mathrm{c}}(B, \mathbb{K}^n)$ is the algebra of polynomial functions on $V^n$.

(c) From Proposition 2.3 (c) it follows that if there is a morphism from $B$ to $A$ then $Z := \mathfrak{M}(B, A) \neq 0$ and also $Z^{\mathrm{c}} := \mathfrak{M}^{\mathrm{c}}(B, A) \neq 0$. Moreover, if $\mathbb{K}$ is a field, it follows from Dedekind's lemma that $\dim_{\mathbb{K}} Z^{\mathrm{c}}$ (and so $\dim_{\mathbb{K}} Z$) is not less than the cardinality of the set of algebra morphisms from $B$ to $A$.

(d) Conversely, suppose that $Z^{\mathrm{c}} \neq 0$. So $Z^{\mathrm{c}}$ has a maximal ideal. If $\mathbb{K}$ is an algebraically closed field and $B$ is finitely generated then Proposition 2.3 (b) implies that $Z^{\mathrm{c}}$ is finitely generated and therefore, it follows from Zariski's lemma that $Z^{\mathrm{c}}$ has a character and so there exits an algebra morphism from $B$ to $A$.

(e) Suppose that $\mathbb{K}$ is an algebraically closed field and $B$ is finitely generated. Moreover, suppose that there are only finitely many morphisms from $B$ to $A$. Then $Z^{\mathrm{c}}/J$ is isomorphic to the algebra $\mathbb{K}^n$ where $J$ denotes the Jacobson radical and $n$ is the number of distinct morphisms from $B$ to $A$. To see this fact, let $f_1, \ldots, f_n$ be the collection of all morphisms from $B$ to $A$ and let $\psi\colon B \to A \otimes \mathbb{K}^n$ be its trivial family. Let $\overline{\psi}\colon Z^{\mathrm{c}} \to \mathbb{K}^n$ be the unique morphism which satisfies $\psi = (i_A \otimes \overline{\psi})\mathfrak{m}^{\mathrm{c}}$. From Proposition 2.3 (c) it follows that $Z^{\mathrm{c}}$ has exactly $n$ distinct characters and these are of the form $p_i\overline{\psi}$, $i = 1, \dots, n$, where $p_i\colon \mathbb{K}^n \to \mathbb{K}$ denotes the canonical projection on the $i$'th component. It follows that any maximal ideal of $Z^{\mathrm{c}}$ is in the form of $\ker(p_i\overline{\psi})$ and so $\overline{\psi}$ induces an injective morphism from $Z^{\mathrm{c}}/J$ to $\mathbb{K}^n$. From Dedekind's Lemma it follows that $\dim_{\mathbb{K}} Z^{\mathrm{c}}/J \geqslant n$. So $Z^{\mathrm{c}}/J$ and $\mathbb{K}^n$are isomorphic.

Let $A$ be a FR algebra. Let $B_1$ and $B_2$ be two algebras and $f\colon B_1 \to B_2$ a morphism. Then the universality of $\mathfrak{M}(B_1, A)$ shows that there is a unique morphism $\mathfrak{M}(f, A) = \mathfrak{M}(f)$ that makes the following diagram commutative:

$$\begin{array}{ccc} B_1 & \xrightarrow{\mathfrak{m}(B_1,A)} & A \otimes \mathfrak{M}(B_1, A) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle i_A \otimes \mathfrak{M}(f)} \\ B_2 & \xrightarrow{\mathfrak{m}(B_2,A)} & A \otimes \mathfrak{M}(B_2, A). \end{array}$$

Analogously, if $g\colon B_2 \to B_3$ is another morphism we find morphisms $\mathfrak{M}(g)\colon \mathfrak{M}(B_2, A) \to \mathfrak{M}(B_3, A)$ and $\mathfrak{M}(gf)\colon \mathfrak{M}(B_1, A) \to \mathfrak{M}(B_3, A)$. Then again the universality of $\mathfrak{M}(B_1, A)$ implies $\mathfrak{M}(gf) = \mathfrak{M}(g)\mathfrak{M}(f)$. Similarly, we have $\mathfrak{M}(i_B) = i_{\mathfrak{M}(B,A)}$ for any algebra $B$. Analogous statements also hold for $\mathfrak{M}^{\mathrm{c}}$. So we have proved the theorem below.

**Theorem 2.7.** *Let $A$ be a* FR *algebra. Then $\mathfrak{M}(\cdot, A)\colon \mathrm{Alg} \to \mathrm{Alg}$ (as well as $\mathfrak{M}^{\mathrm{c}}(\cdot, A)\colon \mathrm{Alg} \to \mathrm{Alg}^{\mathrm{c}}$) is a covariant functor.*

Note that for a fixed algebra $B$, $\mathfrak{M}(B, \cdot)$ and $\mathfrak{M}^{\mathrm{c}}(B, \cdot)$ can be viewed as contravariant functors from the category of FR algebras to Alg and $\mathrm{Alg}^{\mathrm{c}}$, respectively.

**Theorem 2.8.** *Let $A$ be a* FR *commutative algebra. Then $\mathfrak{M}^{\mathrm{c}}(\cdot, A)$ preserves the tensor product.*

P r o o f. Suppose that $B_1$ and $B_2$ are two algebras. Let $C_i = \mathfrak{M}^{\mathrm{c}}(B_i, A)$ and $\phi_i = \mathfrak{m}^{\mathrm{c}}(B_i, A)$ $(i = 1, 2)$ and also let $D = \mathfrak{M}^{\mathrm{c}}(B_1 \otimes B_2, A)$ and $\psi = \mathfrak{m}^{\mathrm{c}}(B_1 \otimes B_2, A)$. We must show that $C_1 \otimes C_2$ and $D$ are isomorphic. Let $\alpha_i\colon B_i \to B_1 \otimes B_2$ be the structural morphisms, i.e. $\alpha_1(b_1) = b_1 \otimes 1$ and $\alpha_2(b_2) = 1 \otimes b_2$ $(b_1 \in B_1,\ b_2 \in B_2)$. Then we find morphisms $\mathfrak{M}(\alpha_i)\colon C_i \to D$. The coproduct structure of $C_1 \otimes C_2$ (in $\mathrm{Alg}^{\mathrm{c}}$) induces a morphism $f\colon C_1 \otimes C_2 \to D$ such that $f\gamma_i = \mathfrak{M}(\alpha_i)$ where $\gamma_i\colon C_i \to C_1 \otimes C_2$ are structural morphisms. Let $g\colon D \to C_1 \otimes C_2$ be the unique morphism that makes the following diagram commutative:

$$\begin{CD}
B_1\otimes B_2 @>\psi>> A\otimes D \\
@. @VV i_A\otimes g V \\
@V\phi\otimes\phi VV A\otimes(C_1\otimes C_2) \\
@. @AA \mu_A\otimes i_{C_1\otimes C_2} A \\
A\otimes C_1\otimes A\otimes C_2 @>i_A\otimes F\otimes i_{C_2}>> (A\otimes A)\otimes(C_1\otimes C_2).
\end{CD}$$

Then it is easily checked that $gf\gamma_i = \gamma_i$. This fact together with the coproduct universal property of $C_1 \otimes C_2$ implies $gf = i_{(C_1\otimes C_2)}$. Also it is easily checked that $(i_A \otimes fg)\psi = \psi$. So by the universal property of $D$ we have $fg = i_D$. This completes the proof. □

**Remark 2.9.** Let $B_1$, $B_2$ and $A$ be algebras such that $A$ is commutative and FR. Then there is a canonical morphism from $\mathfrak{M}(B_1 \otimes B_2, A)$ to $\mathfrak{M}(B_1, A) \otimes \mathfrak{M}(B_2, A)$, analogous to $g$ in the proof of Theorem 2.8.

The following theorem states an *exponential law* in our dual formalism.

