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ON GRACEFUL COLORINGS OF TREES

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Abstract. A proper coloring $c\colon V(G) \to \{1,2,\dots,k\}$, $k \geqslant 2$ of a graph G is called a graceful k-coloring if the induced edge coloring $c'\colon E(G) \to \{1,2,\dots,k-1\}$ defined by $c'(uv) = |c(u) - c(v)|$ for each edge uv of G is also proper. The minimum integer k for which G has a graceful k-coloring is the graceful chromatic number $\chi_g(G)$. It is known that if T is a tree with maximum degree Δ, then $\chi_g(T) \leqslant \lceil \frac{5}{3}\Delta \rceil$ and this bound is best possible. It is shown for each integer $\Delta \geqslant 2$ that there is an infinite class of trees T with maximum degree Δ such that $\chi_g(T) = \lceil \frac{5}{3}\Delta \rceil$. In particular, we investigate for each integer $\Delta \geqslant 2$ a class of rooted trees $T_{\Delta,h}$ with maximum degree Δ and height h. The graceful chromatic number of $T_{\Delta,h}$ is determined for each integer $\Delta \geqslant 2$ when $1 \leqslant h \leqslant 4$. Furthermore, it is shown for each $\Delta \geqslant 2$ that $\lim\limits_{h\to\infty} \chi_g(T_{\Delta,h}) = \lceil \frac{5}{3}\Delta \rceil$.

Keywords: graceful coloring; graceful chromatic numbers; tree

MSC 2010: 05C15, 05C78

1. Introduction

In 1967, Rosa in [7] introduced a vertex labeling of a graph that he called a *β-valuation.* In 1972, Golomb in [5] referred to this labeling as a graceful labeling-terminology that has become standard. As described in [4], graceful labelings, and graph labelings in general, serve as useful models for a broad range of applications, such as coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing and database management. Let G be a graph of order n and size m. A *graceful labeling* of G is a one-to-one function $f\colon V(G) \to \{0,1,\dots,m\}$ that, in turn, assigns to each edge uv of G the label $f'(uv) = |f(u) - f(v)|$ such that no two edges of G are labeled the same. Therefore, if f is a graceful labeling of G, then the set of edge labels is $\{1,2,\dots,m\}$. A graph possessing a graceful labeling is a *graceful graph.* A major problem in this area is that of determining which graphs are graceful.

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Trees constitute one of the simplest yet most important classes of graphs. Trees appeared implicitly in the 1847 work of the German physicist Gustav Kirchhoff in his study of currents in electrical networks, see [6], while Arthur Cayley in [2] used trees in 1857 to count certain types of chemical compounds. Trees are important to the understanding of the structure of graphs and are used to systematically visit the vertices of a graph. Trees and rooted trees are also widely used in computer science as a means to organize and utilize data. One of the best known conjectures dealing with graceful graphs involves trees and is due to Kotzig and Ringel, see [4].

The Graceful Tree Conjecture. *Every nontrivial tree is graceful.*

The *gracefulness* $\mathrm{grac}(G)$ of a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ is the smallest positive integer k for which it is possible to label the vertices of G with distinct elements of the set $\{0, 1, 2, \ldots, k\}$ in such a way that distinct edges receive distinct labels. The gracefulness of every such graph is defined, for if we label v_i by 2^{i-1} for $1 \leqslant i \leqslant n$, then this vertex labeling has the desired property. Thus, if G is a graph of order n and size m, then $m \leqslant \mathrm{grac}(G) \leqslant 2^{n-1}$. If $\mathrm{grac}(G) = m$, then G is graceful. The gracefulness of a graph G can therefore be considered as a measure of how close G is to being graceful—the closer the gracefulness of a graph G is to m, the closer G is to being graceful. The exact value of $\mathrm{grac}(K_n)$ has been determined for $1 \leqslant n \leqslant 10$ (see [5]). For example, $\mathrm{grac}(K_4) = 6 = \binom{4}{2}$, $\mathrm{grac}(K_5) = 11 = \binom{5}{2} + 1$ and $\mathrm{grac}(K_6) = 17 = \binom{6}{2} + 2$. The exact value of $\chi_g(K_n)$ is not known in general, however. On the other hand, Erdős showed that $\mathrm{grac}(K_n) \sim n^2$ (see [5]).

Graceful labelings have also been looked at in terms of colorings. A *rainbow vertex coloring* of a graph G of size m is an assignment f of distinct colors to the vertices of G. If the colors are chosen from the set $\{0, 1, \ldots, m\}$, resulting in each edge uv of G being colored $f'(uv) = |f(u) - f(v)|$ such that the colors assigned to the edges of G are also distinct, then this *rainbow* vertex coloring induces a *rainbow* edge coloring $f'\colon E(G) \to \{1, 2, \ldots, m\}$. So, such a rainbow vertex coloring is a graceful labeling of G.

The colorings of graphs that have received the most attention, however, are *proper vertex colorings* and *proper edge colorings*. In such a coloring of a graph G, every two adjacent vertices or every two adjacent edges are assigned distinct colors. The minimum number of colors needed in a proper vertex coloring of G is its *chromatic number*, denoted by $\chi(G)$, while the minimum number of colors needed in a proper edge coloring of G is its *chromatic index*, denoted by $\chi'(G)$.

Inspired by graceful labelings, we considered vertex colorings that induce edge colorings, both of which are proper rather than rainbow, see [1].

It is useful to describe notation for certain intervals of integers. For positive integers a, b with $a \leqslant b$, let $[a, b] = \{a, a + 1, \ldots, b\}$ and $[b] = [1, b]$. A *graceful* *k-coloring* of a nonempty graph G is a proper vertex coloring $c\colon V(G) \to [k]$, where $k \geqslant 2$, that induces a proper edge coloring $c'\colon E(G) \to [k-1]$ defined by $c'(uv) = |c(u) - c(v)|$. A vertex coloring c of a graph G is a *graceful coloring* if c is a graceful k-coloring for some $k \in \mathbb{N}$. The minimum k for which G has a graceful k-coloring is called the *graceful chromatic number* of G, denoted by $\chi_g(G)$. It was observed in [1] that if G is a nonempty graph of order n, then $\chi_g(G)$ exists and

$$\max\bigl\{\chi(G), \chi'(G)\bigr\} + 1 \leqslant \chi_g(G) \leqslant \operatorname{grac}(G) \leqslant 2^{n-1}.$$

For a graceful k-coloring c of a graph G, the *complementary coloring* $\overline{c}\colon V(G) \to [k]$ of G is a k-coloring defined by $\overline{c}(v) = k + 1 - c(v)$ for each vertex v of G.

Observation 1.1 ([1]). The complementary coloring of a graceful coloring of a graph is also graceful.

If c is a graceful k-coloring of a graph G, then the restriction of c to a subgraph H of G is also a graceful coloring. Thus, we have the following observation.

Observation 1.2 ([1]). If H is a subgraph of a graph G, then $\chi_g(H) \leqslant \chi_g(G)$.

