Mathematica Bohemica

Glenn G. Chappell; John Gimbel

On subgraphs without large components

Mathematica Bohemica, Vol. 142 (2017), No. 1, 85–111

Persistent URL: `http://dml.cz/dmlcz/146011`

ON SUBGRAPHS WITHOUT LARGE COMPONENTS

Glenn G. Chappell, John Gimbel, Fairbanks

Received February 4, 2014. First published November 21, 2016.
Communicated by Riste Škrekovski

Abstract. We consider, for a positive integer k, induced subgraphs in which each component has order at most k. Such a subgraph is said to be k-divided. We show that finding large induced subgraphs with this property is NP-complete. We also consider a related graph-coloring problem: how many colors are required in a vertex coloring in which each color class induces a k-divided subgraph. We show that the problem of determining whether some given number of colors suffice is NP-complete, even for 2-coloring a planar triangle-free graph. Lastly, we consider Ramsey-type problems where graphs or their complements with large enough order must contain a large k-divided subgraph. We study the asymptotic behavior of "k-divided Ramsey numbers". We conclude by mentioning a number of open problems.

Keywords: component; independence; graph coloring; Ramsey number

MSC 2010: 05C69, 05C55

1. k-divided colorings: basic results

All graphs will be finite, simple, and undirected. A *coloring* of a graph is a partition of the vertex set. We will informally say that a set of vertices of a graph has a particular property if the subgraph induced by those vertices has the property. For undefined terms and concepts, the reader is referred to Chartrand, Lesniak, and Zhang [9].

A previously studied generalization of independence is as follows. A set of vertices is *k-dependent* if it induces a graph with maximum degree at most k. So an independent set is precisely a 0-dependent set. We can similarly generalize the concept of a proper coloring. A coloring of a graph is *k-defective* if each color class is k-dependent. By $\chi_k(G)$ we denote the minimum number of colors in a k-defective coloring of G. Clearly, $\chi_0(G) = \chi(G)$, the ordinary chromatic number. The concepts

DOI: 10.21136/MB.2017.0013-14

of k-dependence and k-defective coloring have been studied by a number of authors, e.g., [11], [12], [13], [15], [21].

In this paper we study the notion of a *k-divided* subgraph. For a positive integer k an induced subgraph in which each component has order at most k is said to be k-divided. This concept also generalizes independence, as a set of vertices is independent precisely if it is 1-divided.

A number of works consider the idea of partitioning the vertex set of a graph into k-divided subgraphs for some k. This study seems to have been begun by Kleinberg, Motwani, Raghavan, and Venkatasubramanian [27] and by Alon, Ding, Oporoski, and Vertigan [2]. For example, the latter [2], Corollary 1.1, showed that if a graph G has maximum degree at most 4, then the vertex set of G can be partitioned into two 57-divided induced subgraphs. Haxell, Szabó, and Tardos [26], Theorem 2.7, improved this to two 6-divided induced subgraphs. They also showed that allowing maximum degree 5 gives two 17617-divided induced subgraphs (see their Theorem 2.1, along with the remark after Claim 2.5). Berke [3], Theorem 4.5, has improved this to 1908-divided. On the other hand, Alon et al. [2], Section 2, found, for each k, a graph H of maximum degree 6 such that $V(H)$ cannot be partitioned into two k-divided induced subgraphs. Various other authors, e.g., [18], [28], [31], have also obtained results in this area.

We consider this partition as a graph-coloring parameter. We say a coloring is *k-divided* if each color class is k-divided. By $\chi_k^*(G)$ we denote the minimum number of colors in a k-divided coloring of G. Note that $\chi_1^*(G) = \chi(G)$. Thus, for example, the first result of Haxell, Szabó, and Tardos mentioned above may be restated as saying that, if a graph G has maximum degree at most 4, then $\chi_6^*(G) \leqslant 2$.

By definition, we have the following.

R e m a r k 1.1. For each graph G and each k for which the parameters are defined, the following hold:

$$\chi_{k+1}(G) \leqslant \chi_k(G),$$
$$\chi_{k+1}^*(G) \leqslant \chi_k^*(G).$$

The disjoint union of a collection of k-cliques shows that both inequalities in the above remark can be satisfied with equality. On the other hand, if H is the disjoint union of a collection of cliques, each of order $k(k+1)m$, then $\chi_{k+1}^*(H) = mk$ and $\chi_k^*(H) = m(k+1)$. Thus, χ_{k+1}^* and χ_k^* can be arbitrarily far apart (as can χ_{k+1} and χ_k).

Our next result shows that, while the difference between χ_a^* and χ_b^* may be arbitrarily large for fixed a, b, we can bound the ratio of these two quantities.

Proposition 1.2. *If $j \geqslant k \geqslant 1$, then for every graph G,*

$$\chi_j^*(G) \leqslant \chi_k^*(G) \leqslant \left\lceil \frac{j}{k} \right\rceil \chi_j^*(G).$$

P r o o f. The first inequality follows from Remark 1.1.

For the second inequality, let G be a graph, and let $j \geqslant k \geqslant 1$. Suppose we have a j-divided coloring of G using $\chi_j^*(G)$ colors. Each component of each color class has at most j vertices; these vertices can be partitioned into $\lceil j/k \rceil$ sets, each of cardinality at most k. Thus the vertex set of each color class can be partitioned into $\lceil j/k \rceil$ sets, each of which is k-divided. Using these sets as color classes for a new coloring, we obtain a k-divided coloring of G using $\lceil j/k \rceil \chi_j^*(G)$ colors. □

We note that a k-divided subgraph has maximum degree at most $k-1$. Hence the following.

Proposition 1.3. *For each graph G and each $k \geqslant 1$,*

$$\chi_{k-1}(G) \leqslant \chi_k^*(G).$$

The above is sharp in the sense that any union of complete graphs gives equality between these two parameters. Now let $k \geqslant 2$, and let H be the complete $2k$-partite graph where each part contains $k-1$ vertices. Then $\chi_{k-1}(H) = k$ (each color class is the union of two partite sets), while $\chi_k^*(H) = 2k-2$ (each color class is a partite set together with one vertex from a different partite set). Thus, for large values of k we can push χ_{k-1} and χ_k^* far apart. However, we will soon see that if k and χ_k are fixed, then χ_k^* cannot be arbitrarily large.

We will make use of the following easy result. This follows, for example, from a result of Lovász [30], Theorem 1; we provide a direct proof.

R e m a r k 1.4. If a graph G has maximum degree Δ, then $\chi_2^*(G) \leqslant \lfloor \frac{1}{2}\Delta + 1 \rfloor$.

P r o o f. Our proof uses a well known technique of unknown origin. Given G a graph of maximum degree Δ, let $t = \lfloor \frac{1}{2}\Delta + 1 \rfloor$. Now color the vertices of G with t colors so that the number of monochromatic edges is minimized. If the color on some vertex v is found on two of its neighbors, then, since $2t > \Delta$, some other color is used on at most one neighbor of v. So switch the color of v and further reduce the number of monochromatic edges. Thus we may assume that each vertex is adjacent with at most one other vertex of the same color. And the desired result follows. □

Finding a k-defective coloring of G with the minimum number of colors and then applying Remark 1.4 to each color class yields the following.

Corollary 1.5. *For each graph G and each $k \geqslant 1$,*

$$\chi_2^*(G) \leqslant \left\lfloor \frac{k}{2} + 1 \right\rfloor \chi_k(G).$$

Here is a related bound.

Proposition 1.6. *There exists a constant p such that for sufficiently large k,*

$$\chi_p^*(G) < \frac{k}{3}\,\chi_k(G)$$

for every graph G.

P r o o f. A result of Haxell, Szabó, and Tardos [26], Theorem 3.5, states that there exist positive constants p and ε such that for sufficiently large k, if H is a graph of maximum degree k, then H has a p-divided coloring with at most $k(\frac{1}{3} - \varepsilon)$ colors. So given G, consider a k-defective coloring of G using $\chi_k(G)$ colors. Apply the Haxell-Szabó-Tardos result to each color class to obtain a p-divided coloring of G with the required number of colors. □

We close this section with an open problem.

Q u e s t i o n 1.7. Is the following true?
For each fixed $m \geqslant 1$ there exists k_m such that if $k \geqslant k_m$, then $\chi_k^*(G) \leqslant \chi_k(G)k/m$ for every graph G.

Note that, by Proposition 1.6, the answer to the above question is “yes” for $m = 1, 2, 3$.

2. k-DIVIDED COLORINGS: GRAPHS ON SURFACES

By Remark 1.1, $\chi_k^*(G) \leqslant \chi_2^*(G)$ for all $k \geqslant 2$. By definition, $\chi_2^*(G) = \chi_1(G)$ for every graph G. Cowen, Cowen, and Woodall [11], Theorem 6, showed that if G is planar, then $\chi_1(G) \leqslant 4$. Their proof does not invoke the Four-Color Theorem. Hence, the following can be proved without use of this theorem.