**Theorem 2.10.** *Let $A_1$, $A_2$ and $B$ be algebras such that $A_1$ and $A_2$ are* FR. *Then the algebras $\mathfrak{M}(B, A_1 \otimes A_2)$ and $\mathfrak{M}(\mathfrak{M}(B, A_1), A_2)$ are isomorphic.*

Proof. Let $\phi_i = \mathfrak{m}(B, A_i)$, $\phi_{12} = \mathfrak{m}(B, A_1 \otimes A_2)$ and $\phi = \mathfrak{m}(\mathfrak{M}(B, A_1), A_2)$. Let $\psi$ be the unique morphism such that the diagram

(2.1)
$$\begin{CD}
B @>\phi_{12}>> (A_1\otimes A_2)\otimes\mathfrak{M}(B,A_1\otimes A_2) \\
@V\phi_1VV @VV i_{(A_1\otimes A_2)}\otimes\psi V \\
A_1\otimes\mathfrak{M}(B,A_1) @>i_{A_1}\otimes\phi>> (A_1\otimes A_2)\otimes\mathfrak{M}(\mathfrak{M}(B,A_1),A_2)
\end{CD}$$

is commutative. Let $\varphi$ and $\psi'$ be morphisms that make the following diagrams commutative:

(2.2)
$$\begin{array}{ccc}
B & \xrightarrow{\ \phi_1\ } & A_1\otimes\mathfrak{M}(B,A_1) \\
& \overset{\phi_{12}}{\searrow} & \downarrow{\scriptstyle i_{A_1}\otimes\varphi} \\
& & A_1\otimes(A_2\otimes\mathfrak{M}(B,A_1\otimes A_2))
\end{array}$$

(2.3)
$$\begin{array}{ccc} \mathfrak{M}(B, A_1) & \xrightarrow{\phi} & A_2 \otimes \mathfrak{M}(\mathfrak{M}(B, A_1), A_2) \\ & \searrow^{\varphi} & \downarrow{\scriptstyle i_{A_2} \otimes \psi'} \\ & & A_2 \otimes \mathfrak{M}(B, A_1 \otimes A_2). \end{array}$$

By (2.3), we have $i_{A_1} \otimes \varphi = (i_{(A_1 \otimes A_2)} \otimes \psi')(i_{A_1} \otimes \phi)$ and so by (2.1) and (2.2) we have

$$\begin{aligned} (i_{(A_1 \otimes A_2)} \otimes (\psi'\psi))\phi_{12} &= (i_{(A_1 \otimes A_2)} \otimes \psi')(i_{(A_1 \otimes A_2)} \otimes \psi)\phi_{12} \\ &= (i_{(A_1 \otimes A_2)} \otimes \psi')(i_{A_1} \otimes \phi)\phi_1 \\ &= (i_{A_1} \otimes \varphi)\phi_1 = \phi_{12}. \end{aligned}$$

This equality together with the universality of $\mathfrak{M}(B, A_1 \otimes A_2)$ implies that $\psi'\psi$ is the identity morphism. Let $b \in B$, let $f\colon A_1 \to \mathbb{K}$ be a module homomorphism and $c = (f \otimes i_{\mathfrak{M}(B,A_1)})\phi_1(b)$ (note that by Proposition 2.3 (a) such an element generates $\mathfrak{M}(B, A_1)$). By (2.2) and (2.1) we have

$$\begin{aligned} (i_{A_2} \otimes \psi)\varphi(c) &= (i_{A_2} \otimes \psi)\varphi(f \otimes i_{\mathfrak{M}(B,A_1)})\phi_1(b) \\ &= (i_{A_2} \otimes \psi)(f \otimes i_{(A_2 \otimes \mathfrak{M}(B,A_1 \otimes A_2))})\phi_{12}(b) \\ &= (f \otimes i_{(A_2 \otimes \mathfrak{M}(\mathfrak{M}(B,A_1),A_2))})(i_{A_1} \otimes \phi)\phi_1(b) \\ &= \phi(f \otimes i_{\mathfrak{M}(B,A_1)})\phi_1(b) = \phi(c). \end{aligned}$$

Thus $(i_{A_2} \otimes \psi)\varphi = \phi$. This equality together with (2.3) implies,

$$\begin{aligned} (i_{A_2} \otimes (\psi\psi'))\phi &= (i_{A_2} \otimes \psi)(i_{A_2} \otimes \psi')\phi \\ &= (i_{A_2} \otimes \psi)\varphi = \phi. \end{aligned}$$

Then the above equality together with the universality of $\mathfrak{M}(\mathfrak{M}(B, A_1), A_2)$ shows that $\psi\psi'$ is the identity morphism. This completes the proof. □

An analogue of Theorem 2.10 is also satisfied for the functor $\mathfrak{M}^{\mathrm{c}}$. The following lemma states a fact that will be used in Section 5.

**Lemma 2.11.** *Let $A$ and $B$ be algebras such that $A$ is commutative and* FR. *Then there is a canonical isomorphism $\mathfrak{M}(B^{\mathrm{op}}, A) \cong \mathfrak{M}(B, A)^{\mathrm{op}}$.*

P r o o f. Straightforward. □

## 3. Algebraic families of isomorphisms

In this section we define *algebraic families of isomorphisms* and *algebraic families of all isomorphisms.*

**Definition 3.1.** Let $A$ and $B$ be algebras and let $\psi\colon B \to A \otimes C$ be a family of morphisms from $B$ to $A$. Suppose that there exists a family $\phi\colon A \to B \otimes C$ such that $(i_B \otimes \mu_C)(\phi \circ \psi) = i_B \otimes 1$ and $(i_A \otimes \mu_C)(\psi \circ \phi) = i_A \otimes 1$. Then $\psi$ is called a family of isomorphisms from $B$ onto $A$ or, an invertible family of morphisms from $B$ to $A$. Also, $\phi$ is called an inverse for $\psi$. The class of all invertible families of morphisms from $B$ to $A$ is denoted by $\mathcal{I}(B, A)$.

Suppose that $\psi\colon B \to A \otimes C$ is an invertible family and let $\phi_1, \phi_2\colon A \to B \otimes C$ be two inverses for $\psi$. Then from the identity $(i_A \otimes \mu_C)(\psi \otimes i_C)\phi_2(a) = a \otimes 1$ $(a \in A)$ it is easily seen that $\phi_1 = (i_B \otimes \mu_C)(\phi_1 \otimes i_C)(i_A \otimes \mu_C)(\psi \otimes i_C)\phi_2$. Also it follows from the associativity of $\mu_C$ (i.e. $\mu_C(i_C \otimes \mu_C) = \mu_C(\mu_C \otimes i_C)$) and the identity $(i_B \otimes \mu_C)(\phi_1 \otimes i_C)\psi(b) = b \otimes 1$, $b \in B$, that $i_{B\otimes C} = (i_B \otimes \mu_C)(\phi_1 \otimes i_C)(i_A \otimes \mu_C)(\psi \otimes i_C)$. Thus we conclude $\phi_1 = \phi_2$. So we denote the inverse of an invertible family $\psi$ by $\psi^{-1}$. It is also easily checked that the composition of $\psi$ with another invertible family $\psi'\colon B' \to B \otimes C'$ is an invertible family, too. Indeed, $(\psi \circ \psi')^{-1} = (i_{B'} \otimes F)(\psi'^{-1} \circ \psi^{-1})$.

**Example 3.2.** (a) Suppose that $f_1, \ldots, f_n$ is a finite collection of isomorphisms from $B$ onto $A$. So the trivial family $\psi\colon B \to A \otimes \mathbb{K}^n$, defined by $\psi(b) = \sum_{i=1}^{n} f_i(b) \otimes e_i$, $b \in B$, is an invertible family. Indeed, $\psi^{-1}(a) = \sum_{i=1}^{n} f_i^{-1}(a) \otimes e_i$, $a \in A$. Also, note that the empty family of morphisms is an invertible family.

(b) Let $(B, \Delta, \varepsilon, S)$ be a commutative Hopf-algebra and $V$ a $B$-comodule with the coaction $\varrho\colon V \to V \otimes B$. Let $\varrho' = (i_V \otimes S)\varrho$. We have

$$\begin{aligned}
(i_V \otimes \mu_B)(\varrho' \circ \varrho) &= (i_V \otimes \mu_B)([(i_V \otimes S)\varrho] \otimes i_B)\varrho \\
&= (i_V \otimes \mu_B)(i_V \otimes S \otimes i_B)(\varrho \otimes i_B)\varrho \\
&= (i_V \otimes \mu_B)(i_V \otimes S \otimes i_B)(i_V \otimes \Delta)\varrho \\
&= (i_V \otimes [\mu_B(S \otimes i_B)\Delta])\varrho \\
&= (i_V \otimes \varepsilon 1_B)\varrho = i_V \otimes 1_B.
\end{aligned}$$

Analogously, we have $(i_V \otimes \mu_B)(\varrho \circ \varrho') = i_V \otimes 1_B$. Therefore, $\varrho$ is an invertible family with inverse $\varrho'$.

**Proposition 3.3.** *Let $A$ and $B$ be* FR *algebras. Then there exists an $\mathcal{I}(B,A)$-universal family, say $\phi\colon B \to A\otimes Z$. Moreover, this family has the following additional properties:*

(a) *$Z$ is a finitely generated algebra.*

(b) *There is a canonical one-to-one correspondence between isomorphisms from $B$ onto $A$ and characters of $Z$.*

*Analogues of the above statements also hold for the subclass $\mathcal{I}^{\mathrm{c}}(B,A)$. Indeed, the canonical morphism $\phi^{\mathrm{c}}\colon B\to A\otimes Z/[Z,Z]$, induced by $\phi$, is an $\mathcal{I}^{\mathrm{c}}(B,A)$-universal family.*

We call both $\phi$ and $\phi^{\mathrm{c}}$ *algebraic family of all isomorphisms from $B$ onto $A$*, and sometimes denote them by $\mathfrak{i}(B,A)$ and $\mathfrak{i}^{\mathrm{c}}(B,A)$, and their parameter-algebras by $\mathfrak{I}(B,A)$ and $\mathfrak{I}^{\mathrm{c}}(B,A)$, respectively.