A *caterpillar* is a tree T of order 3 or more, the removal of whose leaves produces a path (called the *spine* of T). The graceful chromatic numbers of all caterpillars were determined in [1].

Theorem 1.3 ([1]). *If T is a caterpillar with maximum degree Δ, then $\Delta + 1 \leqslant \chi_g(T) \leqslant \Delta+2$. Furthermore, $\chi_g(T) = \Delta+2$ if and only if T has a vertex of degree Δ that is adjacent to two vertices of degree Δ in T.*

Also in [1], an upper bound for the graceful chromatic number of a tree in terms of its maximum degree was obtained. An example of a tree was given in [1] to show that the upper bound can be attained.

Theorem 1.4 ([1]). *If T is a nontrivial tree with maximum degree Δ, then $\chi_g(T) \leqslant \lceil \frac{5}{3}\Delta \rceil$.*

We refer to the book [3] for graph theory notation and terminology not described in this paper.

2. Rooted trees $T_{\Delta,h}$ for small h

We now describe a class of trees that will play a central role in this paper. For each integer $\Delta \geqslant 2$, let $T_{\Delta,1}$ be the star $K_{1,\Delta}$. The *central vertex* of $T_{\Delta,1}$ is denoted by v. Thus, $\deg v = \Delta$ and all other vertices of $T_{\Delta,1}$ have degree 1. For each integer $h \geqslant 2$, let $T_{\Delta,h}$ be the tree obtained from $T_{\Delta,h-1}$ by identifying each end-vertex with the central vertex of the star $K_{1,\Delta-1}$. The tree $T_{\Delta,h}$ is therefore a rooted tree (with root v) having height h. The vertex v is then the *central vertex* of $T_{\Delta,h}$. In $T_{\Delta,h}$, every vertex at distance less than h from v has degree Δ; while all remaining vertices are leaves and are at distance h from v. Thus, $T_{2,2} = P_5$ and $T_{2,h}$ is a nontrivial path of odd order $2h+1$, while $T_{3,2}$ and $T_{6,2}$ are shown in Figure 1.

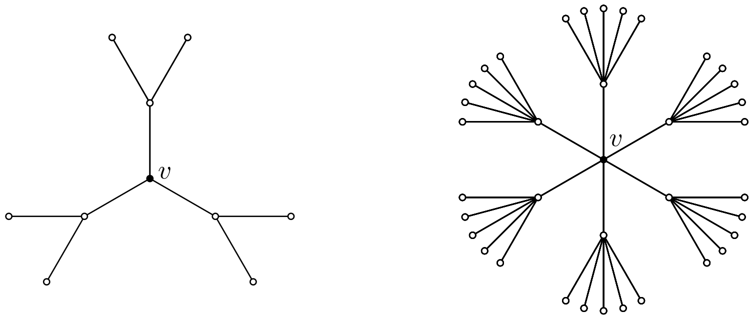

Figure 1. The trees $T_{3,2}$ and $T_{6,2}$.

First, we determine the graceful chromatic number of $T_{\Delta,2}$ for each integer $\Delta \geqslant 2$.

Theorem 2.1. *For each integer $\Delta \geqslant 2$, $\chi_g(T_{\Delta,2}) = \lceil \frac{1}{2}(3\Delta+1)\rceil$.*

Proof. Let $T = T_{\Delta,2}$. Suppose that the central vertex of T is v and $N(v) = \{v_1, v_2, \dots, v_\Delta\}$. For $i = 1, 2, \dots, \Delta$, let $v_{i,1}, v_{i,2}, \dots, v_{i,\Delta-1}$ be the $\Delta - 1$ end-vertices that are adjacent to v_i in T. We first show that $\chi_g(T) \leqslant \lceil \frac{1}{2}(3\Delta+1)\rceil$. There are two cases, according to whether Δ is even or Δ is odd.

Case 1: *Δ is even.* Then $\Delta = 2k$ for some $k \in \mathbb{N}$ and so $\lceil \frac{1}{2}(3\Delta+1)\rceil = 3k+1$. Let $[3k+1] = S_1 \cup S_2 \cup S_3$, where $S_1 = [k+1]$, $S_2 = [k+2, 2k]$ and $S_3 = [2k+1, 3k+1]$. Thus, $|S_1| = |S_3| = k+1$ and $|S_2| = k-1$. Define a coloring $c\colon V(T) \to [3k+1]$ by $c(v) = k+1$, $c(v_i) = i$ for $1 \leqslant i \leqslant k$ and $c(v_i) = i+k+1$ for $k+1 \leqslant i \leqslant 2k$. Hence, $\{c'(vv_i)\colon 1 \leqslant i \leqslant 2k\} = [2k]$. Next, for $1 \leqslant i \leqslant k$, let $\{c(v_{i,j})\colon 1 \leqslant j \leqslant 2k-1\} = [i+1, i+2k] - \{k+1\}$; while for $k+1 \leqslant i \leqslant 2k$, let $\{c(v_{i,j})\colon 1 \leqslant j \leqslant 2k-1\} = [2k] - \{k+1\}$. Thus, if $1 \leqslant i \leqslant k$, then $c'(vv_i) = k+1-i$ and $c'(v_iv_{i,j}) \neq k+1-i$; while if $k+1 \leqslant i \leqslant 2k$, then $c'(vv_i) = i$ and $c'(v_iv_{i,j}) \neq i$. Therefore, c is a graceful coloring using the colors in $[3k+1]$ and so $\chi_g(T) \leqslant 3k+1$.

Case 2: Δ *is odd.* Then $\Delta = 2k+1$ for some $k \in \mathbb{N}$ and so $\lceil \frac{1}{2}(3\Delta+1) \rceil = 3k+2$. Let $[3k+2] = S_1 \cup S_2 \cup S_3$, where $S_1 = [k+1]$, $S_2 = [k+2, 2k+1]$ and $S_3 = [2k+2, 3k+2]$. Thus, $|S_1| = |S_3| = k+1$ and $|S_2| = k$. Define a coloring $c\colon V(T) \to [3k+2]$ by $c(v) = k+1$, $c(v_i) = i$ for $1 \leqslant i \leqslant k$ and $c(v_i) = i+k+1$ for $k+1 \leqslant i \leqslant 2k+1$. Hence, $\{c'(vv_i)\colon 1 \leqslant i \leqslant 2k+1\} = [2k+1]$. Next, for $1 \leqslant i \leqslant k$, let $\{c(v_{i,j})\colon 1 \leqslant j \leqslant 2k\} = [i+1, i+2k+1] - \{k+1\}$; while for $k+1 \leqslant i \leqslant 2k$, let $\{c(v_{i,j})\colon 1 \leqslant j \leqslant 2k\} = [2k+1] - \{k+1\}$. Thus, if $1 \leqslant i \leqslant k$, then $c'(vv_i) = k+1-i$ and $c'(v_iv_{i,j}) \neq k+1-i$; while if $k+1 \leqslant i \leqslant 2k+1$, then $c'(vv_i) = i$ and $c'(v_iv_{i,j}) \neq i$. Therefore, c is a graceful coloring using the colors in $[3k+2]$ and so $\chi_g(T) \leqslant 3k+2$.