Theorem 2.1. *If G is a planar graph and $k \geqslant 2$, then $\chi_k^*(G) \leqslant 4$.*

We shall see that this is best possible.

We remark on a constructive technique that we will use in two proofs below. Suppose G is a graph with $\chi_k^*(G) = m$. Form a new graph G' by taking k disjoint copies of G and then adding a new vertex v that is adjacent to all other vertices. This G' cannot be given a k-divided coloring with m colors. However, if v is given a new color and m colors are used on the remaining vertices, then we can form a k-divided coloring of G' with $m+1$ colors, showing that $\chi_k^*(G') = m+1$.

We note that if G is outerplanar then $\chi_k^*(G) \leqslant 3$ for every k. By the following result (also given without proof by Berke [3], page 16), this is best possible.

Theorem 2.2. *For each $k \geqslant 1$, there exists an outerplanar graph G with $\chi_k^*(G) = 3$.*

Proof. Fix $k \geqslant 1$. Let H be the disjoint union of k paths, each having $k+1$ vertices. Note that $\chi_k^*(H) = 2$. Form a graph G by joining a new vertex v to each vertex of H, as in the above construction. Graph G is outerplanar, and $\chi_k^*(G) = 3$. □

We will use the following in a later proof.

Observation 2.3. Let $k \geqslant 1$, and let G be the outerplanar graph with $\chi_k^*(G) = 3$ constructed in the proof of Theorem 2.2. Then G has a k-divided 3-coloring in which one color class contains exactly one vertex.

The above ideas give a simple alternate proof of the following result, proven by Kleinberg, Motwani, Raghavan, and Venkatasubramanian [27], Theorem 4.2, and also by Alon, Ding, Oporoski, and Vertigan [2], Theorem 6.6.

Theorem 2.4. *For each $k \geqslant 1$, there exists a planar graph G with $\chi_k^*(G) = 4$.*

Proof. Fix k. By Theorem 2.2 there exists a graph, say H, such that H is outerplanar and $\chi_k^*(H) = 3$. The union of k copies of H is also outerplanar. Applying the above constructive technique, we join a single vertex to k copies of H to form a planar graph G with $\chi_k^*(G) = 4$. □

On the other hand, Esperet and Joret [18], Theorem 2, showed that for planar graphs we can limit χ_k^* to 3 if we bound k below by a function of the maximum degree of the graph. (We have restated the following result using our notation.)

Theorem 2.5 (Esperet, Joret [18]). *For each $\Delta \geqslant 0$, there exists k_Δ such that if G is a planar graph with maximum degree at most Δ and $k \geqslant k_\Delta$, then*

$$\chi_k^*(G) \leqslant 3.$$

We may also place limits on χ_k^* based on the girth of a planar graph.

Theorem 2.6. *Let G be a planar graph with girth at least* 11. *Then* $\chi_2^*(G) \leqslant 2$.

P r o o f. Let G be a planar graph with girth at least 11. We proceed by induction on the order of G. If G has order at most 2, then the required coloring obviously exists. Suppose G has order 3 or more.

If G has a vertex v of degree 0 or 1, then we may remove v from G, color the resulting graph by the induction hypothesis, and finally color v using a color not used on its neighbor (if any); this is the required coloring. We may thus assume that G has minimum degree at least 2.

Nešetřil, Raspaud, and Sopena [32], Lemma 5, showed that, for $d \geqslant 1$, if a planar graph has minimum degree at least 2 and girth at least $5d+1$, then the graph contains a path of length $d+1$, each of whose internal vertices has degree 2. Applying this result to graph G with $d = 2$, we see that G must contain two adjacent vertices of degree 2. Call these vertices x and y. Remove x and y from G, color the resulting graph by the induction hypothesis, and finally color each of x and y using the color that is not used on its already-colored neighbor. This is the required coloring. □

Relationships between girth and k-divided coloring seem worthy of further study. Grötzsch [25] showed that every triangle-free planar graph has a proper 3-coloring (also see Thomassen [35]); thus, no planar graph G with girth greater than 3 has $\chi_2^*(G) = 4$. However, we do not know the maximum girth of a planar graph with $\chi_2^*(G) = 3$.

Q u e s t i o n 2.7. What is the greatest girth of a planar graph G with $\chi_2^*(G) = 3$?

Next we consider graphs on other surfaces.

Theorem 2.8. *For every surface S there is a k such that if G embeds on S, then* $\chi_k^*(G) \leqslant 6$.

P r o o f. Thomassen [34], Theorem 5.7, showed that, given a surface S, there is a number w such that if a graph H embeds on S with every noncontractible cycle having length at least w, then $\chi(H) \leqslant 5$.

By induction on the genus of S, we can prove the following corollary: for each surface S there is a k such that, if G embeds on S, then there is a set of k vertices of G whose removal forms a graph G' with $\chi(G') \leqslant 5$. Color these removed vertices with a single color. Then use five more colors to give the remaining vertices a proper coloring. Each of the six sets of vertices is k-divided. Hence, $\chi_k^*(G) \leqslant 6$, our desired result. □

We conjecture that the right-hand side of the inequality in Theorem 2.8 can be replaced by a smaller value.

C o n j e c t u r e 2.9. For every surface S there is a k such that if G embeds on S, then $\chi_k^*(G) \leqslant 5$.

The so-called Albertson Four-Color Conjecture [1] states that given any surface S there is a q such that if G embeds on S, then the removal of at most q vertices from G leaves a 4-colorable graph. By an argument much like that in the proof of Theorem 2.8, if the Albertson Conjecture is true, then Conjecture 2.9 holds.

It seems likely that the right-hand side can be reduced even further.

C o n j e c t u r e 2.10. For every surface S there is a k such that if G embeds on S, then $\chi_k^*(G) \leqslant 4$.

The right-hand side of the above inequality cannot be reduced to 3, by Theorem 2.4.

3. k-divided colorings: computation

Next we consider the complexity of various problems involving k-divided sets.

Theorem 3.1. *The problem of determining whether $\chi_2^*(G) \leqslant 2$, for a given graph G, is NP-complete even when restricted to planar triangle-free graphs with maximum degree* 4.

P r o o f. Note that $\chi_2^*(G) = \chi_1(G)$ for every graph G. Gimbel and Hartman [24], Theorem 9, show the problem $\chi_1(G) \leqslant 2$ to be NP-complete for planar triangle-free graphs with maximum degree 4. □

If "maximum degree 4" is replaced by "maximum degree 3" in the above theorem, then the problem becomes trivial; from Remark 1.4 we see that every graph G with maximum degree 3 has $\chi_2^*(G) \leqslant 2$.

Increasing the required girth can also result in a trivial problem. By Theorem 2.6 every planar graph G with girth at least 11 has $\chi_2^*(G) \leqslant 2$. However, we know nothing about planar graphs with girth from 5 to 10.

Q u e s t i o n 3.2. For each fixed g with $5 \leqslant g \leqslant 10$, is it NP-complete to determine whether $\chi_2^*(G) \leqslant 2$ for a given planar graph G of girth g?

Now we consider χ_k^* for other values of k. Note that the order of the components of a graph can be determined in polynomial time. Thus, for fixed k, m, the following problem lies in NP: for a given graph G, determine whether $\chi_k^*(G) \leqslant m$.

Theorem 3.3. *For fixed $k \geqslant 2$, the problem of determining whether $\chi_k^*(G) \leqslant 2$, for a given graph G, is NP-complete even when restricted to planar triangle-free graphs.*

P r o o f. By Theorem 3.1 we may assume that $k \geqslant 3$. Given such a fixed k and a planar triangle-free graph H, we construct a graph G as follows. Let P be a path on $k+1$ vertices, with the vertices of P numbered 1 to $k+1$. For each pair u, v of adjacent vertices in H, create $k-2$ copies of P. Join one of the adjacent vertices (say u) with an edge to each odd-numbered vertex of each copy of P, and similarly join the other (v in this case) to each even-numbered vertex. Call the resulting graph G. (See Figure 1 for an illustration of the construction of G.)

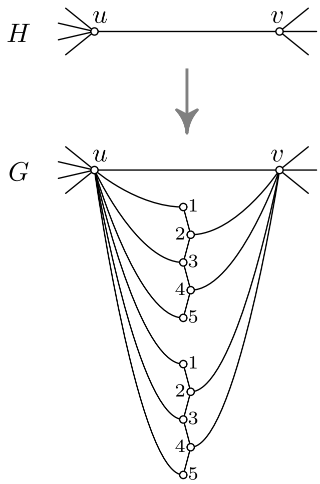

Figure 1. Illustration of the construction of graph G from graph H in the proof of Theorem 3.3. Here $k = 4$. Shown are a single edge uv of H and the corresponding portion of G. For each edge of H we create $k-2=2$ new paths, each with $k+1=5$ vertices.