Proof. Suppose that $A_0=\{a_i\colon i=1,\dots,a_m\}$ and $B_0=\{b_j\colon j=1,\dots,b_n\}$ are, respectively, bases for $A$ and $B$ as free-modules. Let $Z$ be the universal algebra generated by $z_{ji}$ and $z'_{ij}$ such that the following four classes of equations are satisfied:

(i) $q\Big(\sum\limits_{i=1}^{m} a_i\otimes z_{1i},\dots,\sum\limits_{i=1}^{m} a_i\otimes z_{ni}\Big)=0$ for every $q\in I_{B_0}$.

(ii) $p\Big(\sum\limits_{j=1}^{n} b_j\otimes z'_{1j},\dots,\sum\limits_{j=1}^{n} b_j\otimes z'_{mj}\Big)=0$ for every $p\in I_{A_0}$.

(iii) $\sum\limits_{i=1}^{m} z'_{ik}z_{ji}=0$ and $\sum\limits_{i=1}^{m} z'_{ij}z_{ji}=1$ for $1\leqslant j,\ k\leqslant n$ with $j\neq k$.

(iv) $\sum\limits_{j=1}^{n} z_{jl}z'_{ij}=0$ and $\sum\limits_{j=1}^{n} z_{ji}z'_{ij}=1$ for $1\leqslant i,\ l\leqslant m$ with $i\neq l$.

(i) shows that the assignment $b_j\mapsto\sum\limits_{i=1}^{m} a_i\otimes z_{ji}$ defines a morphism $\phi\colon B\to A\otimes Z$. Similarly, (ii) shows that the assignment $a_i\mapsto\sum\limits_{j=1}^{n} b_j\otimes z'_{ij}$ defines a morphism $\phi'\colon A\to B\otimes Z$. (iii) shows that $(i_B\otimes\mu_Z)(\phi'\otimes i_Z)\phi(b_j)=b_j\otimes 1$. Similarly, (iv) shows that $(i_A\otimes\mu_Z)(\phi\otimes i_Z)\phi'(a_i)=a_i\otimes 1$. So $\phi$ is a family of isomorphisms from $B$ onto $A$ with inverse $\phi'$. Now suppose that $\psi\colon B\to A\otimes C$ is a family of isomorphisms and let $c_{ji}$ and $c'_{ij}$ in $C$ be such that $\psi(b_j)=\sum\limits_{i=1}^{m} a_i\otimes c_{ji}$ and $\psi^{-1}(a_i)=\sum\limits_{j=1}^{n} b_j\otimes c'_{ij}$. Then the analogs of the above four classes of equations are satisfied for $c_{ji}$ and $c'_{ij}$. Thus, the universality of $Z$ shows that the assignments $z_{ji}\mapsto c_{ji}$ and $z'_{ij}\mapsto c'_{ij}$ define an algebra morphism $\overline{\psi}\colon Z\to C$ which satisfies $\psi=(i_A\otimes\overline{\psi})\phi$. Uniqueness of $\overline{\psi}$ also follows from universality of $Z$. Therefore $\phi$ is a universal family of isomorphisms. (a) is trivial by the construction of $Z$. Let $\psi\colon B\to A$ be an isomorphism. Then $\psi$ can be viewed as a family of isomorphisms with the parameter-algebra $\mathbb{K}$, and so, $\overline{\psi}$ is a character on $Z$. This proves (b). The other statements follow from Lemma 2.2. $\square$

**Remark 3.4.** Let $A$ and $B$ be FR algebras.

(a) It follows from Proposition 3.3 (b) that if there is an isomorphism from $B$ onto $A$ then $\mathfrak{I}(B,A) \neq 0$ and also $\mathfrak{I}^{c}(B,A) \neq 0$.

(b) Conversely, if $\mathbb{K}$ is an algebraically closed field, then from $\mathfrak{I}^{c}(B,A) \neq 0$ it follows that $B$ and $A$ are isomorphic (see Remark 2.6 (d)).

Let us call two algebras $A$ and $B$ *quasi-isomorphic* if there is a nonempty algebraic family of isomorphisms from $B$ onto $A$. (Note that this notion is very far from the notion of quasi-isomorphic chain complexes in homological algebra.) Then quasi-isomorphism is an equivalence relation on the class of algebras. It is clear that isomorphism (of algebras) implies quasi-isomorphism. But there are non-isomorphic algebras which are quasi-isomorphic: Let $C$ be an algebra which has not IBN, see [8], that is there are natural numbers $n$ and $m$ with $n \neq m$ such that $C^n$ and $C^m$ are isomorphic left free $C$-modules. Equivalently, there are elements $s_{ji}$ and $t_{ij}$ in $C$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ such that $\sum_{i=1}^{m} s_{ji}t_{ij'} = \delta_{jj'}$ and $\sum_{j=1}^{n} t_{ij}s_{ji'} = \delta_{ii'}$. Let $A = \mathbb{K}[x_1,\ldots,x_m]$ and $B = \mathbb{K}[y_1,\ldots,y_n]$ and let the families $\psi\colon A \to B \otimes C$ and $\phi\colon B \to A \otimes C$ be defined by $\psi(x_i) = \sum_{j=1}^{n} y_j \otimes s_{ji}$ and $\phi(y_j) = \sum_{i=1}^{m} x_i \otimes t_{ij}$. Then it is easily checked that $\psi = \phi^{-1}$. So, $A$ and $B$ are quasi-isomorphic. Indeed, this is the case for every $n$ and $m$; this follows from the fact that if $C$ is the algebra of all module endomorphisms on the infinite direct sum $\bigoplus_{i=1}^{\infty} \mathbb{K}$ then $C^n$ and $C^m$ are isomorphic for every $n$ and $m$ as left $C$-modules (see [8], Example 1.4).

We end this section by a list of interesting questions. Is there any relation between quasi-isomorphism and Morita equivalence, or algebraic homotopy equivalence (see [5], [7])? Morita equivalent algebras are characterized by the existence of some specific bimodules, see [8], Section 18C. Is it possible to characterize quasi-isomorphic algebras in this manner? For any fixed algebra $C$, let $\mathrm{Alg}^*$ be a category whose objects are algebras and a typical morphism from $B$ to $A$ is a family $\phi\colon B \to A \otimes C^{\otimes n}$, for some $n \geqslant 0$, together with the composition of families as the composition law of the category. (Note that if $C = \mathbb{K}$ then $\mathrm{Alg}^* = \mathrm{Alg}$.) Is there a Quillen's model category structure (see [3]) on $\mathrm{Alg}^*$ such that algebraic families of isomorphisms play the role of quasi-equivalences of the model? Indeed the term quasi-isomorphism introduced above refers to this (conjectured) property.

Let $A$ and $B$ be fixed algebras. Consider the following four types of algebraic families.

(1) Let $M$ be a $\mathbb{K}$-module and let $\beta\colon M \to B$ and $\alpha\colon M \to A$ be module homomorphisms. $\mathcal{M}_1$ denotes the class of all families $\psi\colon B \to A \otimes C$ satisfying

$$\alpha(m) \otimes 1 = \psi\beta(m), \quad m \in M.$$

(2) Let $M'$ be a $\mathbb{K}$-module and let $\beta'\colon B \to M$ and $\alpha'\colon A \to M$ be module homomorphisms. $\mathcal{M}_2$ denotes the class of all families $\psi\colon B \to A \otimes C$ satisfying

$$\beta'(b) \otimes 1 = (\alpha' \otimes i_C)\psi(b), \quad b \in B.$$

(3) Let $\Delta\colon B \to B \otimes B$ and $\Gamma\colon A \to A \otimes A$ be module homomorphisms. $\mathcal{M}_3$ denotes the class of all families $\psi\colon B \to A \otimes C$ satisfying

$$(\Gamma \otimes i_C)\psi = (i_{A\otimes A} \otimes \mu_C)(i_A \otimes F \otimes i_C)(\psi \otimes \psi)\Delta.$$

(4) Let $N$ be a $\mathbb{K}$-module and let $\Lambda\colon B \to B \otimes N$ and $\Theta\colon A \to A \otimes N$ be module homomorphisms. $\mathcal{M}_4$ denotes the class of all families $\psi\colon B \to A \otimes C$ satisfying

$$(i_A \otimes F)(\Theta \otimes i_C)\psi = (\psi \otimes i_N)\Lambda.$$

The algebraic families introduced in (1)–(4) are respectively analogues of the following four classes of classical families of maps:

(1′) Families of bundle morphisms between two fibre bundles over a classical space.

(2′) Families of continuous maps between two classical spaces which are identity over a common subspace.

(3′) Families of continuous group homomorphisms between two topological groups.

(4′) Families of $G$-maps between two $G$-spaces for a topological group $G$.

All proofs of theorems below are analogous to the proofs of Propositions 2.3 and 3.3, and are omitted for brevity.