Next, we show that $\chi_g(T) \geqslant \lceil \frac{1}{2}(3\Delta+1) \rceil$. Again, we consider two cases, according to whether Δ is even or Δ is odd.

Case 1: Δ *is even.* Then $\Delta = 2k$ for some $k \in \mathbb{N}$ and so $\lceil \frac{1}{2}(3\Delta+1) \rceil = 3k+1$. Assume, to the contrary, that there is graceful coloring c of T using colors from $[3k]$. Let $[3k] = S_1 \cup S_2 \cup S_3$, where $S_1 = [k]$, $S_2 = [k+1, 2k]$ and $S_3 = [2k+1, 3k]$. Thus, $|S_1| = |S_2| = |S_3| = k$. We claim that no vertex having degree $2k$ can be assigned a color in S_2; for otherwise, let $w \in N[v]$ such that $c(w) \in S_2$. Then $k+1 \leqslant c(w) \leqslant 2k$. Since $\deg w = 2k$, there is an edge incident with w, say wx, such that $|c(w) - c(x)| \geqslant 2k$. Hence, either $c(x) - c(w) \geqslant 2k$ or $c(w) - c(x) \geqslant 2k$. That is, either $c(x) \geqslant 2k + c(w) \geqslant 3k+1$ or $c(x) \leqslant c(w) - 2k \leqslant 0$, which is impossible. Therefore, every vertex in $N[v]$ must be assigned a color from $S_1 \cup S_3$. Since $|N[v]| = 2k+1$ and $|S_1 \cup S_3| = 2k$, a contradiction is produced.

Case 2: Δ *is odd.* Then $\Delta = 2k+1$ for some $k \in \mathbb{N}$ and so $\lceil \frac{1}{2}(3\Delta+1) \rceil = 3k+2$. Assume, to the contrary, that there is graceful coloring c of T using colors from $[3k+1]$. Let $[3k+1] = S_1 \cup S_2 \cup S_3$, where $S_1 = [k]$, $S_2 = [k+1, 2k+1]$ and $S_3 = [2k+2, 3k+1]$. Thus, $|S_1| = |S_3| = k$ and $|S_2| = k+1$. We claim that no vertex having degree $2k+1$ can be assigned a color in S_2; for otherwise, let $w \in N[v]$ such that $c(w) \in S_2$. Then $k+1 \leqslant c(w) \leqslant 2k+1$. Since $\deg w = 2k+1$, there is an edge wx such that $|c(w) - c(x)| \geqslant 2k+1$. Hence, either $c(x) - c(w) \geqslant 2k+1$ or $c(w) - c(x) \geqslant 2k+1$. That is, either $c(x) \geqslant 2k+1+c(w) \geqslant 3k+2$ or $c(x) \leqslant c(w) - 2k - 1 \leqslant 0$, which is impossible. Therefore, every vertex in $N[v]$ must be assigned a color from $S_1 \cup S_3$. Since $|N[v]| = 2k+1$ and $|S_1 \cup S_3| = 2k$, a contradiction is produced. $\square$

Next, we determine the graceful chromatic number of $T_{\Delta,3}$ for each integer $\Delta \geqslant 2$.

Theorem 2.2. *For each integer* $\Delta \geqslant 2$, $\chi_g(T_{\Delta,3}) = \lceil \frac{1}{8}(13\Delta+1) \rceil$.

Proof. First, observe that $\chi_g(P_7) = 4$ by Theorem 1.3. Thus, we may assume that $\Delta \geqslant 3$. To simplify notation, let $K = \lceil \frac{1}{8}(13\Delta+1) \rceil$ and let $T = T_{\Delta,3}$ whose central vertex is v. Hence, $\frac{13}{8}\Delta < K < 2\Delta$. Furthermore, $3K \leqslant 5\Delta + 2$ and so

$2K-1-3\Delta \leqslant 2\Delta - K + 1$. Since $K \geqslant \frac{1}{8}(13\Delta+1)$ (or $8K \geqslant 13\Delta + 1$) and $\Delta \geqslant 3$, it follows that

$$3K \geqslant \frac{39\Delta+3}{8} = \frac{32\Delta}{8} + \frac{7\Delta+3}{8} \geqslant 4\Delta + 3$$

and so $2\Delta - K + 1 < 2K - 1 - 2\Delta$.

For each $i = 0, 1, 2, 3$, let V_i be the set of vertices at distance i from v in T. In particular, $V_0 = \{v\}$ and $V_0 \cup V_1 = N[v]$. Thus, $V_0 \cup V_1 \cup V_2$ is the set of vertices of degree Δ in T and V_3 is the set of end-vertices of T.

First, we show that $\chi_g(T) \geqslant K$. Assume, to the contrary, that T has a graceful coloring c using colors from the set $[K-1] = \{1, 2, \ldots, K-1\}$. We first verify two claims.

Claim 1. If $w \in V_0 \cup V_1 \cup V_2$ (that is, if $\deg_T w = \Delta$), then $c(w) \notin [K-\Delta, \Delta]$.

P r o o f of Claim 1. As $\deg_T w = \Delta$, there is $x \in N(w)$ such that $|c(x) - c(w)| \geqslant \Delta$. Thus, either $c(x) - c(w) \geqslant \Delta$ or $c(w) - c(x) \geqslant \Delta$. Since $K - \Delta \leqslant c(w) \leqslant \Delta$, it follows that either $c(x) \geqslant \Delta + c(w) \geqslant K$ or $c(x) \leqslant c(w) - \Delta \leqslant 0$, both of which are impossible. Hence, Claim 1 holds. □

Claim 2. If $w \in V_0 \cup V_1 = N[v]$ (and so each neighbor of w has degree Δ), then

$$c(w) \notin [2K-1-3\Delta, 2\Delta-K+1] \cup [2K-1-2\Delta, 3\Delta-K+1]. \tag{1}$$

P r o o f of Claim 2. Assume, to the contrary, that there is $w \in N[v]$ such that (1) is false, say $c(w) \in [2K-1-3\Delta, 2\Delta-K+1]$. Now, consider the number of colors available for the vertices in $N[w]$.

- ▷ There are at most $K - \Delta - 1$ colors in $[\Delta+1, K-1]$ that are available for the vertices in $N[w]$.
- ▷ Since $c(w) \in [2K-1-3\Delta, 2\Delta-K+1]$, it follows that

$$\max\{c(w), K-\Delta-c(w)\} \leqslant 2\Delta - K + 1.$$

  Hence, there are at most $2\Delta - K + 1$ colors in $[K-\Delta+1]$ that are available for the vertices in $N[w]$.