We note that G is planar and triangle-free and can be constructed in polynomial time from H. Further, G can be given a k-divided coloring with two colors if and only if $\chi_2^*(H) \leqslant 2$. The desired result then follows from Theorem 3.1. □

Theorem 3.4. *For fixed $k \geqslant 1$, the problem of determining whether $\chi_k^*(G) \leqslant 3$, for a given graph G, is NP-complete even when restricted to planar graphs.*

P r o o f. Fix $k \geqslant 1$. By Observation 2.3 there is an outerplanar graph P such that P has no k-divided 2-coloring, and P has a k-divided 3-coloring in which one color class contains a single vertex.

Now suppose H is a planar graph. For each vertex v in H create $k-1$ copies of P and attach v with an edge to each vertex of each copy of P. Call the resulting planar graph G. Note that $\chi(H) \leqslant 3$ if and only if $\chi_k^*(G) \leqslant 3$. But deciding if $\chi(H) \leqslant 3$ for planar graphs is NP-complete by Garey, Johnson, and Stockmeyer [23], Theorem 2.1 and Theorem 2.2. And as the conversion from H to G can be done in polynomial time, the desired result is established. □

Computation of χ_k^* seems to be difficult in general, as the following result suggests. This theorem follows from a result of Farrugia [20], Theorem 2; we provide a short alternate proof.

Theorem 3.5. *For fixed numbers $k, m \geqslant 2$, the problem of determining whether $\chi_k^*(G) \leqslant m$, for a given graph G, is NP-complete.*

P r o o f. From Theorem 3.3 we may assume $m \geqslant 3$. Given a graph H, form G by joining each vertex of H to a complete graph of order $km - 1$. Then $\chi_k^*(G) \leqslant m$ if and only if $\chi(H) \leqslant m$. As we can form G in polynomial steps, and determining whether $\chi(H) \leqslant m$ is NP-complete for $m \geqslant 3$, the desired result follows. □

Now consider the problem of determining whether a given graph contains a large k-divided set.

Theorem 3.6. *For fixed $k \geqslant 1$, the problem of determining whether G contains a k-divided set of m vertices, for given G and m, is NP-complete even when restricted to planar graphs with maximum degree 3 and arbitrarily large girth.*

P r o o f. Let g and k be fixed. (The number g will be the girth of the graph we construct.) Given a graph H with e edges, construct a related graph in the following manner. Let j be the least even number with $j \geqslant g$. Subdivide each edge of H with j additional vertices. Attach to each additional vertex a pendant path of length $k - 1$. Thus, we have added ejk vertices to H; let us refer to these as *new* vertices. Call the resulting graph H'.

Let S be a maximum k-divided set in H' that also maximizes the number of new vertices. If some new vertex of degree one or two is not in S, then we can replace some new vertex of degree three with a new vertex of degree one or two. So we may assume all new vertices of degrees one and two are in S. Let u and v be adjacent vertices in H. Consider the uv-path in H' that consists entirely of new vertices of degree three. If this path contains fewer than $\frac{1}{2}j$ vertices of S, then we did not maximize the number of new vertices in S. Thus, for each edge of H, the set S contains $j(k-1) + \frac{1}{2}j = j(k - \frac{1}{2})$ new vertices. Hence we may assume that S contains exactly $ej(k - \frac{1}{2})$ new vertices. Further, the vertices of S that are not new form an independent set in H.

Clearly the girth of H' is greater than g. Further, given a set of independent vertices of H we can add them to $ej(k-\frac{1}{2})$ new vertices to make a k-divided set in H'.

We see that H contains an independent set of m vertices if and only if H' contains a k-divided set of $ej(k-\frac{1}{2})+m$ vertices. Now, let s be the cardinality of a maximum k-divided set in the g-cycle C_g. Let G be the disjoint union of H' with C_g. (See Figure 2 for an illustration of the construction of G.)

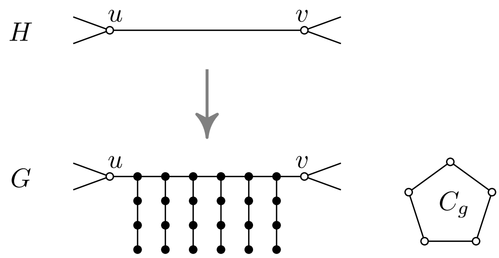

Figure 2. Illustration of the construction of graph G from graph H in the proof of Theorem 3.6. Here $k = 4$ and $g = 5$; therefore $j = 6$. Shown are a single edge of H and the corresponding portion of G, along with the added cycle C_g. Vertices designated in the proof as "new" are shown as non-open.

Then H contains an independent set of m vertices if and only if G contains a k-divided set of $ej(k-\frac{1}{2})+m+s$ vertices.

If graph H is planar with maximum degree 3, then G is planar with maximum degree 3 and girth exactly g. Further, the transformation from H to G can be done in polynomial time. It is known that the problem of determining whether the independence number of a graph is at least some given value is NP-complete for planar graphs of maximum degree 3 (this follows from a result of Garey and Johnson [22], Lemma 1, on vertex covers). The desired result follows. □

While finding large k-divided sets is, as the preceding theorem suggests, difficult, we will see in the next section that if the order of a graph is sufficiently large, then either the graph or its complement must contain a large k-divided set.

4. k-divided Ramsey numbers: basic results

Let $R(a,b)$ denote the ordinary 2-color Ramsey number.

In a previous work [8] we studied *k-defective Ramsey numbers*, a generalization of ordinary Ramsey numbers based on the idea of k-dependent sets. Following Ekim and Gimbel [14], for $k, a, b \geqslant 0$ we defined $R_k(a,b)$ to be the least n such that for each n-vertex graph G, either G contains a k-dependent set of a vertices, or its complement $\overline{G}$ contains a k-dependent set of b vertices.

Now we consider a similar generalization of ordinary Ramsey numbers, based on the idea of k-divided sets. For $k \geqslant 1$ and $a, b \geqslant 0$, let $R^*_k(a,b)$ be the least n such that for each n-vertex graph G, either G contains a k-divided set of a vertices, or $\overline{G}$ contains a k-divided set of b vertices.

We observe that both R_k and R_k^* are well defined; in particular, both $R_k(a,b)$ and $R_k^*(a,b)$ are bounded above by $R(a,b)$ for all values of k, a, b meeting the requirements in the definitions. We also note that R_k and R_k^* generalize the ordinary Ramsey numbers, as $R_0(a,b) = R_1^*(a,b) = R(a,b)$ for all $a, b \geqslant 0$.

Below we show known values of R_k^*. It appears that R_k^* has only been previously studied for $k = 1, 2$. When $k = 1$ these are, as noted above, the ordinary 2-color Ramsey numbers. When $k = 2$ these are defective Ramsey numbers, as $R_2^* = R_1$ (see Lemma 4.1 (b)).

The table below contains known values of $R_2^*(a,b)$ for $2 \leqslant a, b \leqslant 10$. The nontrivial values are from Cockayne and Mynhardt [10]. We also use the facts that $R_2^*(2,b) = 2$ and $R_2^*(3,b) = b$ (see Lemma 4.2 (c) and (d)).

R_2^*	2	3	4	5	6	7	8	9	10
2	2	2	2	2	2	2	2	2	2
3	2	3	4	5	6	7	8	9	10
4	2	4	6	9	11	16	17		
5	2	5	9	15					
6	2	6	11						
7	2	7	16						
8	2	8	17						
9	2	9							
10	2	10							

It should be noted that the values of $R_2^*(4,7)$ and $R_2^*(4,8)$ listed above are not in error. These do indeed only differ by 1; see Cockayne and Mynhardt [10], Corollary 18, Section 5.

In the remainder of this section we will concentrate on R_k^* for $k \geqslant 3$; we will compute no new values in the above table.

We begin our results with a lemma giving relationships between R_k and R_{k+1}^*.

Lemma 4.1. *The following all hold.*

(a) *If* $k, a, b \geqslant 0$, *then* $R_{k+1}^*(a,b) \geqslant R_k(a,b)$.

(b) *If* $k = 0, 1$ *and* $a, b \geqslant 0$, *then* $R_{k+1}^*(a,b) = R_k(a,b)$.

(c) *There exist* $k, a, b \geqslant 0$ *such that* $R_{k+1}^*(a,b) > R_k(a,b)$.

Proof. (a) This follows from the fact that every $(k+1)$-divided set is k-dependent.

(b) This follows from the fact that, for $k = 0, 1$, the properties of being $(k+1)$-divided and k-dependent are identical.