**Theorem 4.1.** *Together with the assumptions of* (1) *suppose that $A$ is* FR. *Then there exists an $\mathcal{M}_1$-universal family, denoted by $\phi_1\colon B \to A \otimes Z_1$. Moreover, there is a canonical one-to-one correspondence between characters of $Z_1$ and morphisms $f\colon B \to A$ satisfying $f\beta = \alpha$.*

**Theorem 4.2.** *Together with the assumptions of* (2) *suppose that $A$ is* FR *and $M'$ is free as module. Then there is an $\mathcal{M}_2$-universal family, denoted by $\phi_2\colon B \to A \otimes Z_2$. Moreover, there is a canonical one-to-one correspondence between characters of $Z_2$ and morphisms $f\colon B \to A$ satisfying $\beta' = \alpha' f$.*

**Theorem 4.3.** *Together with the assumptions of* (3) *suppose that $B$ is free as module and $A$ is* FR. *Then there is an $\mathcal{M}_3$-universal family, denoted by $\phi_3\colon B \to A \otimes Z_3$. Moreover, there is a canonical one-to-one correspondence between characters of $Z_3$ and morphisms $f\colon B \to A$ satisfying $(f \otimes f)\Delta = \Gamma f$.*

**Theorem 4.4.** *Together with the assumptions of* (4) *suppose that $B$ and $N$ are free as modules and $A$ is* FR. *Then there is an $\mathcal{M}_4$-universal family, denoted by $\phi_4\colon B \to A \otimes Z_4$. Moreover, there is a canonical one-to-one correspondence between characters of $Z_4$ and morphisms $f\colon B \to A$ satisfying $(f \otimes i_N)\Lambda = \Theta f$.*

**Remark 4.5.** Under the assumptions of Theorems 4.1–4.4, it follows from Lemma 2.2 that there exists an $\mathcal{M}_i^{\mathrm{c}}$-universal family. Moreover, the parameter-algebra $Z_i^{\mathrm{c}}$ of that family is canonically isomorphic to $Z_i/[Z_i, Z_i]$. Also, if there is a morphism $f\colon B \to A$ which satisfies the conditions of Theorems 4.1–4.4, then $Z_i^{\mathrm{c}} \neq 0$.

Let $\mathcal{I}_i$, $i = 1, \dots, 4$, denote the subclass of $\mathcal{M}_i$ containing invertible families.

**Theorem 4.6.** *Suppose that $A$ and $B$ are* FR *and the assumptions of Theorems* 4.1–4.4 *are satisfied. Then $\mathcal{I}_i$ and $\mathcal{I}_i^{\mathrm{c}}$ have universal families.*

In the next sections, we show that the constructions achieved in this section may be endowed with Hopf-algebra structures in somewhat natural manner.

## 5. The action of $\mathfrak{M}(\cdot, A)$ on Hopf-algebras

This section is devoted to the construction of a Hopf-algebra structure on parameter-algebras of families of morphisms from FR commutative algebras to Hopf-algebras. Most of the results in this section are purely algebraic analogs of the results of [17] and [13]. By using the fact that any commutative finite dimensional C*-algebra is isomorphic to a finite product $\mathbb{C} \oplus \ldots \oplus \mathbb{C}$, Sołtan in [17] showed that this construction in the C*-case is a special case of the Wang free product of compact quantum groups [19], see also Remark 5.7 below.

Let $A$ be a FR commutative algebra and $B$ an arbitrary algebra. Let $\phi = \mathfrak{m}(B, A)$ and $C = \mathfrak{M}(B, A)$. Suppose that $\Delta$ is a comultiplication on $B$ and let $\overline{\Delta}\colon C \to C \otimes C$ be the composition of $\mathfrak{M}(\Delta)$ with the canonical morphism from $\mathfrak{M}(B \otimes B, A)$ to

$C \otimes C$ (see Remark 2.9). More explicitly, $\overline{\Delta}$ is the unique morphism that makes the following diagram commutative:

$$\begin{array}{ccc}
B & \xrightarrow{\quad \phi \quad} & A \otimes C \\
\Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle i_A \otimes \overline{\Delta}} \\
B \otimes B & & A \otimes (C \otimes C) \\
\Big\downarrow{\scriptstyle \phi \otimes \phi} & & \Big\uparrow{\scriptstyle \mu_A \otimes i_{C \otimes C}} \\
A \otimes C \otimes A \otimes C & \xrightarrow{i_A \otimes F \otimes i_C} & (A \otimes A) \otimes (C \otimes C).
\end{array}$$

**Lemma 5.1.** $\overline{\Delta}$ *is a comultiplication on* $C$.

P r o o f. By virtue of the universality of $C$, it is enough to show that the equality

$$(i_A \otimes i_C \otimes \overline{\Delta})(i_A \otimes \overline{\Delta})\phi = (i_A \otimes \overline{\Delta} \otimes i_C)(i_A \otimes \overline{\Delta})\phi$$

is satisfied. We have

$$\begin{aligned}
(5.1)\quad (i_A \otimes i_C \otimes \overline{\Delta})(i_A \otimes \overline{\Delta}) &= (i_A \otimes i_C \otimes \overline{\Delta})(\mu_A \otimes i_{C \otimes C})F(\phi \otimes \phi)\Delta\phi \\
&= (\mu_A \otimes i_C \otimes \overline{\Delta})F(\phi \otimes \phi)\Delta = (\mu_A \otimes i_{C \otimes C \otimes C})F(i_{A \otimes C \otimes A} \otimes \overline{\Delta})(\phi \otimes \phi)\Delta \\
&= (\mu_A \otimes i_{C \otimes C \otimes C})F(i_{A \otimes C \otimes A} \otimes \overline{\Delta})(\phi \otimes i_{A \otimes C})(i_B \otimes \phi)\Delta \\
&= (\mu_A \otimes i_{C \otimes C \otimes C})F(\phi \otimes i_{A \otimes C \otimes C})(i_B \otimes i_A \otimes \overline{\Delta})(i_B \otimes \phi)\Delta \\
&= (\mu_A \otimes i_{C \otimes C \otimes C})F(\phi \otimes i_A \otimes \overline{\Delta})(i_B \otimes \phi)\Delta \\
&= (\mu_A \otimes i_{C \otimes C \otimes C})F(\phi \otimes [(i_A \otimes \overline{\Delta})\phi])\Delta \\
&= (\mu_A \otimes i_{C \otimes C \otimes C})F(\phi \otimes [(\mu_A \otimes i_{C \otimes C})F(\phi \otimes \phi)\Delta])\Delta \\
&= (\mu_A \otimes i_{C \otimes C \otimes C})F(i_{A \otimes C} \otimes \mu_A \otimes i_{C \otimes C})F(\phi \otimes \phi \otimes \phi)(i_B \otimes \Delta)\Delta,
\end{aligned}$$

$$\begin{aligned}
(5.2)\quad (i_A \otimes \overline{\Delta} \otimes i_C)(i_A \otimes \overline{\Delta}) &= (i_A \otimes \overline{\Delta} \otimes i_C)(\mu_A \otimes i_{C \otimes C})F(\phi \otimes \phi)\Delta\phi \\
&= (\mu_A \otimes \overline{\Delta} \otimes i_C)F(\phi \otimes \phi)\Delta \\
&= (\mu_A \otimes i_{C \otimes C \otimes C})F(i_A \otimes \overline{\Delta} \otimes i_A \otimes i_C)(\phi \otimes \phi)\Delta \\
&= (\mu_A \otimes i_{C \otimes C \otimes C})F([(i_A \otimes \overline{\Delta})\phi] \otimes \phi)\Delta \\
&= (\mu_A \otimes i_{C \otimes C \otimes C})F([(\mu_A \otimes i_{C \otimes C})F(\phi \otimes \phi)\Delta] \otimes \phi)\Delta \\
&= (\mu_A \otimes i_{C \otimes C \otimes C})F(\mu_A \otimes i_{C \otimes C \otimes A \otimes C})F(\phi \otimes \phi \otimes \phi)(\Delta \otimes i_B)\Delta.
\end{aligned}$$

It is easily checked that the morphisms

$$(\mu_A \otimes i_{C \otimes C \otimes C})F(i_{A \otimes C} \otimes \mu_A \otimes i_{C \otimes C})F \quad \text{and} \quad (\mu_A \otimes i_{C \otimes C \otimes C})F(\mu_A \otimes i_{C \otimes C \otimes A \otimes C})F$$

appearing, respectively, in the last rows of (5.1) and of (5.2) are equal to each other. Now, we get the result by using the identity $(i_B \otimes \Delta)\Delta = (\Delta \otimes i_B)\Delta$. □

**Lemma 5.2.** $\overline{\Delta}$ *is cocommutative if* $\Delta$ *is.*

Proof. Straightforward. $\square$

Suppose that $\varepsilon\colon B \to \mathbb{K}$ is a left counit for $B$. Let $\tilde{\varepsilon}\colon B \to A$ be defined by $b \mapsto \varepsilon(b)1$ and let $\bar{\varepsilon}$ be the unique character of $C$ satisfying $(i_A \otimes \bar{\varepsilon})\phi = \widetilde{\varepsilon}$ (indeed, using the notation of Theorem 2.7, $\bar{\varepsilon} = \mathfrak{M}(\varepsilon)$).