Thus, there are at most $(K-\Delta-1) + (2\Delta-K+1) = \Delta$ colors in $[K-1]$ that are available for the vertices in $N[w]$. Since $|N[w]| = \Delta + 1$, this is impossible and so Claim 2 holds. □

We now consider the number of colors available for the vertices in $N[v]$. By Claims 1 and 2, if $z \in N[v]$, then $c(z)$ belongs to the set

$$\begin{aligned} S = {} & [2K-3\Delta-2] \cup [2\Delta-K+2, K-\Delta-1] \\ & \cup [\Delta+1, 2K-2\Delta-2] \cup [3\Delta-K+2, K-1]. \end{aligned}$$

Since $K = \lceil \frac{1}{8}(13\Delta + 1) \rceil < \frac{1}{8}(13\Delta + 1) + 1$, it follows that $8K < 13\Delta + 9$ and so

$$|S| = 8K - 12\Delta - 8 < \Delta + 1 = |N[v]|,$$

which is impossible. Therefore, there is no graceful coloring of T using colors from the set $[K - 1]$ and so $\chi_g(T) \geqslant K$.

Next, we show that T has a graceful coloring using colors from $[K]$. Let

$$S^* = S_1 \cup S_2 \cup S_3 \cup S_4,$$

where

$$\begin{aligned} S_1 &= [2K - 3\Delta], \\ S_2 &= [2\Delta - K + 1, K - \Delta], \\ S_3 &= [\Delta + 1, 2K - 2\Delta], \\ S_4 &= [3\Delta - K + 1, K]. \end{aligned}$$

We first verify the following claim.

Claim 3. For each $a \in S^*$, there are at least Δ distinct elements

$$a_1, a_2, \ldots, a_\Delta \in ([K - \Delta] \cup [\Delta + 1, K]) - \{a\}$$

such that all of the Δ integers $|a - a_1|, |a - a_2|, \ldots, |a - a_\Delta|$ are distinct.

Proof of Claim 3. By Observation 1.1, we may assume that $a \in S_1 \cup S_2$. There are two cases.

Case 1: $a \in S_1 = [2K - 3\Delta]$. Observe that $K - \Delta < \Delta$ and

$$|[2K - 3\Delta + 1, K - \Delta] \cup [\Delta + 1, K]| = \Delta.$$

Let $\{a_1, a_2, \ldots, a_\Delta\} = [2K - 3\Delta + 1, K - \Delta] \cup [\Delta + 1, K]$. Since $a \leqslant 2K - 3\Delta$, it follows that $|a - a_1|, |a - a_2|, \ldots, |a - a_\Delta|$ are distinct.

Case 2: $a \in S_2 = [2\Delta - K + 1, K - \Delta]$. Observe that

(1) $\Delta + 1 - a \geqslant \Delta + 1 - (K - \Delta) = 2\Delta - K + 1$,
(2) $|[a - (2\Delta - K), a - 1]| = 2\Delta - K \geqslant 1$ and
(3) $|[a - (2\Delta - K), a - 1] \cup [\Delta + 1, K]| = \Delta$.

Let $\{a_1, a_2, \ldots, a_\Delta\} = [a - (2\Delta - K), a - 1] \cup [\Delta + 1, K]$. It then follows by (1)–(3) that all $|a - a_1|, |a - a_2|, \ldots, |a - a_\Delta|$ are distinct.

Therefore, Claim 3 holds. □

Observe that $|S^*| = 8K - 12\Delta \geqslant (13\Delta+1) - 12\Delta = \Delta+1$. We now define a graceful coloring $c\colon V(T) \to [K]$ as follows. First, let $c(v) = 1$ and assign colors from $S^* - \{1\}$ to the vertices in $V_1 = N(v)$ such that vertices and edges of $T[V_0 \cup V_1]$ are properly colored. For each $w \in V_1$, since $c(w) = a \in S^*$, it follows by Claim 3 that there are $a_1, a_2, \ldots, a_\Delta \in ([K-\Delta] \cup [\Delta+1, K]) - \{a\}$ for which $|a - a_1|, |a - a_2|, \ldots, |a - a_\Delta|$ are distinct. Thus, we can color the vertices in V_2 using colors from the set $[K-\Delta] \cup [\Delta+1, K]$ such that vertices and edges of $T[V_0 \cup V_1 \cup V_2]$ are properly colored. It remains to color the vertices in V_3. For each $x \in V_2$, either $c(x) \in [K - \Delta]$ or $c(x) \in [\Delta+1, K]$, say the former. Since there are Δ colors in $[K-\Delta+1, K]$ that are available for the vertices in $N(x)$, there are at least $\Delta - 1$ colors that are available for the $\Delta - 1$ children of x in V_3.

Therefore, T has a graceful coloring using colors from $[K]$ and so $\chi_g(T) \leqslant K$. $\square$

Employing an approach similar to that used to verify Theorem 2.2, one can obtain the following result.

Theorem 2.3. *For each integer $\Delta \geqslant 2$, $\chi_g(T_{\Delta,4}) = \lceil \frac{1}{32}(53\Delta+1) \rceil$.*

The results obtained in Theorems 2.1–2.3 suggest the following conjecture.

Conjecture 2.4. *For an integer $h \geqslant 2$, let $\sigma_h = 2^{2h-3} + \sum_{i=2}^{h} 2^{2i-4}$. Then*

$$\chi_g(T_{\Delta,h}) = \left\lceil \frac{\sigma_h \Delta + 1}{2^{2h-3}} \right\rceil.$$

3. Rooted trees $T_{\Delta,h}$ for large h

In this section, we show that if $\Delta \geqslant 2$ and $h \geqslant 2 + \lfloor \frac{1}{3}\Delta \rfloor$, then $\chi_g(T_{\Delta,h}) = \lceil \frac{5}{3}\Delta \rceil$. First, we present three lemmas.

Lemma 3.1. *Let Δ and h be integers with $\Delta = 3k$ for some integer $k \geqslant 2$ and $2 \leqslant h \leqslant k+1$ and let v be the central vertex of the tree $T_{\Delta,h}$. If c is a graceful $(5k-1)$-coloring of $T_{\Delta,h}$, then $c(v) \neq k \pm j$ and $c(v) \neq 4k \pm j$ for each j with $0 \leqslant j \leqslant h-2$; that is,*

$$c(v) \notin [k-h+2, k+h-2] \cup [4k-h+2, 4k+h-2].$$

P r o o f. We proceed by induction on $h \geqslant 2$. Consider the tree $T_{\Delta,2}$ whose central vertex is v. Assume, to the contrary, that there is a graceful $(5k-1)$-coloring c of $T_{\Delta,2}$ such that $c(v) \in \{k, 4k\}$. By Observation 1.1, we may assume that $c(v) = k$. First, we claim that

(2) if w is a vertex of $T_{\Delta,2}$ such that $\deg_{T_{\Delta,2}} w = \Delta$, then $c(w) \notin [2k, 3k]$.