(c) Let $k = 2$, $a = 4$, and $b = 4$. It is not hard to see that $R_k(a, b) = R_2(4, 4) = 4$. However, if G is a 4-vertex path, then $|V(G)| = 4$, but $V(G)$ is 3-divided in neither G nor $\overline{G}$. Thus, $R^*_{k+1}(a, b) = R^*_3(4, 4) > 4$. □

The following lemma gives elementary properties of k-divided Ramsey numbers.

Lemma 4.2. *Let k be a positive integer, and let a, b be nonnegative integers. Then the following all hold.*

(a) $R^*_k(a, b) = R^*_k(b, a)$.
(b) $R^*_{k+1}(a, b) \leqslant R^*_k(a, b)$.
(c) *If $a \leqslant k$ or $b \leqslant k$, then $R^*_k(a, b) = \min\{a, b\}$.*
(d) *If $k = 1, 2$, then $R^*_k(k + 1, b) = b$.*

P r o o f. (a) Obvious.

(b) This follows from the fact that every k-divided set is also $(k + 1)$-divided.

(c) This follows from the fact that if an induced subgraph has order at most k, then that subgraph is k-divided.

(d) Suppose k is 1 or 2. Then an induced subgraph of order $k + 1$ is k-divided precisely when it contains an isolated vertex: a vertex with k nonneighbors in the subgraph.

Thus, if a graph G contains no $(k + 1)$-vertex k-divided subset, then no vertex of G has k nonneighbors. In that case, no vertex of $\overline{G}$ has k neighbors in $\overline{G}$, and so $\overline{G}$ is k-divided. Hence $\overline{G}$ contains a k-divided b-set if and only if the order of G is at least b. □

The statement of part (d) above fails to hold when $k \geqslant 3$. An example showing that the statement does not hold for $k = 3$ can be found in our earlier proof of Lemma 4.1 (c).

On the other hand, for defective Ramsey numbers, we have $R_k(k + 2, b) = b$ for all k, b (see Chappell and Gimbel [8], Lemma 3.1 (g)). Thus Lemma 4.2 (d) follows from Lemma 4.1 (b).

The following lemma gives simple bounds for R^*_k. Part (b) generalizes a result of Burr, Erdős, Faudree, and Shelp [6], Theorem 2, who proved it for $k = 1$—ordinary Ramsey numbers.

Lemma 4.3. *Let k, a, b, c be positive integers. Then the following hold.*

(a) *If $a \geqslant k + 1$, then $R^*_k(a, b) \geqslant R^*_k(a, b - 1) + 1$.*
(b) *If $a \geqslant k + 1$, then $R^*_k(a, b + c - 1) \geqslant R^*_k(a, b) + R^*_k(a, c) - 1$.*
(c) $R^*_k(a, b) \leqslant R^*_k(a - 1, b) + R^*_k(a, b - 1)$.

P r o o f. (a) If $b = 1$, then the statement follows from Lemma 4.2 (c).

Suppose $b \geqslant 2$. Let $n = R_k^*(a, b-1) - 1$. Note that $n \geqslant 0$. Let G be a graph of order n, such that G contains no a-vertex k-divided set, and $\overline{G}$ contains no $(b-1)$-vertex k-divided set.

Let G' be obtained from G by adding an additional vertex x, along with an edge from x to each vertex of G. So G' has order $n + 1 = R_k^*(a, b-1)$. Now, G' contains no a-vertex k-divided set, since every set containing x induces a connected subgraph of G', and $a > k$. Furthermore, $\overline{G'}$ contains no b-vertex k-divided set, since $b - 1$ vertices of such a set would constitute a $(b-1)$-vertex k-divided set in $\overline{G}$.

We conclude that $R_k^*(a, b)$ is greater than the order of G'. The statement follows.

(b) Let G_1 be a graph of order $R_k^*(a, b) - 1$ such that G_1 has no k-divided a-set, and $\overline{G_1}$ has no k-divided b-set. Similarly, let G_2 be a graph of order $R_k^*(a, c) - 1$ such that G_2 has no k-divided a-set, and $\overline{G_2}$ has no k-divided c-set. Let G be the graph formed by taking the disjoint union of G_1 and G_2 and adding all edges between vertices in G_1 and vertices in G_2.

Graph G has order $R_k^*(a, b) + R_k^*(a, c) - 2$. Also G has no k-divided a-set. To see this, let $S \subseteq V(G)$ with $|S| = a$. If S lies entirely in either G_1 or G_2, then S is not k-divided. But if S contains vertices of both G_1, G_2, then the subgraph of G induced by S is connected. Since $a \geqslant k + 1$, the set S cannot be k-divided.

Further, $\overline{G}$ has no k-divided $(b+c-1)$-set, since any $(b+c-1)$-set in $V(G)$ must contain either b vertices of $\overline{G_1}$ or c vertices of $\overline{G_2}$, in which case it is not k-divided in $\overline{G}$.

(c) Let $n = R_k^*(a-1, b) + R_k^*(a, b-1)$, and let G be a graph of order n. Let $x \in V(G)$. Then either x has at least $R_k^*(a-1, b)$ nonneighbors or x has at least $R_k^*(a, b-1)$ neighbors. We consider the former case; the other is handled similarly.

Let T be the set of nonneighbors of x. If T has a b-vertex subset that is k-divided in $\overline{G}$, then we are done. Otherwise, T must have an $(a-1)$-vertex subset U that is k-divided in G. Then $U \cup \{x\}$ is an a-vertex set that is k-divided in G. □

Using inductive arguments based on Lemma 4.3, we can find nontrivial bounds on R_k^*.

Proposition 4.4. *Let $k \geqslant 1$, and let $a, b \geqslant 0$. Then the following hold.*

(a) *If $a, b \geqslant 1$, then $R_k^*(k+a, k+b) \geqslant k + a + b - 1$.*
(b) *$R_k^*(k+a, k+b) \leqslant \binom{a+b}{a} k$.*

Proof. (a) We proceed by induction, first on a, and then on b. If $a = b = 1$, then

$$\begin{aligned} R_k^*(k+a, k+b) &= R_k^*(k+1, k+1) \\ &\geqslant R_k^*(k, k+1) + 1 && \text{by Lemma 4.3 (a)} \\ &= k + 1 && \text{by Lemma 4.2 (c)} \\ &= k + a + b - 1. \end{aligned}$$

If $a = 1$ and $b \geqslant 2$, then

$$\begin{aligned} R_k^*(k+a, k+b) &\geqslant R_k^*(k+a, k+b-1) + 1 && \text{by Lemma 4.3 (a)} \\ &\geqslant (k + a + [b-1] - 1) + 1 && \text{by the induction hypothesis} \\ &= k + a + b - 1. \end{aligned}$$

The statement similarly holds when $a \geqslant 2$.

(b) We proceed by induction, first on a, and then on b. If $a = 0$, then we need to show that $R_k^*(k, k+b) \leqslant k$. This follows from Lemma 4.2 (c). The statement similarly holds when $b = 0$.

Now assume that $a, b \geqslant 1$, and that the statement holds for all smaller values of a and, with the given value of a for all smaller values of b. Apply Lemma 4.3 (c) to obtain

$$\begin{aligned} R_k^*(k+a, k+b) &\leqslant R_k^*(k+a-1, k+b) + R_k^*(k+a, k+b-1) \\ &\leqslant \binom{a+b-1}{a-1} k + \binom{a+b-1}{a} k \\ &= \binom{a+b}{a} k. \end{aligned}$$

□

5. k-divided Ramsey numbers: asymptotics

In this section, we study the asymptotic behavior of the values of R_k^* in three ways.

(i) First, we consider $R_k^*(k+a, k+b)$ for fixed a, b and increasing k.
(ii) Second, we consider $R_k^*(a, a)$ for fixed k and increasing a.
(iii) Third, we consider $R_k^*(a, b)$ for fixed k, a and increasing b.

We will see that, in the first situation above, the values are $\Theta(k)$, while in the second, they are exponential in a. In the third situation, the values are polynomial in b, although in general we have not determined the degree of this polynomial. Along the way, we will compute the exact values of two infinite families of our Ramsey numbers.

We begin with the first asymptotic situation: $R_k^*(k+a, k+b)$ for fixed a, b and increasing k.

It follows from Proposition 4.4 that $R_k^*(k+a, k+b)$ is $\Theta(k)$ for fixed a, b. In a previous work [8] we proved this also for R_k. We were actually able to prove ([8], Corollary 4.3) a stronger bound: $R_k(k+a, k+b)$ is $k + O(1)$ for fixed a, b. However, as we will see, this does not hold for R_k^*.

This fact will follow immediately from our next proposition, in which we determine some diagonal values of R_k^*. We will make use of a theorem on circulant graphs due to Boesch and Tindell [4], Theorem 1.