**Lemma 5.3.** *The character* $\bar{\varepsilon}$ *is a left counit on* $C$.

Proof. By the universal property of $C$ it is enough to show that

$$(i_A \otimes [(\bar{\varepsilon} \otimes i_C)\overline{\Delta}])\phi = \phi.$$

For every $b \in B$ we have

$$\begin{aligned}
(i_A \otimes [(\bar{\varepsilon} \otimes i_C)\overline{\Delta}])\phi(b) &= (i_A \otimes \bar{\varepsilon} \otimes i_C)(i_A \otimes \overline{\Delta})\phi(b)\\
&= (i_A \otimes \bar{\varepsilon} \otimes i_C)(m \otimes i_{C\otimes C})(i_A \otimes F \otimes i_C)(\phi \otimes \phi)\Delta(b)\\
&= (\mu_A \otimes \bar{\varepsilon} \otimes i_C)(i_A \otimes F \otimes i_C)(\phi \otimes \phi)\Delta(b)\\
&= (\mu_A \otimes i_C)(i_A \otimes \bar{\varepsilon} \otimes i_A \otimes i_C)(\phi \otimes \phi)\Delta(b)\\
&= (\mu_A \otimes i_C)([(i_A \otimes \bar{\varepsilon})\phi] \otimes \phi)\Delta(b)\\
&= (\mu_A \otimes i_C)(\widetilde{\varepsilon} \otimes \phi)\Delta(b)\\
&= (\mu_A \otimes i_C)(i_A \otimes \phi)(\widetilde{\varepsilon} \otimes i_B)\Delta(b)\\
&= (\mu_A \otimes i_C)(i_A \otimes \phi)(1_A \otimes b) = \phi(b).
\end{aligned}$$

$\square$

The analogue of the above result holds for a right counit. In particular, if $\varepsilon$ is a counit for $B$ then $\bar{\varepsilon}$ is a counit for $C$.

**Remark 5.4.** Let $B$ and $D$ be algebras such that $D$ is commutative. Then any bialgebra structure $(B, \Delta, \varepsilon)$ induces a monoidal structure ([9], Chapter 9) $\widehat{\otimes}$ on the category $\mathcal{M}(B, D)$ as follows. For any two families $\psi_1\colon B \to D \otimes E_1$ and $\psi_2\colon B \to D \otimes E_2$ of morphisms from $B$ to $D$ let

$$\psi_1 \widehat{\otimes} \psi_2 = (\mu_D \otimes i_{E_1 \otimes E_2})(i_D \otimes F \otimes i_{E_2})(\psi_1 \otimes \psi_2)\Delta.$$

The associativity of the bifunctor $\widehat{\otimes}$ follows from the coassociativity of $\Delta$, and the unit object of the monoidal structure is a family $B \to D$ defined by $b \mapsto \varepsilon(b)1$. If $\Delta$ is cocommutative then $\widehat{\otimes}$ is a symmetric monoidal structure. Moreover, suppose that $B$ is a cocommutative Hopf-algebra. Then for any character $\xi$ of $B$, the family $B \to D$, defined by $b \mapsto \xi(b)1$, is an invertible object of $\mathcal{M}(B, D)$. Moreover, any invertible object is of this form. It follows that the Picard group (see [6], [10]) of $\mathcal{M}(B, D)$ is isomorphic to the group of characters of $B$ (with group operation induced by $\Delta$).

Suppose now that $(B, \Delta, \varepsilon, S)$ is a Hopf-algebra. Let $\overline{S} = \mathfrak{M}(S)\colon C \to C^{\mathrm{op}}$, where $C^{\mathrm{op}}$ is identified with $\mathfrak{M}(B^{\mathrm{op}}, A)$ (see Lemma 2.11). We shall show that $\overline{S}$ is an antipode for the bialgebra $(C, \overline{\Delta}, \bar{\varepsilon})$. Suppose $b \in B$ and $f\colon A \to \mathbb{K}$ is a $\mathbb{K}$-module homomorphism. Let $c = (f \otimes i_C)\phi(b)$, and note that, by Proposition 2.3 (a), such elements generate $C$. Then

$$\begin{aligned}
\mu_C(i_C \otimes \overline{S})\overline{\Delta}(c) &= \mu_C(i_C \otimes \overline{S})\overline{\Delta}(f \otimes i_C)\phi(b) \\
&= (f \otimes \mu_C)(i_A \otimes i_C \otimes \overline{S})(i_A \otimes \overline{\Delta})\phi(b) \\
&= (f \otimes \mu_C)(i_A \otimes i_C \otimes \overline{S})(\mu_A \otimes i_{C\otimes C})(i_A \otimes F \otimes i_C)(\phi \otimes \phi)\Delta(b) \\
&= (f\mu_A \otimes \mu_C)(i_A \otimes i_C \otimes \overline{S})(\phi \otimes \phi)\Delta(b) \\
&= (f\mu_A \otimes \mu_C)(i_A \otimes F \otimes i_C)(i_{A\otimes C} \otimes i_A \otimes \overline{S})(\phi \otimes \phi)\Delta(b) \\
&= (f\mu_A \otimes \mu_C)(i_A \otimes F \otimes i_C)(\phi \otimes [(i_A \otimes \overline{S})\phi])\Delta(b) \\
&= (f\mu_A \otimes \mu_C)(i_A \otimes F \otimes i_C)(\phi \otimes \phi S)\Delta(b) \\
&= (f\mu_A \otimes \mu_C)(i_A \otimes F \otimes i_C)(\phi \otimes \phi)(i_B \otimes S)\Delta(b) \\
&= (f \otimes i_C)\mu_{A\otimes C}(\phi \otimes \phi)(i_B \otimes S)\Delta(b) = (f \otimes i_C)\phi\mu_B(i_B \otimes S)\Delta(b) \\
&= (f \otimes i_C)\phi(\varepsilon(b)1_B) = \varepsilon(b)f(1_A)1_C = \bar{\varepsilon}(c)1_C.
\end{aligned}$$

Analogously, we have $\mu_C(\overline{S} \otimes i_C)\overline{\Delta}(c) = \bar{\varepsilon}(c)1_C$. Thus $\overline{S}$ is an antipode for $C$. We have proved the following theorem.

**Theorem 5.5.** *Let $A$ be a* FR *commutative algebra and let $B$ be a (cocommutative) bialgebra. Then $\mathfrak{M}(B, A)$ has a canonical (cocommutative) bialgebra structure. Moreover, $\mathfrak{M}(B, A)$ is a Hopf-algebra if $B$ is.*

The analogue of this result is satisfied for $\mathfrak{M}^{\mathrm{c}}$:

**Theorem 5.6.** *Let $A$ be a* FR *commutative algebra and let $B$ be a Hopf-algebra or bialgebra. Then $\mathfrak{M}^{\mathrm{c}}(B, A)$ has, respectively, a canonical Hopf-algebra or bialgebra structure.*

Proof. For every commutative algebra $D$, the set $\mathrm{Alg}(B, A \otimes D)$ has a canonical group or monoid structure and thus the functor $\mathrm{Alg}(B, A \otimes \cdot)$ is group or monoid valued, respectively. By the analog of Proposition 2.3 (d) for $\mathfrak{M}^{\mathrm{c}}$, the algebra $\mathfrak{M}^{\mathrm{c}}(B, A)$ represents $\mathrm{Alg}(B, A \otimes \cdot)$, and thus has a canonical Hopf-algebra or bialgebra structure (see [11], Chapter I).

The theorem can of course be proved by another method: By Theorem 5.5, $\mathfrak{M}(B, A)$ is a Hopf-algebra or bialgebra. On the other hand, the algebra $\mathfrak{M}^{\mathrm{c}}(B, A)$ is the quotient of $\mathfrak{M}(B, A)$ by the commutator ideal. Since the commutator ideal is also a Hopf-ideal or biideal it follows that $\mathfrak{M}^{\mathrm{c}}(B, A)$ is a quotient Hopf-algebra or bialgebra, respectively.

In the case when $B$ is commutative, there is still another proof: $B$ is a group or monoid object in $\mathrm{Alg}^{\mathrm{c}}$. By Theorem 2.8, $\mathfrak{M}^{\mathrm{c}}(\cdot, A)$ transforms coproducts to coproducts and thus transforms group or monoid objects to group or monoid objects, respectively. □

**Remark 5.7.** (a) It follows from Remark 2.6 (b) that the free or tensor product of finitely many copies of a (commutative) Hopf-algebra or bialgebra has, respectively, a canonical Hopf-algebra or bialgebra structure. For example, the usual bialgebra structures on $\mathbb{K}[x_1, \ldots, x_n] = \mathfrak{M}(\mathbb{K}[x], \mathbb{K}^n)$ and $\mathbb{K}^{\mathrm{c}}[x_1, \ldots, x_n] = \mathfrak{M}^{\mathrm{c}}(\mathbb{K}[x], \mathbb{K}^n)$ ([18], Section 3.2) are induced by the usual bialgebra structure on $\mathbb{K}[x]$.

(b) It has been shown by Wang [19] that the free product of finitely many (C*-algebraic) compact quantum groups is a compact quantum group.