Suppose that (2) is false. Then there is a vertex w in $T_{\Delta,2}$ such that $\deg_{T_{\Delta,2}} w = \Delta$ and $2k \leqslant c(w) \leqslant 3k$. Necessarily, there is $x \in N(w)$ such that $|c(w) - c(x)| \geqslant 3k$. Thus, either $c(x) \leqslant c(w) - 3k \leqslant 0$ or $c(x) \geqslant 3k + c(w) \geqslant 5k$, both of which are impossible. Thus, $c(w) \notin [2k, 3k]$ and, as claimed, (2) holds.

Next, we consider the number of colors that are available for the vertices in $N(v)$. If $x \in N(v)$, then $\deg_{T_{\Delta,2}} x = \Delta$ and so by (2),

$$c(x) \in [2k-1] \cup [3k+1, 5k-1].$$

By symmetry, we may assume that $c(x) \in [2k-1]$. Observe that $|[3k+1, 5k-1]| = 2k-1$. Since $c(v) = k$, there are at most

$$\max\{c(v) - 1, 2k - 1 - c(v)\} = \max\{k-1, 2k-1-k\} = k-1$$

colors in $[2k-1]$ that are available for the vertices in $N(v)$. Hence, there are at most $(2k-1) + (k-1) = 3k - 2 = \Delta - 2$ colors available for the vertices in $N(v)$, which is impossible. Thus, $c(v) \notin \{k, 4k\}$, establishing the base step.

Next, assume for some integer h with $3 \leqslant h \leqslant k+1$ that if c^* is a graceful $(5k-1)$-coloring of the tree $T_{\Delta,h-1}$ with central vertex v, then $c^*(v) \neq k \pm j$ and $c^*(v) \neq 4k \pm j$ for each j with $0 \leqslant j \leqslant h-3$; that is,

$$c^*(v) \notin [k-h+3, k+h-3] \cup [4k-h+3, 4k+h-3].$$

Suppose that c is a graceful $(5k-1)$-coloring of $T_{\Delta,h}$ with central vertex v. We show that $c(v) \neq k \pm j$ and $c(v) \neq 4k \pm j$ for each j with $0 \leqslant j \leqslant h-2$. Since

(i) v is the center of the subtree $T' = T_{\Delta,h-1}$ obtained by removing all leaves from $T_{\Delta,h}$ and

(ii) the restriction of c to T' is a graceful $(5k-1)$-coloring of T',

it follows by the induction hypothesis that $c(v) \neq k \pm j$ and $c(v) \neq 4k \pm j$ for each j with $0 \leqslant j \leqslant h-3$. Hence, it suffices to show that $c(v) \neq k \pm (h-2)$ and $c(v) \neq 4k \pm (h-2)$. Furthermore, by Observation 1.1, it suffices to show that $c(v) \neq k \pm (h-2)$. We consider two cases, according to whether $c(v) = k + h - 2$ or $c(v) = k - h + 2$.

Case 1: $c(v) = k+h-2$. Let $x \in N(v)$. Since x is the center of a subtree of $T_{\Delta,h}$ that is isomorphic to $T_{\Delta,h-1}$, it follows by the induction hypothesis that

$$c(x) \notin [k-h+3, k+h-3] \cup [4k-h+3, 4k+h-3]. \tag{3}$$

It then follows by (2) and (3) that $c(x)$ belongs to the set

$$S = [k-h+2] \cup [k+h-1, 2k-1] \cup [3k+1, 4k-h+2] \cup [4k+h-2, 5k-1].$$

Observe that

$$|[3k+1, 4k-h+2] \cup [4k+h-2, 5k-1]| = 2k-2h+4.$$

Since $h \geqslant 3$, it follows that $k+h-3 > k-h+1$ and so $|[k-h+3]| > |[k+h-1, 2k-1]|$. Because at most one of the two colors $c(v)-t$ and $c(v)+t$ ($1 \leqslant t \leqslant k+h-3$) can be used for a vertex in $N(v)$, it follows that at most $(2k-2h+4)+(k+h-3) = 3k-h+1$ colors in $[5k-1]$ are available for the vertices in $N(v)$. However, $3k-h+1 < 3k-1 = \Delta-1$ and $|N(v)| = \Delta$, which is impossible.

Case 2: $c(v) = k-h+2$. Let $x \in N(v)$. Since x is the center of a subtree isomorphic to $T_{\Delta,h-1}$, it follows by the induction hypothesis that $c(x)$ satisfies (3). Now by (2), $c(x)$ belongs to

$$S = [k-h+1] \cup [k+h-2, 2k-1] \cup [3k+1, 4k-h+2] \cup [4k+h-2, 5k-1].$$

Observe that

$$|[3k+1, 4k-h+2] \cup [4k+h-2, 5k-1]| = 2k-2h+4.$$

Since $h \geqslant 3$, it follows that $k+h-3 > k-h+1$ and so $|[k-h+3, 2k-1]| > |[k-h+1]|$. Because at most one of the two colors $c(v)-t$ and $c(v)+t$ ($1 \leqslant t \leqslant k+h-3$) can be used for a vertex in $N(v)$, it follows that at most $(2k-2h+4)+(k+h-3) = 3k-h+1$ colors in $[5k-1]$ are available for the vertices in $N(v)$. However, $3k-h+1 < 3k-1 = \Delta-1$ and $|N(v)| = \Delta$, which is impossible. $\square$

Lemma 3.2. *Let Δ and h be integers with $\Delta = 3k+1$ for some integer $k \geqslant 2$ and $2 \leqslant h \leqslant k+1$ and let v be the central vertex of the tree $T_{\Delta,h}$. If c is a graceful $(5k+1)$-coloring of $T_{\Delta,h}$, then $c(v) \neq k \pm j$ and $c(v) \neq 4k+2 \pm j$ for each j with $0 \leqslant j \leqslant h-2$; that is,*

$$c(v) \notin [k-h+2, k+h-2] \cup [4k-h+4, 4k+h].$$

P r o o f. We proceed by induction on $h \geqslant 2$. Consider the tree $T_{\Delta,2}$ whose central vertex is v. Assume, to the contrary, that there is a graceful $(5k+1)$-coloring c of $T_{\Delta,2}$ such that $c(v) \in \{k, 4k+2\}$. By Observation 1.1, we may assume that $c(v) = k$. First, we claim that

(4) if w is a vertex of $T_{\Delta,2}$ such that $\deg_{T_{\Delta,2}} w = \Delta$, then $c(w) \notin [2k+1, 3k+1]$.

Suppose that (4) is false. Then there is a vertex w in $T_{\Delta,2}$ such that $\deg_{T_{\Delta,2}} w = \Delta$ and $2k+1 \leqslant c(w) \leqslant 3k+1$. Necessarily, there is $x \in N(w)$ such that $|c(w) - c(x)| \geqslant 3k+1$. Thus, either $c(x) \leqslant c(w) - (3k+1) \leqslant 0$ or $c(x) \geqslant (3k+1) + c(w) \geqslant 5k+2$, both of which are impossible. Thus, $c(w) \notin [2k+1, 3k+1]$ and, as claimed, (4) holds.