Let n be a positive integer, and let S be a set of positive integers, each of which is less than n. The *circulant graph* of order n with *distance set* S is defined to be the simple graph G with vertex set $\{1, 2, \ldots, n\}$ and an edge between distinct vertices i, j if there exists some $a \in S$ with $|i - j| \equiv a \pmod{n}$.

Theorem 5.1 (Boesch, Tindell [4]). *Let G be a circulant graph with order n and distance set $S = \{a_1, a_2, \ldots, a_t\}$. Say each vertex of G has degree d. Then the connectivity of G is less than d if and only if there is some proper divisor r of n so that the number of distinct positive residues modulo r of the numbers $a_1, a_2, \ldots, a_t, n - a_1, n - a_2, \ldots, n - a_t$ is less than the minimum of $r - 1$ and dr/n.*

We use the above result to determine some diagonal values of R_k^*.

Theorem 5.2. *Let $k \geqslant 1$. Then*

$$R_k^*(k+1, k+1) = \begin{cases} 2k - 1, & \text{if } k \text{ is even,} \\ 2k, & \text{if } k \text{ is odd.} \end{cases}$$

P r o o f. If $k \leqslant 2$, then the result is easily checked. Therefore, suppose that $k \geqslant 3$. We consider two cases: when k is even, and when k is odd.

Case I. Suppose k is even—For the lower bound, let G be $K_{k-1,k-1}$ with a perfect matching removed. Denote the partite sets of G by A, B. Let S be a $(k+1)$-subset of $V(G)$. We wish to show that S is k-divided in neither G nor $\overline{G}$. Since $|S| = k + 1$, it suffices to show that S induces a connected subgraph in both G and $\overline{G}$.

Based on the cardinalities of A, B, and S, we see that S intersects each of A, B in at least 2 vertices. Because $k \geqslant 3$ and k is even, we have $k+1 > 4$, and so S intersects one of A, B in at least 3 vertices. Without loss of generality, say $|S \cap A| \geqslant 3$, while $|S \cap B| \geqslant 2$. Then, in the subgraph of G induced by S, any two vertices in $S \cap B$ must have at least one common neighbor in $S \cap A$; thus these two vertices must lie in the same component. Further, each vertex in $S \cap A$ has a neighbor in $S \cap B$, and so it is also in this same component. We see that the subgraph of G induced by S is connected.

Now consider $\overline{G}$. The set $S \cap A$ induces a complete subgraph of $\overline{G}$, and so all the vertices in $S \cap A$ lie in the same component of the subgraph of $\overline{G}$ induced by S, and similarly for $S \cap B$. The edges between A, B in $\overline{G}$ form a (perfect) matching having size $k - 1$. Since $|V(\overline{G}) - S| = k - 3$, there exists at least one edge between $S \cap A$ and $S \cap B$. We see that the subgraph of $\overline{G}$ induced by S is connected.

Since $|V(G)| = 2k-2$, we have $R_k^*(k+1, k+1) \geqslant 2k-1$ for even k.

For the upper bound when k is even, let H be a graph of order $2k-1$ with no k-divided set of order $k+1$. Then $\overline{H}$ has no $K_{1,k}$ subgraph, and so the maximum degree of $\overline{H}$ is at most $k-1$. Since $\overline{H}$ has odd order, it cannot be regular of odd degree; there must be $x \in V(H)$ with degree at most $k-2$ in $\overline{H}$. Then x has at least $(2k-1)-1-(k-2) = k$ nonneighbors in $\overline{H}$. Vertex x, together with k of its nonneighbors, forms a k-divided set of order $k+1$ in $\overline{H}$.

Thus, $R_k^*(k+1, k+1) = 2k-1$ for even k.

Case II. Suppose k is odd—Now suppose that $k \geqslant 3$ is an odd integer.

We first consider the lower bound. Since k is odd, we can find t such that $k = 2t+1$. Let $n = 2k-1$, and let G be a circulant graph on n vertices with distance set $\{1, \ldots, t\}$. (For example, when $k = 3$, we have $t = 1$ and $n = 5$; graph G is a 5-cycle.) We wish to show that for each $T \subseteq V(G)$ with $|T| = k+1$, the set T is k-divided in neither G nor $\overline{G}$. Such a set fails to be k-divided if and only if it induces a connected subgraph. Thus, we wish to show that both G and $\overline{G}$ have connectivity at least $n-(k+1)+1 = k-1 = 2t$. Since G is regular of degree $2t$, we may make use of Theorem 5.1 to prove this.

If n is prime, then the required connectivity follows immediately from Theorem 5.1. Otherwise, let r be a proper divisor of n. Because $n = 2k-1$ is odd, we have $r \leqslant \frac{1}{3}n \leqslant k-1$. The integers $n-t, n-(t-1), \ldots, n-1, 0, 1, 2, \ldots, t$ form a sequence of $2t+1 = k$ consecutive values modulo n. Since $r \leqslant k-1$, all residues modulo r must be represented in this sequence. The set of distances between adjacent vertices in G is $\{1, 2, \ldots, t, n-1, n-2, \ldots, n-t\}$. These values are the same as those in the above sequence, with the exception of zero, and so all $r-1$ *positive* residues modulo r are represented in the set. Hence, by Theorem 5.1, the connectivity of G is its vertex degree: $2t = k-1$.

We handle $\overline{G}$ similarly. The set of distances between adjacent vertices in $\overline{G}$ is $\{t+1, \ldots, n-(t+1)\}$, a set of $2t = k-1$ consecutive values. Since $r \leqslant k-1$, all $r-1$ positive residues modulo r must be represented in this set. Hence, by Theorem 5.1, the connectivity of $\overline{G}$ is its vertex degree: $2t = k-1$.

For the upper bound when k is odd, let H be a graph of order $2k$ with no k-divided set of order $k+1$. Then $\overline{H}$ has no $K_{1,k}$ subgraph, and so the maximum degree of $\overline{H}$ is at most $k-1$. Let $x \in V(H)$. Then x has at least $2k-1-(k-1) = k$ nonneighbors in $\overline{H}$. Vertex x, together with k of its nonneighbors, forms a k-divided set of order $k+1$ in $\overline{H}$.

Thus, $R_k^*(k+1, k+1) = 2k$ for odd k. □

By Theorem 5.2, it is not the case that $R_k^*(k+a, k+b)$ is $k+O(1)$ for all fixed a, b.

Now we turn to our second asymptotic situation: $R_k^*(a,a)$ for fixed k and increasing a.

The following theorem generalizes a result of Erdős [16], Theorem 1, who proved it for $k=1$—ordinary Ramsey numbers—with $t=\sqrt{2}$ for $a\geqslant 3$. (Erdős attributes the special case of part (b) when $k=1$ to Szekeres, citing a paper of Erdős and Szekeres [17].)

Theorem 5.3. *Let k be a positive integer.*

(a) *There exists a constant $t=t(k)$ such that if $a\geqslant 2$, then $R_k^*(a,a)>t^a$.*
(b) *If $a\geqslant k+1$, then $R_k^*(a,a)<4^{a-k}k$.*

Proof. (a) In a previous work we proved the same bound for R_k (see [8], Theorem 6.4 (a)). The required statement then follows from Lemma 4.1 (a).

(b) We can apply Proposition 4.4 (b) to show that

$$R_k^*(a,a)\leqslant\binom{2a-2k}{a-k}k.$$

The statement then follows from the fact that $\binom{2s}{s}<4^s$ when $s\geqslant 1$ (this bound can be proven using a simple inductive argument). $\square$

We see that, for fixed k, the values of $R_k^*(a,a)$—and $R_k^*(k+a,k+a)$—grow exponentially, as is also true for R_k (see Chappell and Gimbel [8], Theorem 6.4, Corollary 6.5).

Corollary 5.4. *For fixed k, the value of $\log R_k^*(a,a)$ is $\Theta(a)$.*

Next we study our third asymptotic situation: $R_k^*(a,b)$ for fixed k, a and increasing b.

It follows from Proposition 4.4 (b) that $b\leqslant R_k^*(k+1,b)\leqslant kb+k-k^2$ for all $k,b\geqslant 1$; thus these values are $\Theta(b)$ for each fixed k. For some k we can compute the values exactly. For example, it follows from Lemma 4.2 (d) that $R_2^*(3,b)=b$ for all b. We have also been able to determine these values for $k=3$, although the computation is rather less trivial.

Theorem 5.5. *Let b be a nonnegative integer. Then*

$$R_3^*(4,b)=\begin{cases}0, & \text{if } b=0,\\ b+2\left\lfloor\frac{1}{3}(b-1)\right\rfloor, & \text{otherwise.}\end{cases}$$

P r o o f. We proceed by induction on b. If $b \leqslant 3$, then the result follows from Lemma 4.2 (c). Suppose that $b \geqslant 4$, and write $b = 3s + t + 1$, where s, t are integers with $s \geqslant 1$ and $0 \leqslant t \leqslant 2$. It is not hard to show that $b + 2\lfloor \frac{1}{3}(b-1) \rfloor = 5s + t + 1$. It suffices, therefore, to prove that $R_3^*(4, b) = 5s + t + 1$.

