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Mathematica Bohemica, Vol. 142 (2017), No. 1, 1–7

SOME FIXED POINT THEOREMS IN LOGARITHMIC CONVEX STRUCTURES

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Received November 18, 2014. First published October 17, 2016.
Communicated by Pavel Pyrih

Abstract. In this paper, we introduce the concept of a logarithmic convex structure. Let X be a set and $D\colon X\times X\to[1,\infty)$ a function satisfying the following conditions:

(i) For all $x,y\in X$, $D(x,y)\geqslant 1$ and $D(x,y)=1$ if and only if $x=y$.
(ii) For all $x,y\in X$, $D(x,y)=D(y,x)$.
(iii) For all $x,y,z\in X$, $D(x,y)\leqslant D(x,z)D(z,y)$.
(iv) For all $x,y,z\in X$, $z\neq x,y$ and $\lambda\in(0,1)$,

$$D(z,W(x,y,\lambda))\leqslant D^{\lambda}(x,z)D^{1-\lambda}(y,z),$$
$$D(x,y)=D(x,W(x,y,\lambda))D(y,W(x,y,\lambda)),$$

where $W\colon X\times X\times[0,1]\to X$ is a continuous mapping. We name this the logarithmic convex structure. In this work we prove some fixed point theorems in the logarithmic convex structure.

Keywords: fixed point; logarithmic convex structure; convex metric space

MSC 2010: 47H09, 47H10, 54H25

1. Introduction

In 1970, Takahashi [6] introduced the notion of the convexity in metric spaces and studied some fixed point theorems for nonlinear mappings in such spaces. He proved that all norm spaces and their convex subsets are convex metric spaces and also gave some examples of convex metric spaces which are not embedded in any normed Banach spaces. Subsequently, Guay, Singh and Whitfield [3], Machado [4], Tallman [7], Shimizu [5], Ćirić [2], Chang et al. [1] and others proved many kinds of fixed point theorems in convex metric spaces and probabilistic convex metric spaces.

DOI: 10.21136/MB.2017.0074-14

In this paper, we introduce the concept of a logarithmic convex structure and prove some fixed point theorems in logarithmic convex structure.

2. Definitions and preliminaries

Let (X,d) be a metric space. A continuous mapping $W\colon X\times X\times[0,1]\to X$ is said to be a *convex structure* on X if, for all $x,y\in X$ and $\lambda\in[0,1]$,

$$d(u,W(x,y,\lambda))\leqslant \lambda d(u,x)+(1-\lambda)d(u,y)$$

holds for all $u\in X$. The metric space (X,d) with the convex structure is called a *convex metric space*. A Banach space and each of its convex subsets are convex metric spaces, but a Fréchet space is not necessarily a convex metric space (see [6]).

Definition 2.1. Let X be a set and $D\colon X\times X\to[1,\infty)$ a function satisfying the following conditions:

(i) For all $x,y\in X$, $D(x,y)\geqslant 1$ and $D(x,y)=1$ if and only if $x=y$.
(ii) For all $x,y\in X, D(x,y)=D(y,x)$.
(iii) For all $x,y,z\in X$, $D(x,y)\leqslant D(x,z)D(z,y)$.
(iv) For all $x,y,z\in X$, $z\neq x,y$ and $\lambda\in(0,1)$,

$$D(z,W(x,y,\lambda))\leqslant D^{\lambda}(x,z)D^{1-\lambda}(y,z) \tag{2.1}$$

and

$$D(x,y)=D(x,W(x,y,\lambda))D(y,W(x,y,\lambda)), \tag{2.2}$$

where $W\colon X\times X\times[0,1]\to X$ is a continuous mapping. We name this the *logarithmic convex structure*. A subset K of a logarithmic convex structure (X,d,W) is said to be *log-convex* if $W(x,y,\lambda)\in K$ for all $x,y\in K$ and λ $(0\leqslant\lambda\leqslant 1)$.