Next, we consider the number of colors that are available for the vertices in $N(v)$. If $x \in N(v)$, then $\deg_{T_{\Delta,2}} x = \Delta$ and so by (4),

$$c(x) \in [2k] \cup [3k+2, 5k+1].$$

Observe that $|[3k+2, 5k+1]| = 2k$ and that at most

$$\max\{c(v) - 1, 2k - c(v)\} = 2k - c(v) = k$$

colors in $[2k]$ are available for the vertices in $N(v)$. Hence, there are at most $2k + k = 3k = \Delta - 1$ colors available for the vertices in $N(v)$, which is impossible. Thus, $c(v) \notin \{k, 4k+2\}$, establishing the base step.

Next, assume for some integer h with $3 \leqslant h \leqslant k+1$ that if c^* is a graceful $(5k+1)$-coloring of the tree $T_{\Delta,h-1}$ with central vertex v, then $c^*(v) \neq k \pm j$ and $c^*(v) \neq 4k+2 \pm j$ for each j with $0 \leqslant j \leqslant h-3$; that is,

$$c^*(v) \notin [k-h+3, k+h-3] \cup [4k-h+5, 4k+h-1].$$

Suppose that c is a graceful $(5k+1)$-coloring of $T_{\Delta,h}$ with central vertex v. We show that $c(v) \neq k \pm j$ and $c(v) \neq 4k+2 \pm j$ for each j with $0 \leqslant j \leqslant h-2$. Since

(i) v is the center of the subtree $T' = T_{\Delta,h-1}$ obtained by removing all leaves from $T_{\Delta,h}$ and

(ii) the restriction of c to T' is a graceful $(5k+1)$-coloring of T',

it follows by the induction hypothesis that $c(v) \neq k \pm j$ and $c(v) \neq 4k+2 \pm j$ for each j with $0 \leqslant j \leqslant h-3$; that is,

$$c(v) \notin [k-h+3, k+h-3] \cup [4k-h+5, 4k+h-1].$$

Hence, it suffices to show that $c(v) \neq k \pm (h-2)$ and $c(v) \neq 4k+2 \pm (h-2)$. Furthermore, by Observation 1.1, it suffices to show that $c(v) \neq k \pm (h-2)$. We consider two cases, according to whether $c(v) = k+h-2$ or $c(v) = k-h+2$.

Case 1: $c(v) = k + h - 2$. Let $x \in N(v)$. Since x is the center of a subtree isomorphic to $T_{\Delta,h-1}$, it follows by the induction hypothesis that

$$c(x) \notin [k - h + 3, k + h - 3] \cup [4k - h + 5, 4k + h - 1]. \tag{5}$$

It then follows by (4) and (5) that $c(x)$ belongs to the set

$$S = [k - h + 2] \cup [k + h - 1, 2k] \cup [3k + 2, 4k - h + 4] \cup [4k + h, 5k + 1].$$

Observe that

$$|[3k + 2, 4k - h + 4] \cup [4k + h, 5k + 1]| = 2k - 2h + 5.$$

Since $h \geqslant 3$, it follows that $k+h-3 > k-h+2$ and so $|[k-h+3]| > |[k+h-1, 2k]|$. Because at most one of the two colors $c(v)-t$ and $c(v)+t$ $(1 \leqslant t \leqslant k+h-3)$ can be used for a vertex in $N(v)$, it follows that at most $(2k-2h+5)+(k+h-3) = 3k-h+2$ colors in $[5k + 1]$ are available for the vertices in $N(v)$. However, $3k - h + 2 < 3k = \Delta - 1$ and $|N(v)| = \Delta$, which is impossible.

Case 2: $c(v) = k - h + 2$. Let $x \in N(v)$. Since x is the center of a subtree isomorphic to $T_{\Delta,h-1}$, it follows by the induction hypothesis that $c(x)$ satisfies (5). Now by (4), $c(x)$ belongs to

$$S = [k - h + 1] \cup [k + h - 2, 2k] \cup [3k + 2, 4k - h + 4] \cup [4k + h, 5k + 1].$$

Observe that

$$|[3k + 2, 4k - h + 4] \cup [4k + h, 5k + 1]| = 2k - 2h + 5.$$

Since $h \geqslant 3$, it follows that $k+h-2 > k-h+1$ and so $|[k-h+3, 2k]| > |[k+h-1]|$. Hence, at most $(2k - 2h + 5) + (k+h-2) = 3k-h+3$ colors in $[5k+1]$ are available for the vertices in $N(v)$. However, $3k - h + 3 \leqslant 3k = \Delta - 1$ and $|N(v)| = \Delta$, which is impossible. □

Lemma 3.3. *Let Δ and h be integers with $\Delta = 3k + 2$ for some integer $k \geqslant 2$ and $2 \leqslant h \leqslant k + 1$ and let v be the central vertex of the tree $T_{\Delta,h}$. If c is a graceful $(5k + 3)$-coloring of $T_{\Delta,h}$, then $c(v) \neq (k + 1) \pm j$ and $c(v) \neq (4k + 3) \pm j$ for each j with $0 \leqslant j \leqslant h - 2$; that is,*

$$c(v) \notin [k - h + 3, k + h - 1] \cup [4k - h + 5, 4k + h + 1].$$

P r o o f. We proceed by induction on $h \geqslant 2$. Consider the tree $T_{\Delta,2}$ whose central vertex is v. Assume, to the contrary, that there is a graceful $(5k+3)$-coloring c of $T_{\Delta,2}$ such that $c(v) \in \{k+1, 4k+3\}$. By Observation 1.1, we may assume that $c(v) = k+1$. First, we claim that

(6) if w is a vertex of $T_{\Delta,2}$ such that $\deg_{T_{\Delta,2}} w = \Delta$, then $c(w) \notin [2k+2, 3k+2]$.

Suppose that (6) is false. Then there is a vertex w in $T_{\Delta,2}$ such that $\deg_{T_{\Delta,2}} w = \Delta$ and $2k+2 \leqslant c(w) \leqslant 3k+2$. Necessarily, there is $x \in N(w)$ such that $|c(w) - c(x)| \geqslant 3k+2$. Thus, either $c(x) \leqslant c(w) - (3k+2) \leqslant 0$ or $c(x) \geqslant (3k+2) + c(w) \geqslant 5k+4$, both of which are impossible. Thus, $c(w) \notin [2k+2, 3k+2]$ and, as claimed, (6) holds.

Next, we consider the number of colors that are available for the vertices in $N(v)$. If $x \in N(v)$, then $\deg_{T_{\Delta,2}} x = \Delta$ and so by (6),

$$c(x) \in [2k+1] \cup [3k+3, 5k+3].$$

Observe that $|[3k+3, 5k+3]| = 2k+1$ and that at most

$$\max\{c(v) - 1, 2k+1-c(v)\} = (2k+1) - (k+1) = k$$

colors in $[2k+1]$ are available for the vertices in $N(v)$. Hence, there are at most $(2k+1) + k = 3k+1 = \Delta - 1$ colors available for the vertices in $N(v)$, which is impossible. Thus, $c(v) \notin \{k+1, 4k+3\}$, establishing the base step.