We first consider the lower bound. If $t = 0$, then $b = 3s + 1$ with $s \geqslant 1$. Let G be a graph such that $\overline{G}$ is the disjoint union of s copies of C_5. Then G has order $5s$. This graph G contains no 4-vertex 3-divided set, since the subgraph of $\overline{G}$ induced by such a set would have a subgraph isomorphic either to C_4 or to $K_{1,3}$, which is impossible. Now consider a 3-divided set T in $\overline{G}$. Set T contains at most 3 vertices of each copy of C_5, and so $|T| \leqslant 3s < b$. Thus, $\overline{G}$ contains no b-vertex 3-divided set.

We see that, when $t = 0$, $R_3^*(4, b)$ is greater than the order of G; that is, $R_3^*(4, b) \geqslant 5s + t + 1$.

Now, consider the lower bound when $t = 1, 2$. We have

$$\begin{aligned} R_3^*(4, b) &\geqslant R_3^*(4, b-1) + 1 && \text{by Lemma 4.3 (a)} \\ &= R_3^*(4, 3s + [t-1] + 1) + 1 \\ &= (5s + [t-1] + 1) + 1 && \text{by the induction hypothesis, since } t > 0 \\ &= 5s + t + 1. \end{aligned}$$

Next we consider the upper bound. Let H be a graph of order $5s+t+1$ containing no 4-vertex 3-divided set. We show that $\overline{H}$ contains a 3-divided set with at least $3s + t + 1$ vertices.

Note that we can always form a 3-divided set by taking 3 vertices from each component of order at least 3, and all vertices from all smaller components. Thus, if every component of $\overline{H}$ has order 4 or less, then $\overline{H}$ contains a 3-divided set of cardinality at least

$$\left\lceil \frac{3}{4}(5s + t + 1) \right\rceil \geqslant 3s + \left\lceil \frac{3}{4}t + \frac{3}{4} \right\rceil = 3s + t + 1, \quad 0 \leqslant t \leqslant 2.$$

We therefore assume that $\overline{H}$ has a component of order at least 5. Thus, $s \geqslant 1$. The graph H contains no 4-vertex 3-divided set, and so $\overline{H}$ has no $K_{1,3}$ subgraph. Thus the maximum degree of $\overline{H}$ is at most 2, and so $\overline{H}$ is a disjoint union of paths and cycles. Let v_1, v_2, v_3, v_4, v_5 be consecutive vertices along some path or cycle with at least 5 vertices. Remove these five vertices from H, note that the resulting graph has order $5(s-1)+t+1$, with $s - 1 \geqslant 0$, and apply the induction hypothesis. We obtain a 3-divided set in $\overline{H}$ having order $3(s-1) + t + 1$. Add v_2, v_3, and v_4 to this set, to obtain a 3-divided set in G with order $3s + t + 1$. Thus,

$$R_3^*(4, b) = R_3^*(4, 3s + t + 1) \leqslant 5s + t + 1.$$

By induction, the result is proven. □

In general we can show, using Proposition 4.4, that for fixed $k, a \geqslant 1$, the values of $R_k^*(k+a, b)$ are $\Omega(b)$ and $O(b^a)$. These values are thus polynomial in b. But that allows for quite a wide range of possibilities, particularly for large values of a.

P r o b l e m 5.6. For fixed k, a, determine the asymptotic behavior of $R_k^*(a, b)$.

6. k-divided Ramsey numbers: individual values

Next we consider values of individual k-divided Ramsey numbers. We will also look at relationships between these and certain graph Ramsey numbers involving sets of complete multipartite graphs.

We have been able to compute two otherwise unknown values of R_4^* using a computer program.

Proposition 6.1. *The following both hold.*

(a) $R_4^*(5, 6) = 9$, *with exactly* 12 *isomorphism classes of extremal graphs.*
(b) $R_4^*(5, 7) = 11$, *with exactly* 14 *isomorphism classes of extremal graphs.*

The upper bounds and the enumerations of extremal graphs were all verified using a computer program [7]. We give proofs for the lower bounds.

P r o o f of lower bounds. (a) [To show: $R_4^*(5, 6) \geqslant 9$] For the lower bound, we can use the following 8-vertex graph G, which is an extremal graph for $R_4^*(6, 5)$. Begin with a 5-cycle, with vertices numbered 1 through 5. Add three additional vertices a, b, c, forming an induced 3-vertex path in that order. Make vertex a adjacent to 1 and 2, vertex b adjacent to 3, and vertex c adjacent to 4 and 5. Let G be the resulting graph. (See Figure 3.)

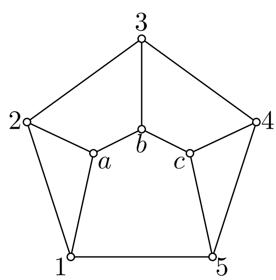

Figure 3. Graph G from the proof of Proposition 6.1 (a). Graph G has order 8, G contains no 4-divided 6-set, and $\overline{G}$ contains no 4-divided 5-set. Thus $R_4^*(6, 5) \geqslant 9$.

Graph G has no 6-vertex 4-divided set, and $\overline{G}$ has no 5-vertex 4-divided set; we conclude that $R_4^*(6, 5) \geqslant 9$.

(b) [To show: $R_4^*(5,7) \geqslant 11$] For the lower bound, we can use the following 10-vertex graph G, which is an extremal graph for $R_4^*(7,5)$. Begin with a 10-cycle, with vertices numbered 1 through 10. Add five additional edges: between vertices 1 and 4, vertices 3 and 6, vertices 5 and 8, vertices 7 and 10, and vertices 2 and 9. Let G be the resulting graph. (See Figure 4.)

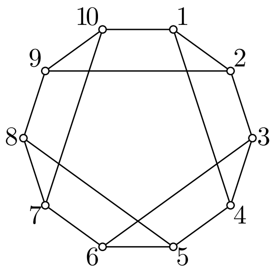

Figure 4. Graph G from the proof of Proposition 6.1 (b). Graph G has order 10, G contains no 4-divided 7-set, and $\overline{G}$ contains no 4-divided 5-set. Thus $R_4^*(7,5) \geqslant 11$.

Graph G has no 7-vertex 4-divided set, and $\overline{G}$ has no 5-vertex 4-divided set; we conclude that $R_4^*(7,5) \geqslant 11$. □

Another method for determining individual values makes use of a correspondence between k-divided Ramsey numbers and certain graph Ramsey numbers involving collections of complete multipartite graphs.

Observe that a graph G contains a k-divided set of cardinality a if and only if $\overline{G}$ has a (not necessarily induced) subgraph isomorphic to some complete multipartite graph $K_{p_1,p_2,\ldots}$, where $p_1 + p_2 + \ldots = a$ is a partition of the integer a in which no part is greater than k.

For example, suppose we are interested in whether a graph G has a 3-divided set of cardinality 4. Set $k = 3$ and $a = 4$ in the previous paragraph. There are four partitions of the integer 4 in which no part is greater than 3: $1+3$, $2+2$, $1+1+2$, and $1+1+1+1$. Look at the first such partition: $1+3 = 4$. If graph G has a set of 4 vertices inducing a subgraph with a component of order 1 and another component of order 3, then these same 4 vertices form the vertex set of a (not necessarily induced) $K_{1,3}$ subgraph in $\overline{G}$; the converse holds as well. Similarly, for the second partition $(2+2=4)$, G has a set of 4 vertices inducing a subgraph with two components of order 2 if and only if $\overline{G}$ has a subgraph isomorphic to $K_{2,2}$, and so on for the other partitions of the integer 4.

Similar reasoning holds for other values of k, a; our observation above is correct. In fact, a stronger statement holds: the observation remains true if we restrict our attention to those partitions that are not refinements of other partitions under consideration. For example, the partition $1+1+2$ is a refinement of $1+3$. We need not consider the former when looking for subgraphs of $\overline{G}$, since, if $\overline{G}$ has a $K_{1,1,2}$ subgraph, then it necessarily has a $K_{1,3}$ subgraph. The partition $1+1+1+1$ is also a refinement of $1+3$; again, we need not consider the former.

We see that, if we wish to determine whether a graph G contains a 3-divided set of cardinality 4, then we need consider only two partitions: $1+3=4$ and $2+2=4$. We conclude that a graph G contains a 3-divided set of 4 vertices if and only if $\overline{G}$ has a subgraph isomorphic to either $K_{1,3}$ or $K_{2,2}$.

