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ON MULTISET COLORINGS OF GENERALIZED CORONA GRAPHS

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Abstract. A vertex k-coloring of a graph G is a *multiset k-coloring* if $M(u) \neq M(v)$ for every edge $uv \in E(G)$, where $M(u)$ and $M(v)$ denote the multisets of colors of the neighbors of u and v, respectively. The minimum k for which G has a multiset k-coloring is the *multiset chromatic number* $\chi_m(G)$ of G. For an integer $l \geqslant 0$, the *l-corona* of a graph G, $\mathrm{cor}^l(G)$, is the graph obtained from G by adding, for each vertex v in G, l new neighbors which are end-vertices. In this paper, the multiset chromatic numbers are determined for *l-coronas* of all complete graphs, the regular complete multipartite graphs and the Cartesian product $K_r \,\square\, K_2$ of K_r and K_2. In addition, we show that the minimum l such that $\chi_m(\mathrm{cor}^l(G)) = 2$ never exceeds $\chi(G) - 2$, where G is a regular graph and $\chi(G)$ is the chromatic number of G.

Keywords: multiset coloring; multiset chromatic number; generalized corona of a graph; neighbor-distinguishing coloring

MSC 2010: 05C15

1. Introduction

Proper vertex coloring of a graph G is a well-known method to distinguish adjacent vertices of G. However, in the past decades, a large number of coloring methods emerged for the purpose of distinguishing adjacent vertices or all the vertices of a graph G. In [11], Zhang et al. presented the concept of adjacent vertex-distinguishing edge coloring of graphs. Based on proper vertex coloring of a graph G, Radcliffe and Zhang considered the multiset of colors of the neighboring vertices of each vertex to distinguish all vertices of G, see [9] for details. Further, in [4], Chartrand et al. studied the situation when the vertex coloring may not be proper. Please refer to [1], [2], [3], [7], [10], [12] for more related literatures.

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A coloring c of the vertices of a graph G (where adjacent vertices may be assigned the same color) is called *neighbor-distinguishing* if every two adjacent vertices of G are distinguished from each other in some manner by the coloring c. With no doubt, a proper vertex coloring is neighbor-distinguishing and the minimum number of colors needed for a proper vertex coloring of G is the *chromatic number* $\chi(G)$. In [5], a new coloring, called the *multiset coloring*, was put forward and studied that never requires more than $\chi(G)$ colors.

Given a graph $G = (V, E)$, let $c\colon V(G) \to \{1, 2, \ldots, k\}$ be a vertex coloring (which need not be proper) of G. Consider such a coloring c; for each vertex v of G, let $M(v)$ be the multiset of colors in $N(v)$, where $N(v)$ denotes the set of vertices adjacent to v. If $M(u) \neq M(v)$ for every pair of adjacent vertices $uv \in E(G)$, then c is called a *multiset k-coloring* of G. The minimum number k such that G has a multiset k-coloring is the *multiset chromatic number* $\chi_m(G)$ of G. For example, a multiset 2-coloring of C_6^2 is depicted in Figure 1.

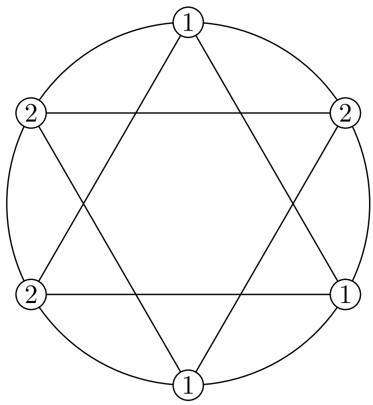

Figure 1. A multiset 2-coloring of C_6^2.

It is obvious that every proper vertex coloring of G is a multiset coloring of G, thus

$$\chi_m(G) \leqslant \chi(G). \tag{1.1}$$

By the definition of the multiset coloring of a graph G, if u and v are two adjacent vertices of G with $d_G(u) \neq d_G(v)$, then necessarily $M(u) \neq M(v)$. Thus the following observation is easily obtained.