Next, assume for some integer h with $3 \leqslant h \leqslant k+1$ that if c^* is a graceful $(5k+3)$-coloring of the tree $T_{\Delta,h-1}$ with central vertex v, then $c^*(v) \neq (k+1) \pm j$ and $c^*(v) \neq (4k+3) \pm j$ for each j with $0 \leqslant j \leqslant h-3$; that is,

$$c^*(v) \notin [k-h+4, k+h-2] \cup [4k-h+6, 4k+h].$$

Suppose that c is a graceful $(5k+3)$-coloring of $T_{\Delta,h}$ with central vertex v. We show that $c(v) \neq (k+1) \pm j$ and $c(v) \neq (4k+3) \pm j$ for each j with $0 \leqslant j \leqslant h-2$. Since

(i) v is the center of the subtree $T' = T_{\Delta,h-1}$ obtained by removing all leaves from $T_{\Delta,h}$ and

(ii) the restriction of c to T' is a graceful $(5k+3)$-coloring of T',

it follows by the induction hypothesis that $c(v) \neq (k+1) \pm j$ and $c(v) \neq (4k+3) \pm j$ for each j with $0 \leqslant j \leqslant h-3$; that is,

$$c(v) \notin [k-h+4, k+h-2] \cup [4k-h+6, 4k+h].$$

Hence, it suffices to show that $c(v) \neq (k+1) \pm (h-2)$ and $c(v) \neq (4k+3) \pm (h-2)$. Furthermore, by Observation 1.1, it suffices to show that $c(v) \neq (k+1) \pm (h-2)$. We consider two cases, according to whether $c(v) = k+h-1$ or $c(v) = k-h+3$.

Case 1: $c(v) = k+h-1$. Let $x \in N(v)$. Since x is the center of a subtree isomorphic to $T_{\Delta,h-1}$, it follows by the induction hypothesis that

$$c(x) \notin [k-h+4, k+h-2] \cup [4k-h+6, 4k+h]. \tag{7}$$

It then follows by (6) and (7) that $c(x)$ belongs to the set

$$S = [k-h+3] \cup [k+h, 2k+1] \cup [3k+3, 4k-h+5] \cup [4k+h+1, 5k+3].$$

Observe that

$$|[3k+3, 4k-h+5] \cup [4k+h+1, 5k+3]| = 2k-2h+6.$$

Since $h \geqslant 3$, it follows that $k+h-2 > k-h+2$ and so $|[k+h-2]| > |[k+h, 2k+1]|$. Hence, at most $(2k-2h+6)+(k+h-2) = 3k-h+4$ colors in $[5k+3]$ are available for the vertices in $N(v)$. However, $3k-h+4 = (3k+1)-h+3 \leqslant \Delta-1$ and $|N(v)| = \Delta$, which is impossible.

Case 2: $c(v) = k-h+3$. Let $x \in N(v)$. Since x is the center of a subtree isomorphic to $T_{\Delta,h-1}$, it follows by the induction hypothesis that $c(x)$ satisfies (7). Now by (6), $c(x)$ belongs to

$$S = [k-h+2] \cup [k+h-1, 2k+1] \cup [3k+3, 4k-h+5] \cup [4k+h+1, 5k+3].$$

Observe that

$$|[3k+3, 4k-h+5] \cup [4k+h+1, 5k+3]| = 2k-2h+6.$$

Since $h \geqslant 3$, it follows that $k+h-2 > k-h+2$ and so $|[k-h+4, 2k+1]| > |[k-h+2]|$. Hence, at most $(2k-2h+6)+(k+h-2) = 3k-h+4$ colors in $[5k+3]$ are available for the vertices in $N(v)$. However, $3k-h+4 = (3k+1)-h+3 \leqslant \Delta-1$ and $|N(v)| = \Delta$, which is impossible. $\square$

We are now prepared to show that for every integer $\Delta \geqslant 2$, the graceful chromatic number of the tree $T_{\Delta,h}$ is $\lceil \frac{5}{3}\Delta \rceil$ if its height h is sufficiently large.

Theorem 3.4. *Let $\Delta \geqslant 2$ be an integer. If h is an integer such that $h \geqslant 2+\lfloor \frac{1}{3}\Delta \rfloor$, then*

$$\chi_g(T_{\Delta,h}) = \left\lceil \frac{5\Delta}{3} \right\rceil.$$

Proof. By Theorems 2.1 and 2.2, we may assume that $\Delta \geqslant 6$. Furthermore, by Theorem 1.4, it suffices to show that $\chi_g(T_{\Delta,h}) \geqslant \lceil \frac{5}{3}\Delta \rceil$. Let $\Delta = 3k+r$ for integers k and r where $k \geqslant 3$ and $r = 0, 1, 2$. Then $2 + \lfloor \frac{1}{3}\Delta \rfloor = 2+k$. Since $h \geqslant 2+k$, it follows that $T_{\Delta,2+k} \subseteq T_{\Delta,h}$ and so $\chi_g(T_{\Delta,h}) \geqslant \chi_g(T_{\Delta,2+k})$. Hence, it remains only to show that $\chi_g(T_{\Delta,2+k}) \geqslant \lceil \frac{5}{3}\Delta \rceil$. Let $T = T_{\Delta,2+k}$ whose central vertex is v. We consider three cases, according to the values of Δ as integers modulo 3.

Case 1: $\Delta \equiv 0 \pmod 3$. Then $\Delta = 3k$ for some integer $k \geqslant 2$ and $\lceil \frac{5}{3}\Delta \rceil = 5k$. We show that $\chi_g(T) \geqslant 5k$. Assume, to the contrary, that T has a graceful coloring c whose colors are in the set $[5k-1]$. By Lemma 3.1, it follows that $c(v) \notin [2k] \cup [3k, 5k-1]$. Furthermore, it follows by (2) in the proof of Lemma 3.1 that, in addition, $c(v) \notin [2k, 3k]$ and so $c(v) \notin [5k-1]$, a contradiction. Hence, $\chi_g(T_{\Delta,2+k}) \geqslant 5k = \lceil \frac{5}{3}\Delta \rceil$. Therefore, $\chi_g(T_{\Delta,h}) = \lceil \frac{5}{3}\Delta \rceil$ for all $h \geqslant 2 + \frac{1}{3}\Delta$ when $\Delta \equiv 0 \pmod 3$.