Using such reasoning, we can show that each k-divided Ramsey number is equal to some graph Ramsey number, for two collections of complete multipartite graphs. For example, for $R_3^*(4,5)$, the partitions of the integer 4 are $1+3=4$ and $2+2=4$, as above, and the partition of the integer 5 is $2+3=5$. Thus, $R_3^*(4,5) = R(\{K_{1,3},K_{2,2}\},K_{2,3})$.

Applying this idea to various of the k-divided Ramsey numbers from Theorem 5.2, Theorem 5.5, and Proposition 6.1, we obtain values for the following graph Ramsey numbers, all of which we believe to be previously unknown.

Proposition 6.2. *The following all hold.*

(a) $R(\{K_{1,3},K_{2,2}\},\{K_{1,3},K_{2,2}\}) = R_3^*(4,4) = 6$.
(b) $R(\{K_{1,4},K_{2,3}\},\{K_{1,4},K_{2,3}\}) = R_4^*(5,5) = 7$.
(c) $R(\{K_{1,5},K_{2,4},K_{3,3}\},\{K_{1,5},K_{2,4},K_{3,3}\}) = R_5^*(6,6) = 10$.
(d) $R(\{K_{1,6},K_{2,5},K_{3,4}\},\{K_{1,6},K_{2,5},K_{3,4}\}) = R_6^*(7,7) = 11$.
(e) $R(\{K_{1,3},K_{2,2}\},K_{2,3}) = R_3^*(4,5) = 7$.
(f) $R(\{K_{1,3},K_{2,2}\},\{K_{3,3},K_{2,2,2}\}) = R_3^*(4,6) = 8$.
(g) $R(\{K_{1,3},K_{2,2}\},\{K_{1,3,3},K_{2,2,3}\}) = R_3^*(4,7) = 11$.
(h) $R(\{K_{1,4},K_{2,3}\},\{K_{2,4},K_{3,3}\}) = R_4^*(5,6) = 9$.
(i) $R(\{K_{1,4},K_{2,3}\},K_{3,4}) = R_4^*(5,7) = 11$.

We can also reverse the above idea, determining values of k-divided Ramsey numbers, or bounds on these, using known values or bounds for graph Ramsey numbers. Few of the relevant graph Ramsey numbers are known, but we do have the following.

Proposition 6.3. *The following both hold.*

(a) $R_3^*(5,5) = R(K_{2,3},K_{2,3}) = 10$.
(b) $R_4^*(7,7) = R(K_{3,4},K_{3,4}) \leqslant 30$.

P r o o f. (a) The first equality follows from reasoning much like that discussed earlier, before Proposition 6.2. The second equality was proven by Burr [5], Item 50; also see Radziszowski [33], Section 3.3.1.

(b) Once again, the equality follows from reasoning like that discussed earlier. The inequality is due to Lortz and Mengersen [29], Theorem 6; also see Radziszowski [33], Section 3.3.1. □

The following table shows the values of $R_3^*(a,b)$ that we have found for $3 \leqslant a, b \leqslant 10$. These are from Lemma 4.2 (c), Theorem 5.5, and Proposition 6.3.

R_3^*	3	4	5	6	7	8	9	10
3	3	3	3	3	3	3	3	3
4	3	6	7	8	11	12	13	16
5	3	7	10					
6	3	8						
7	3	11						
8	3	12						
9	3	13						
10	3	16						

The following table shows the values of $R_4^*(a,b)$ that we have found for $4 \leqslant a, b \leqslant 8$. These are from Lemma 4.2 (c), Theorem 5.2, and Proposition 6.1.

R_4^*	4	5	6	7	8
4	4	4	4	4	4
5	4	7	9	11	
6	4	9			
7	4	11			
8	4				

The following table shows the values of $R_5^*(a,b)$ that we have found for $5 \leqslant a, b \leqslant 8$. These are from Lemma 4.2 (c) and Theorem 5.2.

R_5^*	5	6	7	8
5	5	5	5	5
6	5	10		
7	5			
8	5			

The following theorem allows us to compute lower bounds for R_k^* based on lower bounds for ordinary Ramsey numbers, by "blowing up" graphs in a manner we will describe.

Theorem 7.1. *Let $k \geqslant 1$ and $a, b \geqslant 0$. Then*

$$R_k^*(ka+1, kb+1) - 1 \geqslant k[R(a+1, b+1) - 1].$$

Proof. This result is easier to state if we use different notation. Let $S_k^*(a, b)$ be the greatest order of a graph G such that each k-divided set in G contains at most a vertices, and each k-divided set in $\overline{G}$ contains at most b vertices. Then, for all $k, a, b \geqslant 0$, we have

$$S_k^*(a, b) + 1 = R_k^*(a+1, b+1).$$

Our desired result becomes

$$S_k^*(ka, kb) \geqslant kS_1^*(a, b).$$

Let G be a graph of order $S_1^*(a, b)$, such that no set of more than a vertices is 1-divided (i.e., independent) in G, and no set of more than b vertices is 1-divided in $\overline{G}$.

Let H, H' be graphs defined as follows. For each vertex x of G, let V_x be a set of k vertices, so that the sets V_x are all pairwise disjoint. Let $V(H) = V(H')$ be the union of all the V_x sets, so that H and H' both have order $kS_1^*(a, b)$. If $u \in V_x$ and $v \in V_y$, with $x \neq y$, then u, v are adjacent in H (and similarly in H') if and only if x, y are adjacent in G. If $u, v \in V_x$, with $u \neq v$, then u, v are nonadjacent in H and are adjacent in H'.

Let $T \subseteq V(H)$ be k-divided in H. We claim that T is k-divided in H'. If T is not k-divided in H', then the subgraph of H' induced by T must have a component with more than k vertices. This component must therefore include vertices in more than one V_x set, and it is not hard to show that its vertex set induces a connected subgraph of H with more than k vertices. By contradiction, the claim is proven.

Let $U \subseteq T$ include one vertex from each component of the subgraph of H' induced by T. Then U is an independent set containing at most one vertex from each V_x set. The set of all x such that U meets V_x forms an independent set in G; thus, $|U| \leqslant a$, and so $|T| \leqslant ka$.

We have shown than each k-divided set in H contains at most ka vertices. We can similarly show than each k-divided set in $\overline{H}$ contains at most kb vertices. We conclude that $S_k^*(ka, kb)$ is at least the order of H, and the result is proven. □

Since R_2^* has been previously studied (in the form of R_1; see Cockayne and Mynhardt [10], and Ekim and Gimbel [14]), the special case of Theorem 7.1 with $k = 2$ is of particular interest. We close with a table of the lower bounds on values of R_2^* obtained from Theorem 7.1. Note that $R_2^*(5, 5)$ is known to be 15; the other bounds shown below are for unknown values of R_2^*.

$R_2^* = R_1 \geqslant$	5	7	9	11	13	15	17
5	11	17	27	35	45	55	71
7	17	35	49				
9	27	49					
11	35						
13	45						
15	55						
17	71						

8. Future work

A number of open questions were brought up in earlier sections.

Question 1.7. Is the following true?
For each fixed $m \geqslant 1$ there exists k_m such that if $k \geqslant k_m$, then $\chi_k^*(G) \leqslant \chi_k(G)k/m$ for every graph G.

By Proposition 1.6, the answer to the above question is "yes" for $m = 1, 2, 3$.

In Theorem 2.6 we established that every planar graph with girth at least 11 has $\chi_2^* \leqslant 2$. On the other hand, a construction of Gimbel and Hartman [24], Theorem 9, gives planar graphs with girth 4 and $\chi_2^* = 3$. We do not know the status of planar graphs with girth strictly between 5 and 11.

Question 2.7. What is the greatest girth of a planar graph G with $\chi_2^*(G) = 3$?

By the earlier comments, the answer must be in the range 4 to 10, inclusive.

For an integer g which allows for a planar graph of girth g to have $\chi_2^* = 3$, we considered the problem of determining whether $\chi_2^* \leqslant 2$; we ask whether this problem is NP-complete.

Question 3.2. For each fixed g with $5 \leqslant g \leqslant 10$, is it NP-complete to determine whether $\chi_2^*(G) \leqslant 2$ for a given planar graph G of girth g?

In Theorem 2.8 we showed that, if a graph G embeds on a surface S, then there is k—depending only on S—such that $\chi_k^*(G) \leqslant 6$. We conjecture that this "6" can be reduced.

C o n j e c t u r e 2.9. For every surface S there is a k such that if G embeds on S, then $\chi_k^*(G) \leqslant 5$.

C o n j e c t u r e 2.10. For every surface S there is a k such that if G embeds on S, then $\chi_k^*(G) \leqslant 4$.

By Theorem 2.4, the above value cannot be reduced to 3.