Observation 1.1. For any graph G, $\chi_m(G) = 1$ if and only if every two adjacent vertices of G have different degrees.

Since every nonempty bipartite graph has chromatic number 2, the following is a natural consequence of (1.1) and Observation 1.1.

Proposition 1.1. *If G is a bipartite graph, then*

$$\chi_m(G) = \begin{cases} 1, & \text{if every two adjacent vertices of } G \text{ have different degrees;} \\ 2, & \text{otherwise.} \end{cases}$$

By Proposition 1.1, for the complete bipartite graph $K_{s,t}$ we have

$$\chi_m(K_{s,t}) = \begin{cases} 1 & \text{if } s \neq t; \\ 2 & \text{if } s = t. \end{cases}$$

For a vertex u of G, the multiset $M(u)$ can be represented by the k-vector

$$\mathrm{code}(u) = (a_1, a_2, \ldots, a_k) := a_1 a_2 \ldots a_k,$$

where a_i, $1 \leqslant i \leqslant k$ denotes the number of neighbors of u colored i. The k-vector code(u) is called the *multiset color code* of u, or *color code* of u for short. For example, code$(u) = (2, 3)$ means that the number of neighbors of u colored 1 is 2 and the number of neighbors of u colored 2 is 3.

The following observation is often useful.

Observation 1.2. If u and v are two adjacent vertices in a graph G such that $N(u) - v = N(v) - u$, then $c(u) \neq c(v)$ for every multiset coloring c of G.

It is a consequence of Observation 1.2 that $\chi_m(K_n) = n$, where K_n is a complete graph of order n, $n \geqslant 2$. The multiset chromatic number of every complete multipartite graph was determined in [5]. Besides, the multiset chromatic numbers for cycles and their squares, cubes, and fourth powers were also determined in [5]. In the meantime, a conjecture was proposed as follows.

Conjecture 1.1 ([5]). *For every integer $r \geqslant 3$, there exists an integer $f(r)$ such that $\chi_m(C_n^r) = 3$ for all $n \geqslant f(r)$.*

This conjecture was solved by Feng and Lin, please refer to [6] for details.

There are further studies concerning multiset colorings of graphs. In [8], it was shown that for every positive integer N there is an r-regular graph G such that $\chi(G) - \chi_m(G) = N$ and for every pair k, r of integers with $2 \leqslant k \leqslant r - 1$ there exists an r-regular graph with multiset chromatic number k.

The *corona* cor(G) of a graph G is the graph obtained from G by adding, for each vertex v in G, a new vertex v' and the edge vv'. It is trivial that $\chi_m(\mathrm{cor}(G)) \leqslant \chi_m(G)$

for every connected graph G. In [8], the multiset chromatic numbers were determined for the coronas of all complete graphs and regular complete multipartite graphs. We list them below.

Theorem 1.1 ([8]). *For every integer $n \geqslant 2$,*

$$\chi_m(\mathrm{cor}(K_n)) = \left\lceil \frac{1+\sqrt{4n-3}}{2} \right\rceil.$$

For $k \geqslant 2$ and $n \geqslant 1$, the regular complete k-partite graph, each partite set of which contains n vertices, is denoted by $K_{k(n)}$. Thus, $K_{k(1)} = K_k$. The following theorem determines the multiset chromatic number of the corona of $K_{k(n)}$. To this end, for positive integers l and n, define

$$g(l,n) = \binom{l+n-2}{n-1} + l\binom{l+n-2}{n}.$$

Theorem 1.2 ([8]). *For integers $n, k \geqslant 2$, the multiset chromatic number of $\mathrm{cor}(K_{k(n)})$ is the unique positive integer l such that*

$$g(l-1,n) < k \leqslant g(l,n).$$

In this paper, the concept of corona of a graph is generalized and the generalized corona of a graph is naturally defined as follows.