Case 2: $\Delta \equiv 1 \pmod 3$. Then $\Delta = 3k+1$ for some integer $k \geqslant 2$ and $\lceil \frac{5}{3}\Delta \rceil = 5k+2$. We show that $\chi_g(T) \geqslant 5k+2$. Assume, to the contrary, that T has a graceful coloring c whose colors are in the set $[5k+1]$. By Lemma 3.2, it follows that $c(v) \notin [2k] \cup [3k+2, 5k+1]$. Furthermore, it follows by (4) in the proof of Lemma 3.2 that, in addition, $c(v) \notin [2k+1, 3k+1]$ and so $c(v) \notin [5k+1]$, a contradiction. Hence, $\chi_g(T_{\Delta,2+k}) \geqslant 5k+2 = \lceil \frac{5}{3}\Delta \rceil$. Therefore, $\chi_g(T_{\Delta,h}) = \lceil \frac{5}{3}\Delta \rceil$ for all $h \geqslant 2 + \lfloor \frac{1}{3}\Delta \rfloor$ when $\Delta \equiv 1 \pmod 3$.

Case 3: $\Delta \equiv 2 \pmod 3$. Then $\Delta = 3k+2$ for some integer $k \geqslant 2$ and $\lceil \frac{5}{3}\Delta \rceil = 5k+4$. We show that $\chi_g(T) \geqslant 5k+4$. Assume, to the contrary, that T has a graceful coloring c whose colors are in the set $[5k+3]$. By Lemma 3.3, it follows that $c(v) \notin [2k+1] \cup [3k+3, 5k+3]$. Furthermore, it follows by (6) in the proof of Lemma 3.3 that, in addition, $c(v) \notin [2k+2, 3k+2]$ and so $c(v) \notin [5k+3]$, a contradiction. Hence, $\chi_g(T_{\Delta,2+k}) \geqslant 5k+4 = \lceil \frac{5}{3}\Delta \rceil$. Therefore, $\chi_g(T_{\Delta,h}) = \lceil \frac{5}{3}\Delta \rceil$ for all $h \geqslant 2 + \lfloor \frac{1}{3}\Delta \rfloor$ when $\Delta \equiv 2 \pmod 3$. □

The following two results are consequences of Theorem 3.4.

Corollary 3.5. *For each integer* $\Delta \geqslant 2$, $\lim_{h\to\infty} \chi_g(T_{\Delta,h}) = \lceil \frac{5}{3}\Delta \rceil$.

Corollary 3.6. *If* T *is a tree with maximum degree* $\Delta \geqslant 2$ *containing a vertex* v *such that every vertex of* T *within distance* $2 + \lfloor \frac{1}{3}\Delta \rfloor$ *of* v *also has degree* Δ*, then* $\chi_g(T) = \lceil \frac{5}{3}\Delta \rceil$.

With the aid of Theorem 3.4, we present a lower bound for the graceful chromatic number of a connected graph. For a vertex coloring c of a graph G and a set X of vertices of G, denote the set of colors of the vertices of X by $c(X) = \{c(x) \colon x \in X\}$.

Corollary 3.7. *If* G *is a connected graph with minimum degree* $\delta \geqslant 2$*, then* $\chi_g(G) \geqslant \lceil \frac{5}{3}\delta \rceil$.

P r o o f. Assume, to the contrary, that there is a connected graph G with $\delta(G) = \delta \geqslant 2$ such that $\chi_g(G) \leqslant \lceil \frac{5}{3}\delta \rceil - 1$ and so G has a graceful $(\lceil \frac{5}{3}\delta \rceil - 1)$-coloring $c\colon V(G) \to [\lceil \frac{5}{3}\delta \rceil - 1]$. By Theorem 3.4, there exists a tree T with $\Delta(T) = \delta$ such that $\chi_g(T) = \lceil \frac{5}{3}\delta \rceil$. Let v be the central vertex (or root) of T. For $0 \leqslant i \leqslant e(v)$, let $V_i = \{x \in V(T)\colon d(v,x) = i\}$. Thus, $V_0 = \{v\}$ and $V_1 = N_T(v)$. Furthermore, let u be any vertex of G.

We now define a coloring $c_T\colon V(T) \to [\lceil \frac{5}{3}\delta \rceil - 1]$ of T from the graceful coloring c of G as follows. First, let $c_T(v) = c(u)$. Since c is a graceful coloring of G and

$$|N_T(v)| \leqslant \Delta(T) = \delta = \delta(G) \leqslant |c(N_G(u))|,$$

we can assign the colors from the set $c(N_G(u)) \subseteq [\lceil \frac{5}{3}\delta \rceil - 1]$ to the vertices in $N_T(v)$ such that the vertices and edges in the tree $T_1 = T[V_0 \cup V_1]$ are properly colored. Suppose then, for some integer i where $1 \leqslant i < e(v)$, that the vertices in the tree $T_i = T\Big[\bigcup\limits_{j=0}^{i} V_j\Big]$ have been assigned colors from $[\lceil \frac{5}{3}\delta \rceil - 1]$ such that

(i) for each $x \in V(T_i)$, there is $u_x \in V(G)$ for which $c_T(x) = c(u_x)$ and $c_T(N_{T_i}(x)) \subseteq c(N_G(u_x))$ and

(ii) all vertices and edges of T_i are properly colored.

Next, we define the colors of vertices in V_{i+1}. Let $y \in V_i$ that is not an end-vertex of T and let $z \in V_{i-1}$ such that $yz \in E(T)$. Then there is a vertex $u_y \in V(G)$ such that $c_T(y) = c(u_y)$ and $c_T(z) \in c_T(N_{T_i}(y)) \subseteq c(N_G(u_y))$. Since c is a graceful coloring and $|N_T(y) \cap V_{i+1}| \leqslant \delta - 1 \leqslant |c(N_G(u_y)) - \{c_T(z)\}|$, we can assign the colors from the set $c(N_G(u_y)) - \{c_T(z)\} \subseteq [\lceil \frac{5}{3}\delta \rceil - 1]$ to the vertices in $N_T(y) \cap V_{i+1}$ such that the vertices and edges of the tree $T_{i+1} = T\Big[\bigcup\limits_{j=0}^{i+1} V_j\Big]$ are properly colored. Therefore, c_T is a graceful coloring of T using colors from the set $[\lceil \frac{5}{3}\delta \rceil - 1]$. However, then $\chi_g(T) \leqslant \lceil \frac{5}{3}\delta \rceil - 1$, which is a contradiction. $\square$

The lower bound for the graceful chromatic number of a graph presented in Corollary 3.7 is best possible. For example, the graph G of Figure 2 has $\delta(G) = \delta = 2$ and graceful chromatic number $\chi_g(G) = \lceil \frac{5}{3}\delta \rceil = 4$. A graceful 4-coloring of G is shown in the figure.

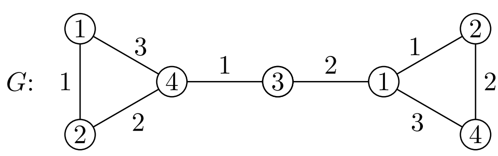

Figure 2. A graph G with $\chi_g(G) = \lceil \frac{5}{3}\delta \rceil$.

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