Note that for every convex metric space (X,d), by defining $D(x,y)=\mathrm{e}^{d(x,y)}$, we obtain a log-convex structure. In the following, we construct a log-convex structure which is not of the above form.

Example 2.2. Let $D(x,y)=1+\mathrm{e}^{-|x-y|}$ for $x\neq y$ and 1 for the case $x=y$. Then $(\mathbb{R},D,W)$ is a log-convex structure, where $W(x,y,\lambda)=\lambda x+(1-\lambda)y$. For inequality in (iv), one may apply the inequality

$$1+a^{\lambda}b^{1-\lambda}\leqslant(1+a)^{\lambda}(1+b)^{1-\lambda}\quad(0<a,\ b\leqslant 1).$$

Since the function $f(a,b)=(1+a)^{\lambda}(1+b)^{1-\lambda}-a^{\lambda}b^{1-\lambda}$ attains its minimum at points (a,a), the above inequality holds.

Theorem 2.3. *Let $\{K_\alpha\colon\ \alpha \in I\}$ be a family of log-convex subsets of a logarithmic convex structure X. Then $\bigcap\limits_{\alpha\in I} K_\alpha$ is a log-convex subset of X.*

Proposition 2.4. *The open balls $B_r(x)$ and the closed balls $\overline{B_r(x)}$ are log-convex subsets of a logarithmic convex structure X.*

Proof. Let $y, z \in B_r(x)$ be different from $x \in X$. Then, by (2.1), we have

$$d(x, W(y,z,\lambda) \leqslant d^\lambda(x,y) d^{1-\lambda}(x,z) < r^\lambda r^{1-\lambda} = r.$$

If $y = x$ or $z = x$, then the equation (2.2) implies that $W(x,y,\lambda)$ belongs to $B_r(x)$. □

Definition 2.5. Let X be a log-convex structure and A a nonempty log-convex bounded set in X. For any $x \in X$, set

$$r_x(A) = \sup_{y\in A} d(x,y)$$

and

$$r(A) = \inf_{x\in A} r_x(A).$$

(1) We define $A_c = \{x \in A\colon\ r_x(A) = r(A)\}$ to be the *center* of A. We denote the *diameter* of a subset A of X by

$$\delta(A) = \sup\{d(x,y)\colon\ x, y \in A\}.$$

(2) A point $x \in A$ is called a *diametral point* of A if

$$\sup_{y\in A} d(x,y) = \delta(A).$$

(3) A log-convex structure X is said to have *Property* (LC) if every bounded decreasing net of nonempty closed log-convex subsets of X has a nonempty intersection.

Proposition 2.6. *If X has Property* (LC), *then A_c is nonempty closed and log-convex.*

Proof. Let $A_n(x) = \{y \in A\colon\ d(x,y) \leqslant r(A) + 1/n\}$ for each $n \in \mathbb{N}$ and $x \in X$. By defining $C_n = \bigcap\limits_{x\in X} A_n(x)$, C_n is a decreasing sequence of nonempty closed log-convex sets. In fact, if $y, z \in C_n$, then $y, z \in A_n(x)$ for all $x \in X$. So we have $d(x,y) \leqslant r(A) + 1/n$, $d(x,z) \leqslant r(A) + 1/n$. If $x \neq y, z$, then we have

$$d(x, W(y,z,\lambda)) \leqslant d^\lambda(x,y) d^{1-\lambda}(x,z) \leqslant r(A) + \frac{1}{n}.$$

Also, in the case $x = y$ or $x = z$, the equation (2.2) implies that

$$d(x, W(y, z, \lambda)) \leqslant r(A) + \frac{1}{n}.$$

Therefore, in each case, $W(y, z, \lambda) \in C_n$. Now, from $A_c = \bigcap_{n=1}^{\infty} C_n$, it follows that the conclusion is proved. This completes the proof. □

Proposition 2.7. *Let M be a nonempty compact subset of X and let K be the least closed log-convex set containing M. If the diameter $\delta(M)$ is positive, then there exists $u \in K$ such that* $\sup\{d(x, u)\colon\ x \in M\} < \delta(M)$.