## 6. Quantum group of gauge transformations

In this short section we apply the construction of the preceding section to obtain a notion for Hopf-algebra of gauge transformations. We refer the reader to [4] for basic ideas and notions on the gauge theory on noncommutative spaces. Let $B$ be a Hopf-algebra with comultiplication $\Delta$ and counit $\varepsilon$ and let $E$ be a trivial left quantum vector bundle with fiber $V$ and base $A$ in the sense of [4]. This means that $A$ is an algebra and $V$ is a left $B$-comodule algebra with a coaction $\varrho$ and $E$ is (isomorphic to) the algebra $V \otimes A$. We slightly change the definitions in [4] and define $\gamma$ to be a quantum gauge transformation of $E$ if $\gamma$ is a morphism from $B$ to $A$. Then it is natural to define a gauge field just to be a morphism from $V$ to $A$. Now, it is clear that if $A$ is commutative and FR then the Hopf-algebra $C = \mathfrak{M}(B, A)$ and the algebra $H = \mathfrak{M}(V, A)$, respectively, play the role of the *quantum group of gauge transformations* and the *quantum space of gauge fields* in our dual setting. In the classical case, there is a canonical action of gauge transformations on gauge fields. This is also the case in our setting: Let $\overline{\Delta}, \bar{\varepsilon}$ and $\phi$ be as in the preceding section and let $\varphi = \mathfrak{m}(V, A)$. Let $\bar{\varrho}\colon H \to C \otimes H$ be the unique morphism satisfying $(i_A \otimes \bar{\varrho})\varphi = (\mu_A \otimes i_{C\otimes H})F(\phi \otimes \varphi)\varrho$. We show that $\bar{\varrho}$ is a coaction:

$$
\begin{aligned}
(i_A \otimes [(\overline{\Delta} \otimes i_H)\bar{\varrho}])\varphi &= (i_A \otimes \overline{\Delta} \otimes i_H)(i_A \otimes \bar{\varrho})\varphi \\
&= (i_A \otimes \overline{\Delta} \otimes i_H)(\mu_A \otimes i_{C\otimes H})F(\phi \otimes \varphi)\varrho \\
&= (\mu_A \otimes i_{C\otimes C\otimes H})F([(i_A \otimes \overline{\Delta})\phi] \otimes \varphi)\varrho \\
&= (\mu_A \otimes i_{C\otimes C\otimes H})F([(\mu_A \otimes i_{C\otimes C})F(\phi \otimes \phi)\Delta] \otimes \varphi)\varrho \\
&= (\mu_A \otimes i_{C\otimes C\otimes H})F(\phi \otimes \phi \otimes \varphi)(\Delta \otimes i_V)\varrho,
\end{aligned}
$$

$$
\begin{aligned}
(i_A \otimes [(i_C \otimes \bar{\varrho})\bar{\varrho}])\varphi &= (i_A \otimes i_C \otimes \bar{\varrho})(i_A \otimes \bar{\varrho})\varphi \\
&= (i_{A\otimes C} \otimes \bar{\varrho})(\mu_A \otimes i_{C\otimes H})F(\phi \otimes \varphi)\varrho \\
&= (\mu_A \otimes i_{C\otimes C\otimes H})F(i_{A\otimes C} \otimes i_A \otimes \bar{\varrho})(\phi \otimes \varphi)\varrho \\
&= (\mu_A \otimes i_{C\otimes C\otimes H})F(\phi \otimes [(i_A \otimes \bar{\varrho})\varphi])\varrho \\
&= (\mu_A \otimes i_{C\otimes C\otimes H})F(\phi \otimes [(\mu_A \otimes i_{C\otimes H})F(\phi \otimes \varphi)\varrho])\varrho \\
&= (\mu_A \otimes i_{C\otimes C\otimes H})F(\phi \otimes \phi \otimes \varphi)(i_B \otimes \varrho)\varrho.
\end{aligned}
$$

Thus $(i_A \otimes [(\overline{\Delta} \otimes i_H)\bar{\varrho}])\varphi = (i_A \otimes [(i_C \otimes \bar{\varrho})\bar{\varrho}])\varphi$ and it follows that $(\overline{\Delta} \otimes i_H)\bar{\varrho} = (i_C \otimes \bar{\varrho})\bar{\varrho}$. It remains to prove the counit identity $(\bar{\varepsilon} \otimes i_H)\bar{\varrho} = i_H$. It follows from the calculation below:

$$
\begin{aligned}
(i_A \otimes [(\bar{\varepsilon} \otimes i_H)\bar{\varrho}])\varphi &= (i_A \otimes \bar{\varepsilon} \otimes i_H)(i_A \otimes \bar{\varrho})\varphi \\
&= (i_A \otimes \bar{\varepsilon} \otimes i_H)(\mu_A \otimes i_{C\otimes H})F(\phi \otimes \varphi)\varrho \\
&= (\mu_A \otimes i_H)F([(i_A \otimes \bar{\varepsilon})\phi] \otimes \varphi)\varrho \\
&= \varphi(\varepsilon \otimes i_V)\varrho = \varphi.
\end{aligned}
$$

## 7. Pontryagin dual of a FR Hopf-algebra

In this section we consider, in our dual formalism, a construction analogous to the construction of the (semi)group of homomorphisms from a (semi)group to an abelian (semi)group, and as a consequence we obtain a notion of Pontryagin dual for FR Hopf-algebras.

Let $(B, \Delta, \varepsilon)$ be a cocommutative bialgebra and let $A$ be a FR commutative algebra together with an arbitrary morphism $\Gamma\colon A \to A \otimes A$. Suppose that $B$ is free as module. Then, by Theorem 4.3, the class $\mathcal{M}_3$ has a universal family, which we denote by $\psi\colon B \to A \otimes D$. Let $\chi\colon B \to A \otimes (D \otimes D)$ be the composition

$$
(\mu_A \otimes i_{D\otimes D})(i_A \otimes F \otimes i_D)(\psi \otimes \psi)\Delta.
$$

It follows from the coassociativity and cocommutativity of $\Delta$ that,

$$
\begin{aligned}
&(i_{A\otimes A} \otimes \mu_{D\otimes D})(i_A \otimes F \otimes i_{D\otimes D})(\chi \otimes \chi)\Delta \\
&\qquad = (\mu_{A\otimes A} \otimes i_{D\otimes D})(i_{A\otimes A} \otimes F \otimes i_D)([(\Gamma \otimes i_D)\psi] \otimes [(\Gamma \otimes i_D)\psi])\Delta \\
&\qquad = (\Gamma \otimes i_{D\otimes D})\chi.
\end{aligned}
$$

So, $\chi$ is in $\mathcal{M}_3$. Since $\psi$ is the universal family of $\mathcal{M}_3$, there is a unique morphism $\widetilde{\Delta}\colon D \to D \otimes D$ for which $\chi = (i_A \otimes \widetilde{\Delta})\psi$. In a manner analogous to the proofs of

Lemma 5.1 and Lemma 5.2, it is shown that $\widetilde{\Delta}$ is a cocommutative comultiplication. It is easily checked that the family $\varepsilon'\colon B \to A \otimes \mathbb{K}$ defined by $b \mapsto \varepsilon(b)1$ belongs to $\mathcal{M}_3$. So, by the universal property of $D$, there is a unique character $\widetilde{\varepsilon}$ on $D$ for which $(i_A \otimes \widetilde{\varepsilon})\psi = \varepsilon'$. A proof analogous to the proof of Lemma 5.3 shows that $\widetilde{\varepsilon}$ is a counit for $\widetilde{\Delta}$. Moreover, if the bialgebra $B$ has an antipode $S$, then the appropriately modified versions of Lemma 2.11 and the above proofs show that $D$ has an antipode. So, we have proved the theorem below.

**Theorem 7.1.** *Let $B$ be a cocommutative bialgebra with comultiplication $\Delta$, let $A$ be a* FR *commutative algebra, and let $\Gamma\colon A \to A \otimes A$ be a morphism. Suppose that $B$ is free as $\mathbb{K}$-module and let $D$ be the parameter-algebra of the universal family of the class of families $\psi'\colon B \to A \otimes D'$ for which $(\Gamma \otimes i_{D'})\psi' = (i_{A\otimes A} \otimes \mu_{D'})(i_A \otimes F \otimes i_{D'})(\psi' \otimes \psi')\Delta$. Then $D$ has a canonical cocommutative bialgebra structure. Moreover, $D$ is a Hopf-algebra if $B$ is.*

A similar result also holds for the universal family of $\mathcal{M}_3^{\mathrm{c}}$.