We determined the asymptotic behavior of k-divided Ramsey numbers in various situations. It follows from Proposition 4.4 that, for fixed a, b, the value of $R_k^*(k+a, k+b)$ is $\Theta(k)$. Corollary 5.4 states that, for fixed k, the value of $\log R_k^*(a,a)$ is $\Theta(a)$, so $R_k^*(a,a)$ is exponential in a for fixed k.

An apparently more difficult problem is to determine the asymptotic behavior of $R_k^*(a,b)$ for fixed k, a. It follows from Proposition 4.4 that, for fixed $k, a \geqslant 1$, the value of $R_k^*(k+a, b)$ is $\Omega(b)$ and $O(b^a)$. It thus seems likely that, for fixed k, a, the value of $R_k^*(a,b)$ is $\Theta(b^t)$, where t depends on k and a.

P r o b l e m 5.6. For fixed k, a, determine the asymptotic behavior of $R_k^*(a,b)$.

As for specific values of $R_k^*(a,b)$, various values of $R_1^*(a,b)$—ordinary Ramsey numbers—are known. Cockayne and Mynhardt [10] computed a number of values of $R_2^*(a,b)$ (see Section 4). And in Theorem 5.5 and Proposition 6.3 (a), we determined values of $R_3^*(a,b)$.

However, much less is known about $R_k^*(a,b)$ for $k \geqslant 4$. From Theorem 5.2 we know the value of $R_k^*(k+1,k+1)$ for all k, but other than that, we have just two nontrivial values: $R_4^*(5,6) = 9$ and $R_4^*(5,7) = 11$ (Proposition 6.1).

P r o b l e m 8.1. Compute $R_k^*(a,b)$ for other values of k, a, b.

A c k n o w l e d g e m e n t. The authors would like to thank Jing Huang for suggesting the technique used in the proof of Theorem 2.6.

N o t e a d d e d i n P r o o f. After this paper was accepted for publication, the authors learned that Conjecture 2.9 has been proven by Esperet and Ochem [19], Theorem 2.

References

[1] *M. Albertson*: Open Problem 2. The Theory and Applications of Graphs. Proc. 4th Int. Conf., Kalamazoo, 1980. John Wiley & Sons, New York, 1981. zbl MR
[2] *N. Alon, G. Ding, B. Oporowski, D. Vertigan*: Partitioning into graphs with only small components. J. Comb. Theory, Ser. B *87* (2003), 231–243. zbl MR doi
[3] *R. Berke*: Coloring and Transversals of Graphs. Ph.D. thesis. ETH, Zurich, 2008.
[4] *F. Boesch, R. Tindell*: Circulants and their connectivities. J. Graph Theory *8* (1984), 487–499. zbl MR doi

[5] *S. A. Burr*: Diagonal Ramsey numbers for small graphs. J. Graph Theory *7* (1983), 57–69. zbl MR doi
[6] *S. A. Burr, P. Erdős, R. J. Faudree, R. H. Schelp*: On the difference between consecutive Ramsey numbers. Util. Math. *35* (1989), 115–118. zbl MR
[7] *G. G. Chappell*: GraphR [computer software]. August 26, 2016. Available at `https://www.cs.uaf.edu/users/chappell/public_html/papers/graphr/`.
[8] *G. G. Chappell, J. Gimbel*: On defective Ramsey numbers. Avaible at `https://www.cs.uaf.edu/users/chappell/public_html/papers/defram/`.
[9] *G. Chartrand, L. Lesniak, P. Zhang*: Graphs & Digraphs. CRC Press, Boca Raton, 2011. zbl MR
[10] *E. J. Cockayne, C. M. Mynhardt*: On 1-dependent Ramsey numbers for graphs. Discuss. Math., Graph Theory *19* (1999), 93–110. zbl MR doi
[11] *L. J. Cowen, R. H. Cowen, D. R. Woodall*: Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency. J. Graph Theory *10* (1986), 187–195. zbl MR doi
[12] *L. Cowen, W. Goddard, C. E. Jesurum*: Defective coloring revisited. J. Graph Theory *24* (1997), 205–219. zbl MR doi
[13] *N. Eaton, T. Hull*: Defective list colorings of planar graphs. Bull. Inst. Combin. Appl. *25* (1999), 79–87. zbl MR
[14] *T. Ekim, J. Gimbel*: Some defective parameters in graphs. Graphs Combin. *29* (2013), 213–224. zbl MR doi
[15] *H. Era, M. Urabe*: On the k-independent sets of graphs. Proc. Fac. Sci. Tokai Univ. *26* (1991), 1–4. zbl MR
[16] *P. Erdős*: Some remarks on the theory of graphs. Bull. Am. Math. Soc. *53* (1947), 292–294. zbl MR doi
[17] *P. Erdős, G. Szekeres*: A combinatorial problem in geometry. Compos. Math. *2* (1935), 463–470. zbl MR
[18] *L. Esperet, G. Joret*: Colouring planar graphs with three colours and no large monochromatic components. Comb. Probab. Comput. *23* (2014), 551–570. zbl MR doi
[19] *L. Esperet, P. Ochem*: Islands in graphs on surfaces. SIAM J. Discrete Math. *30* (2016), 206–219. zbl MR doi
[20] *A. Farrugia*: Vertex-partitioning into fixed additive induced-hereditary properties is NP-hard. Electron. J. Comb. *11* (2004), research paper R46, 9 pages. zbl MR
[21] *J. F. Fink, M. S. Jacobson*: On n-domination, n-dependence and forbidden subgraphs. Graph Theory with Applications to Algorithms and Computer Science. Proc. 5th Quadr. Int. Conf. on the Theory and Applications of Graphs with special emphasis on Algorithms and Computer Science Applications, Kalamazoo, 1984 (Y. Alavi et al., eds.). Wiley-Interscience Publication, John Wiley & Sons, New York, 1985, pp. 301–311. zbl MR
[22] *M. R. Garey, D. S. Johnson*: The rectilinear Steiner tree problem is NP-complete. SIAM J. Appl. Math. *32* (1977), 826–834. zbl MR doi
[23] *M. R. Garey, D. S. Johnson, L. Stockmeyer*: Some simplified NP-complete graph problems. Theor. Comput. Sci. *1* (1976), 237–267. zbl MR doi
[24] *J. Gimbel, C. Hartman*: Subcolorings and the subchromatic number of a graph. Discrete Math. *272* (2003), 139–154. zbl MR doi
[25] *H. Grötzsch*: Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg, Math.-Natur. Reihe *8* (1959), 109–120. zbl MR
[26] *P. Haxell, T. Szabó, G. Tardos*: Bounded size components - partitions and transversals. J. Comb. Theory, Ser. B *88* (2003), 281–297. zbl MR doi
[27] *J. Kleinberg, R. Motwani, P. Raghavan, S. Venkatasubramanian*: Storage management for evolving databases. Proc. 38th IEEE Symposium on Foundations of Computer Science (FOCS 97). 1997, pp. 353–362.

[28] *N. Linial, J. Matoušek, O. Sheffet, G. Tardos*: Graph colouring with no large monochromatic components. Comb. Probab. Comput. *17* (2008), 577–589. zbl MR doi

[29] *R. Lortz, I. Mengersen*: Bounds on Ramsey numbers of certain complete bipartite graphs. Result. Math. *41* (2002), 140–149. zbl MR doi

[30] *L. Lovász*: On decomposition of graphs. Stud. Sci. Math. Hung. *1* (1966), 237–238. zbl MR

[31] *J. Matoušek, A. Přívětivý*: Large monochromatic components in two-colored grids. SIAM J. Discrete Math. *22* (2008), 295–311. zbl MR doi

[32] *J. Nešetřil, A. Rapaud, E. Sopena*: Colorings and girth of oriented planar graphs. Discrete Math. *165/166* (1997), 519–530. zbl MR doi

[33] *S. P. Radziszowski*: Small Ramsey numbers. Revision # 14: January 12, 2014. Electron. J. Comb. DS1, Dynamic Surveys (electronic only) (1996), 94 pages. zbl MR

[34] *C. Thomassen*: Five-coloring maps on surfaces. J. Comb. Theory, Ser. B *59* (1993), 89–105. zbl MR doi

[35] *C. Thomassen*: A short list color proof of Grötzsch's theorem. J. Comb. Theory, Ser. B *88* (2003), 189–192. zbl MR doi

Authors' addresses: *Glenn G. Chappell*, Department of Computer Science, University of Alaska, 202 Chapman Building, 513 Ambler Lane, Fairbanks, Alaska 99775-6670, USA, e-mail: `chappellg@member.ams.org`; *John Gimbel*, Department of Mathematical Sciences, University of Alaska, 101 Chapman Building, 513 Ambler Lane, Fairbanks, Alaska 99775-6660, USA, e-mail: `jggimbel@alaska.edu`.