Proof. By the assumption, M is compact and so one may find $x_1, x_2 \in M$ such that $d(x_1, x_2) = \delta(M)$. Let $M_0 \subseteq M$ be maximal so that $\{x_1, x_2\} \subseteq M_0$ and $d(x, y) = 0$ or $\delta(M)$ for all $x, y \in M_0$.

Now, we claim that M_0 is finite. If M_0 is not finite, then it will have a cluster point, say z_0. Now, the continuity of the metric function d implies that $z_0 \in M_0$, which contradicts the maximality of M_0. Let us assume $M_0 = \{x_1, x_2, \ldots, x_n\}$. Since X is a log-convex structure, we can define

$$\begin{aligned} y_1 &= W\Big(x_1, x_2, \frac{1}{2}\Big), \\ y_2 &= W\Big(x_3, y_1, \frac{1}{3}\Big), \\ &\vdots \\ y_{n-2} &= W\Big(x_{n-1}, y_{n-3}, \frac{1}{n-1}\Big), \\ y_{n-1} &= W\Big(x_n, y_{n-2}, \frac{1}{n}\Big) = u. \end{aligned}$$

On the other hand, since M is compact, there exists $y_0 \in M$ such that

$$d(y_0, u) = \sup\{d(x, u)\colon\ x \in M\}.$$

Now, by applying (2.1), we obtain

$$\begin{aligned} d(y_0, u) &\leqslant d^{1/n}(y_0, x_n) d^{(n-1)/n}(y_0, y_{n-2}) \\ &\leqslant d^{1/n}(y_0, x_n) \big(d^{1/(n-1)}(y_0, x_{n-1}) d^{(n-2)/(n-1)}\big)^{(n-1)/n} \\ &\vdots \\ &\leqslant \prod_{1}^{n} d^{1/n}(y_0, x_k) \\ &\leqslant \delta(M). \end{aligned}$$

Thus, if $d(y_0, u) = \delta(M)$, then $\prod_1^n d^{1/n}(y_0, x_k) = \delta(M)$. Therefore, $d(y_0, x_k) = \delta(M)$ for all $k = 1, 2, \ldots, n$ and so $y_0 \in M_0$, which implies that $y_0 = x_k$ for some $k = 1, 2, \ldots, n$, which is a contradiction. Therefore, we have $d(y_0, u) < \delta(M)$. This completes the proof. □

Definition 2.8. A log-convex structure X is said to have the *normal structure* if, for each closed bounded log-convex subset A of X which contains at least two points, there exists $x \in A$ which is not a diametral point of A.

Let K be a nonempty subset of a log-convex structure X. A mapping $T\colon K \to K$ is said to have *Property* (G) if there exists $\lambda \in [0, 1]$ such that

$$d(TX, Ty) \leqslant d^{\lambda}(x, Tx) d^{1-\lambda}(y, Ty)$$

for all $x, y \in K$. The mapping T is said to have *Property* (B) on K if, for every closed and log-convex subset F of K which has nonzero diameter and is invariant under T, there exists some $x \in F$ such that

$$d(x, Tx) < \sup_{y \in F} d(y, Ty).$$

3. Main results

Let X be a metric space and K a subset of X. A mapping $T\colon K \to X$ is said to be *nonexpansive* if

$$d(Tx, Ty) \leqslant d(x, y)$$

for all $x, y \in K$.