We want now to offer a notion for *Pontryagin dual* of a FR commutative Hopf-algebra. For clarity of the discussion, we first recall some standard facts. Given a FR Hopf-algebra $A$ there is another FR Hopf-algebra which we call *algebraic dual* of $A$ and denote by $A^*$. The underlying module of $A^*$ is $\mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$, the set of all module homomorphisms from $A$ to $\mathbb{K}$. (Co)multiplication of $A^*$ is induces by that of $A$ via canonical isomorphism $\mathrm{Hom}_{\mathbb{K}}(A \otimes A, \mathbb{K}) \simeq \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K}) \otimes \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$. (Co)unit and antipode are induced by the usual duality between $A$ and $\mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$. Note that $A^{**} \simeq A$ as Hopf-algebras. For a finite group $G$ the group algebra $\mathbb{K}G$ (with convolution multiplication) is also a FR Hopf-algebra with comultiplication, counit and antipode given, respectively, by $1_g \mapsto 1_g \otimes 1_g$, $1_g \mapsto 1$ and $1_g \mapsto 1_{g^{-1}}$, where $1_g\colon G \to \mathbb{K}$ is defined by $1_g(g) = 1$ and $1_g(g') = 0$ for $g \neq g'$. Then the algebraic dual of $\mathbb{K}G$ is canonically isomorphic to the function algebra on $G$, denoted by $\mathbb{K}(G)$, with pointwise multiplication and with comultiplication, counit and antipode given, respectively, by $1_g \mapsto \sum_{h\in G} 1_h \otimes 1_{h^{-1}g}$, $1_g \mapsto 1_g(e)$ and $1_g \mapsto 1_{g^{-1}}$.

Recall that, in classical harmonic analysis, to any compact or discrete abelian group $G$ there corresponds its Pontryagin dual $\widehat{G}$ which is, respectively, a discrete or compact abelian group. $\widehat{G}$ is the pointwise multiplication group of all group homomorphisms from $G$ to the multiplicative group of complex numbers of absolute value 1. Similarly to the algebraic dual of Hopf-algebras we have Pontryagin's duality: $G \simeq \widehat{\widehat{G}}$. It is not hard to see that if $G$ is finite then $G \simeq \widehat{G}$, $\mathbb{C}G \simeq \mathbb{C}(\widehat{G})$ and $\mathbb{C}(G) \simeq \mathbb{C}\widehat{G}$ (see [9]). Also note that in this case any group homomorphisms from $G$ to $\mathbb{C} - \{0\}$ takes values automatically on the unit circle.

A purely algebraic analog of the group $\mathbb{C} - \{0\}$ is the Hopf-algebra $(\mathbb{K}^\dagger, \Delta, \varepsilon, S)$ that represents the affine group scheme of invertible elements of $\mathbb{K}$-algebras (see [11]). The underlying algebra of $\mathbb{K}^\dagger$ is $\mathbb{K}^{\mathrm{c}}[x,y]/(xy-1)$ and the Hopf-algebra operations are defined by $\Delta(x) = x \otimes x$, $\Delta(y) = y \otimes y$, $\varepsilon(x) = \varepsilon(y) = 1$, $S(x) = y$ and $S(y) = x$. Note that $\mathbb{K}^\dagger$ is a free module.

Now we are in a position to introduce the notion of the Pontryagin dual: Let $A$ be a FR commutative Hopf-algebra. Denote by $\mathfrak{p}\colon \mathbb{K}^\dagger \to A \otimes \mathfrak{P}(A)$ the universal family defined by Theorem 7.1 with $B = \mathbb{K}^\dagger$. The corresponding universal family with commutative parameter-algebra is denoted by $\mathfrak{p}^{\mathrm{c}}\colon \mathbb{K}^\dagger \to A \otimes \mathfrak{P}^{\mathrm{c}}(A)$. Then, it is reasonable to call both the Hopf-algebras $\mathfrak{P}(A)$ and $\mathfrak{P}^{\mathrm{c}}(A)$ the *Pontryagin dual* of $A$.

It is not clear to the author how in general $A^*$ and $\mathfrak{P}(A)$ are related. But we have the following observation in the special case of finite groups. Let $G$ be a finite abelian group and suppose that $\mathbb{K}$ is a field (or more generally, a ring for which any equation $z^n = 1$ has only finitely many solutions where $n$ is any divisor of $|G|$). Then the multiplicative group $\widehat{G}_{\mathbb{K}}$ of all group homomorphisms from $G$ to the group of units of $\mathbb{K}$ is finite. Let $A = \mathbb{K}(G)$ and let the family $\psi\colon \mathbb{K}^\dagger \to A \otimes \mathbb{K}(\widehat{G}_{\mathbb{K}})$ be defined by $\psi(x) = \sum \alpha(g) 1_g \otimes 1_\alpha$ and $\psi(y) = \sum \alpha(g)^{-1} 1_g \otimes 1_\alpha$ where the sums are taken over all $g \in G$ and $\alpha \in \widehat{G}_{\mathbb{K}}$. It is easily checked that $\psi$ satisfies the condition of Theorem 7.1 and so there is a unique morphism $\varphi\colon \mathfrak{P}^{\mathrm{c}}(A) \to \mathbb{K}(\widehat{G}_{\mathbb{K}})$ for which $\psi = (i_{\mathbb{K}(G)} \otimes \varphi)\mathfrak{p}^{\mathrm{c}}$. Now if $\mathbb{K}$ is an algebraically closed field then a deduction analogous to Remark 2.6 (e) shows that $\mathfrak{P}^{\mathrm{c}}(A)/J \simeq \mathbb{K}(\widehat{G}_{\mathbb{K}})$ and so if $\mathbb{K} = \mathbb{C}$ then $\mathfrak{P}^{\mathrm{c}}(A)/J \simeq A^*$ as Hopf-algebras.

## 8. Hopf-algebra structure of the family of all automorphisms

In this section we consider a canonical bialgebra structure on the parameter-algebra of the family of all endomorphisms of a FR algebra. This construction is analogous to the semigroup of self-maps on an ordinary space.

Let $A$ be a FR algebra and let $C = \mathfrak{M}(A,A)$ and $\phi = \mathfrak{m}(A,A)$. Let $\Gamma\colon C \to C \otimes C$ be the unique morphism satisfying $(i_A \otimes \Gamma)\phi = \phi \circ \phi$. Then

$$\begin{aligned}(i_A \otimes [(\Gamma \otimes i_C)\Gamma])\phi &= (i_A \otimes \Gamma \otimes i_C)(i_A \otimes \Gamma)\phi = (i_A \otimes \Gamma \otimes i_C)(\phi \circ \phi) \\ &= (i_A \otimes \Gamma \otimes i_C)(\phi \otimes i_C)\phi = ([(i_A \otimes \Gamma)\phi] \otimes i_C)\phi \\ &= ([(\phi \otimes i_C)\phi] \otimes i_C)\phi = (\phi \otimes i_{C\otimes C})(\phi \otimes i_C)\phi,\end{aligned}$$

and also,

$$\begin{aligned}(i_A \otimes [(i_C \otimes \Gamma)\Gamma])\phi &= (i_A \otimes i_C \otimes \Gamma)(i_A \otimes \Gamma)\phi = (i_A \otimes i_C \otimes \Gamma)(\phi \otimes i_C)\phi \\ &= (\phi \otimes \Gamma)\phi = (\phi \otimes i_{C\otimes C})(i_A \otimes \Gamma)\phi = (\phi \otimes i_{C\otimes C})(\phi \otimes i_C)\phi.\end{aligned}$$

So, $(i_A\otimes[(\Gamma\otimes i_C)\Gamma])\phi = (i_A\otimes[(i_C\otimes\Gamma)\Gamma])\phi$. This identity together with the universal property of $C$ implies $(\Gamma\otimes i_C)\Gamma = (i_C\otimes\Gamma)\Gamma$. Thus, $\Gamma$ is a comultiplication. (One may also apply the method of the proof of [15], Theorem 4.1, using Proposition 2.3 (a).) Now, let $\varepsilon\colon\ C \to \mathbb{K}$ be the unique character satisfying $(i_A \otimes \varepsilon)\phi = i_A \otimes 1$. Then

$$\begin{aligned}(i_A \otimes [(i_C \otimes \varepsilon)\Gamma])\phi &= (i_A \otimes i_C \otimes \varepsilon)(i_A \otimes \Gamma)\phi = (i_A \otimes i_C \otimes \varepsilon)(\phi \otimes i_C)\phi \\ &= (\phi \otimes i_{\mathbb{K}})(i_A \otimes \varepsilon)\phi = \phi \otimes 1 = (i_A \otimes [i_C \otimes 1])\phi.\end{aligned}$$

This equality together with the universal property of $C$ implies $(i_C \otimes \varepsilon)\Gamma = i_C \otimes 1$. Analogously, $(\varepsilon \otimes i_C)\Gamma = i_C \otimes 1$. So, $\varepsilon$ is a counit for $\Gamma$ and $(C, \Gamma, \varepsilon)$ is a bialgebra. Also, it is easily seen that $A$ is a $C$-comodule via $\phi$. We now show that $\phi$ is a universal coaction among all coactions of bialgebras on $A$. Let $D$ be a bialgebra with a comultiplication $\Theta$, and suppose that we are given a coaction $\varphi\colon\ A \to A \otimes D$ of $D$ on $A$. Let $\psi\colon\ C \to D$ be the unique morphism satisfying $(i_A \otimes \psi)\phi = \varphi$. Then

$$\begin{aligned}(i_A \otimes [(\psi \otimes \psi)\Gamma])\phi &= (i_A \otimes \psi \otimes \psi)(i_A \otimes \Gamma)\phi = (i_A \otimes \psi \otimes \psi)(\phi \otimes i_C)\phi \\ &= ([(i_A \otimes \psi)\phi] \otimes \psi)\phi = (\varphi \otimes \psi)\phi = (\varphi \otimes i_D)(i_A \otimes \psi)\phi \\ &= (\varphi \otimes i_D)\varphi = (i_A \otimes \Theta)\varphi = (i_A \otimes \Theta)(i_A \otimes \psi)\phi = (i_A \otimes \Theta\psi)\phi.\end{aligned}$$

The above identity together with the universal property of $C$ implies $\Theta\psi = (\psi\otimes\psi)\Gamma$, which means $\psi$ is a bialgebra morphism. So, we have proved the theorem below.