Theorem 3.1. *Suppose that X has Property* (LC). *Let K be a nonempty bounded closed log-convex subset of X with the normal structure. If T is a nonexpansive mapping from K into itself, then T has a fixed point in K.*

Proof. Let $\mathcal{A}$ be a family of all nonempty closed, T-invariant and log-convex subsets of K. By Property (LC) and Zorn's lemma, $\mathcal{A}$ has a minimal element A. We show that A is singleton. Let $x \in A_c$. Then it follows that, for all $y \in A$,

$$d(Tx, Ty) \leqslant d(x, y) \leqslant r_x(A)$$

and so

$$T(A) \subseteq \overline{B_{r(A)}(T(x))} \doteq \overline{B}.$$

Since $T(A \cap \overline{B}) \subseteq A \cap \overline{B}$, the minimality of A implies that $A \subseteq \overline{B}$. Thus $r_{T(x)}(A) \leqslant r(A)$. Since $r(A) \leqslant r_x(A)$ for all $x \in A$, $r_{T(x)}(A) = r(A)$. Hence $T(x) \in A_c$ and $T(A_c) \subseteq A_c$. By Proposition 2.6, $A_c \in \mathcal{A}$. If $z, w \in A_c$, then $d(z, w) \leqslant r_x(A) = r(A)$ and so, due to the normal structure,

$$\delta(A_c) \leqslant r(A) < \delta(A),$$

which contradicts the minimality of A and so $\delta(A) = 0$. Therefore, A is a singleton. This completes the proof. □

Theorem 3.2. *Let T be a self-mapping on a nonempty bounded closed and log-convex subset K of a log-convex structure X having Property* (LC)*. Let T have Property* (G)*. If $\sup\limits_{y \in F} d(y, Ty) < \delta(F)$, ($\delta(F)$ being the diameter of F) for every nonempty bounded closed and log-convex subset F of K which has nonzero diameter and is mapped into itself by T, then T has a unique fixed point in K.*

Proof. Let $\mathcal{A}$ be the family of all bounded closed, T-invariant and log-convex subsets of K. By Zorn's lemma and Property (LC), $\mathcal{A}$ is nonempty and has a minimal element A. If $\delta(A) = 0$, then the point in A is a fixed point of T. Suppose $\delta(A) > 0$. Then, for any $x, y \in A$,

$$d(Tx, Ty) \leqslant d^{\lambda}(x, Tx) d^{1-\lambda}(y, Ty) \leqslant r,$$

where $r = \sup\limits_{y \in A} d(y, Ty)$. So we have

$$T(A) \subseteq \overline{B_r(Tx)} \doteq B_0.$$

On the other hand, $A \cap B_0$ is invariant under T. So, by the minimality of A, $A \subseteq B_0$. Hence we have

$$\sup_{y \in A} d(y, Tx) \leqslant r \tag{3.1}$$

for all $x \in A$. Let $A_1 = \{z \in A \colon \sup\limits_{y \in A} d(z, y) \leqslant r\}$. Obviously, A_1 is nonempty. Moreover, we have

$$d(y, W(u, v, \lambda) \leqslant d^{\lambda}(y, u) d^{1-\lambda}(y, v) \leqslant r$$

for all $u, v \in A_1$ and $\lambda \in [0, 1]$ and so $W(u, v, \lambda) \in A_1$, which implies that A_1 is log-convex. Now, assume that $z \in \overline{A}_1$. Then there exists a sequence $\{z_n\}$ in A_1 such that $z_n \to z$ and, for all $y \in A$,

$$d(z_n, y) \leqslant \sup_{y \in A} d(z_n, y) \leqslant r.$$

By letting n tend to infinity, we have $d(z,y) \leqslant r$ and so $z \in A_1$. Thus A_1 is closed.

On the other hand, the equation (3.1) implies that, for all $z \in A_1$,

$$\sup_{y \in A} d(Tz, y) \leqslant r.$$

So $Tz \in A_1$ for all $z \in A_1$. Thus A_1 is invariant under T. Also, $\delta(A_1) \leqslant r < \delta(A)$. Hence A_1 is a proper closed and log-convex subset of A, which contradicts the minimality of A.

For the uniqueness of the fixed point, note that, if T has two fixed points x and y, then we have

$$d(x,y) = d(Tx,Ty) \leqslant d^{\lambda}(x,Tx) d^{1-\lambda}(y,Ty) = 0$$

and so $x = y$. This completes the proof. □

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