**Theorem 8.1.** *Let $A$ be a* FR *algebra. Then $\mathfrak{M}(A, A)$ is a bialgebra in a canonical way and $A$ is an $\mathfrak{M}(A, A)$-comodule via $\mathfrak{m}(A, A)$. Moreover, $\mathfrak{m}(A, A)$ is a universal coaction in the following sense: For any bialgebra $D$ and any coaction $\varphi\colon\ A \to A\otimes D$ of $D$ on $A$, the unique morphism $\psi\colon\ \mathfrak{M}(A, A) \to D$ satisfying $(i_A \otimes \psi)\mathfrak{m}(A, A) = \varphi$ is also a bialgebra morphism and is compatible with the comodule structures on $A$.*

An analogue of this result also holds for $\mathfrak{M}^{\mathrm{c}}(A, A)$.

Now, we consider the family of isomorphisms. Let $G = \mathfrak{I}(A, A)$, and $\xi = \mathfrak{i}(A, A)$. Since $\xi \circ \xi\colon\ A \to A \otimes (G \otimes G)$ and $i_A \otimes 1\colon\ A \to A \otimes \mathbb{K}$ are families of isomorphisms from $A$ onto $A$, there are unique morphisms $\Lambda\colon\ G \to G \otimes G$ and $\delta\colon\ G \to \mathbb{K}$ which satisfy $(i_A \otimes \Lambda)\xi = \xi \circ \xi$ and $(i_A \otimes \delta)\xi = i_A \otimes 1$. A proof analogous to the above proof shows that,

$$(i_A \otimes [(i_G \otimes \Lambda)\Lambda])\xi = (\xi \otimes i_{G\otimes G})(\xi \otimes i_G)\xi = (i_A \otimes [(i_G \otimes \Lambda)\Lambda])\xi.$$

Since the middle term of the above equality is an invertible family, the universal property of $G$ shows that $(\Lambda \otimes i_G)\Lambda = (i_G \otimes \Lambda)\Lambda$. Also, we may similarly conclude $(\delta \otimes i_G)\Lambda = i_G \otimes 1$. So, we have proved that $(G, \Lambda, \delta)$ is a bialgebra. Analogously, if we use $G^{\mathrm{c}} = \mathfrak{I}^{\mathrm{c}}(A, A)$ and $\xi^{\mathrm{c}} = \mathfrak{i}^{\mathrm{c}}(A, A)$ instead of $G$ and $\xi$ above, we find a bialgebra $(G^{\mathrm{c}}, \Lambda^{\mathrm{c}}, \delta^{\mathrm{c}})$ which is also a quotient of $G$. Actually, $G^{\mathrm{c}}$ is a Hopf-algebra algebra: Let $S\colon G^{\mathrm{c}} \to G^{\mathrm{c}}$ be the unique morphism satisfying $(\xi^{\mathrm{c}})^{-1} = (i_A \otimes S)\xi^{\mathrm{c}}$. Then

$$\begin{aligned}(i_A \otimes [\mu_{G^{\mathrm{c}}}(S \otimes i_{G^{\mathrm{c}}})\Lambda^{\mathrm{c}}])\xi^{\mathrm{c}} &= (i_A \otimes \mu_{G^{\mathrm{c}}})(i_A \otimes S \otimes i_{G^{\mathrm{c}}})(i_A \otimes \Lambda^{\mathrm{c}})\xi^{\mathrm{c}} \\ &= (i_A \otimes \mu_{G^{\mathrm{c}}})(i_A \otimes S \otimes i_{G^{\mathrm{c}}})(\xi^{\mathrm{c}} \circ \xi^{\mathrm{c}}) \\ &= (i_A \otimes \mu_{G^{\mathrm{c}}})(i_A \otimes S \otimes i_{G^{\mathrm{c}}})(\xi^{\mathrm{c}} \otimes i_{G^{\mathrm{c}}})\xi^{\mathrm{c}} \\ &= (i_A \otimes \mu_{G^{\mathrm{c}}})((\xi^{\mathrm{c}})^{-1} \otimes i_{G^{\mathrm{c}}})\xi^{\mathrm{c}} \\ &= i_A \otimes 1 = (i_A \otimes \delta^{\mathrm{c}} 1)\xi^{\mathrm{c}}.\end{aligned}$$

Since $i_A \otimes 1\colon A \to A \otimes G^{\mathrm{c}}$ is an invertible family, the above identity together with the universal property of $G^{\mathrm{c}}$ shows that $\mu_{G^{\mathrm{c}}}(S \otimes i_{G^{\mathrm{c}}})\Lambda^{\mathrm{c}} = \delta^{\mathrm{c}}$. Similarly we prove that, $\mu_{G^{\mathrm{c}}}(i_{G^{\mathrm{c}}} \otimes S)\Lambda^{\mathrm{c}} = \delta^{\mathrm{c}}$. So, $S$ is an antipode and $(G^{\mathrm{c}}, \Lambda^{\mathrm{c}}, \delta^{\mathrm{c}}, S)$ is a Hopf-algebra. Also, note that $A$ is a $G^{\mathrm{c}}$-comodule via the coaction $\xi^{\mathrm{c}}$. If $D$ is another commutative Hopf-algebra with a coaction $\varphi\colon A \to A \otimes D$ on $A$, then Example 3.2 (b) shows that $\varphi$ is an invertible family and so there is a canonical morphism from $G^{\mathrm{c}}$ to $D$ which is, analogously to the above, a Hopf-algebra morphism. So, we have proved the next theorem.

**Theorem 8.2.** *Let $A$ be a* FR *algebra. Then $\mathfrak{I}^{\mathrm{c}}(A, A)$ has a canonical Hopf-algebra structure and $A$ is a $\mathfrak{I}^{\mathrm{c}}(A, A)$-comodule via $\mathfrak{i}^{\mathrm{c}}(A, A)$. Moreover, $\mathfrak{i}^{\mathrm{c}}(A, A)$ is a universal coaction in the following sense: If $D$ is a commutative Hopf-algebra with a coaction $\varphi\colon A \to A \otimes D$ on $A$, then the unique morphism $\psi\colon \mathfrak{I}^{\mathrm{c}}(A, A) \to D$ satisfying $\varphi = (i_A \otimes \psi)\mathfrak{i}^{\mathrm{c}}(A, A)$ is also a Hopf-algebra morphism and is compatible with the coactions on $A$.*

At the end of this note, we consider a notion dual to the notion of the Galois group. Let $A$ be a FR algebra and let $B$ be a subalgebra of $A$. The results of Section 4 (see (1), Theorem 4.1 and Theorem 4.6 in the case $i = 1$) show that there is commutative algebra $\mathfrak{G}$ together with a morphism $\mathfrak{g}\colon A \to A \otimes \mathfrak{G}$ satisfying $\mathfrak{g}(b) = b \otimes 1$ for every $b \in B$, and with the following universal property: For every commutative algebra $D$ and any morphism $\varphi\colon A \to A \otimes D$ satisfying $\varphi(b) = b \otimes 1$ for every $b \in B$, there is a unique morphism $\psi\colon \mathfrak{G} \to D$ such that $\varphi = (i_A \otimes \psi)\mathfrak{g}$. A proof, analogous to the proof of Theorem 8.2, shows that $\mathfrak{G}$ has a canonical Hopf-algebra structure and also $A$ is a $\mathfrak{G}$-comodule via $\mathfrak{g}$. Moreover, if in the above mentioned universal property, $D$ is a Hopf-algebra and $\varphi$ is a coaction then $\psi$ is a Hopf-algebra morphism. A result analogous to the second part of Theorem 4.1 shows that there is a one-to-one correspondence between the characters on $\mathfrak{G}$ and the algebra automorphisms of $A$ which preserve $B$. Indeed, the usual Galois group

Gal$(A/B)$ of the algebra extension $A/B$ is isomorphic to the character group of the Hopf-algebra $\mathfrak{G}$. Also, an expression, analogous to Remark 2.6 (e), shows that if $\mathbb{K}$ is an algebraically closed field then $\mathfrak{G}/J$ is isomorphic to $\mathbb{K}$(Gal(A/B)) as Hopf-algebras, where $J$ denotes the Jacobson radical. Summarizing, it is reasonable to call $\mathfrak{G} = \mathfrak{Gal}(A/B)$ the Galois-Hopf-algebra of the algebra extension $A/B$.

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