Definition 1.1. For an integer $l \geqslant 0$, the *l-corona* of a graph G, denoted by $\mathrm{cor}^l(G)$, is the graph obtained from G by adding, for each vertex v in G, l new vertices $v_1, v_2, \ldots, v_l$ and l new edges $vv_1, vv_2, \ldots, vv_l$. Let $J^l(v) = \{v_1, v_2, \ldots, v_l\}$, then each vertex v_i, $1 \leqslant i \leqslant l$ in $J^l(v)$ is called a leaf vertex, or an end-vertex of v.

If $l = 0$, then $\mathrm{cor}^0(G)$ represents the graph G, i.e., $\mathrm{cor}^0(G) = G$. Obviously, $\mathrm{cor}^1(G) = \mathrm{cor}(G)$. So in this sense, the *l-corona* of a graph G defined above is viewed as the generalized corona of a graph. If no confusion occurs, sometimes we simply use generalized corona graphs or generalized corona instead of generalized corona of a graph.

The multiset chromatic numbers of generalized corona graphs are mainly discussed in this paper. First, we present some properties of the multiset chromatic numbers of generalized corona graphs. Next, the multiset chromatic numbers are determined for the l-coronas of all complete graphs and the regular complete multipartite graphs. These results are generalizations of Theorem 1.1 and Theorem 1.2. In addition, the multiset chromatic number of the Cartesian product $K_r \,\square\, K_2$ is also determined in this work. We conclude the paper by showing that the minimum l such that $\chi_m(\mathrm{cor}^l(G)) = 2$ never exceeds $\chi(G) - 2$, where G is a regular graph.

The following notation will be used in the paper.

Given a graph $G = (V, E)$, let $X \subseteq V(G)$. For a not necessarily proper vertex coloring c of G, let $M_c(X)$ (or simply $M(X)$) be the multiset of colors of the vertices in X. Please note that for $v \in V(G)$, $M(v)$ stands for the multiset of colors in $N(v)$, while $M(\{v\})$ stands for the multiset of color of v.

For integers z_1 and z_2, z_1 modulo z_2 is denoted by $[z_1]_{z_2}$.

2. Properties for generalized corona graphs

According to the definition of generalized corona graphs, the following observations are easily obtained.

Observation 2.1. For every graph G, let l_1 and l_2 be two non-negative integers with $l_1 \geqslant l_2$, then

$$\chi_m(\mathrm{cor}^{l_1}(G)) \leqslant \chi_m(\mathrm{cor}^{l_2}(G)).$$

Proof. If $l_1 = l_2$, then the conclusion holds trivially. Now let $l_1 > l_2$, then the graph $\mathrm{cor}^{l_1}(G)$ can be obtained from $\mathrm{cor}^{l_2}(G)$ by adding, for each vertex v in G, $l_1 - l_2$ new vertices $v_1, v_2, \ldots, v_{l_1-l_2}$ and $l_1 - l_2$ new edges $vv_1, vv_2, \ldots, vv_{l_1-l_2}$. Let c be a multiset coloring of $\mathrm{cor}^{l_2}(G)$. The coloring c can be extended to a multiset coloring c' of $\mathrm{cor}^{l_1}(G)$ as follows:

$$c'(v) = \begin{cases} c(v), & v \in \mathrm{cor}^{l_2}(G); \\ 1, & \text{otherwise.} \end{cases}$$

□

Observation 2.2. For $l \geqslant 0$ and $n \geqslant 2$,

$$\chi_m(\mathrm{cor}^l(K_n)) \geqslant \chi_m(\mathrm{cor}^l(K_{n-1})).$$

Proof. If $l = 1$ and $n = 2$, then $\chi_m(\mathrm{cor}^l(K_{n-1})) = \chi_m(P_2) = 2$ while $\chi_m(\mathrm{cor}^l(K_n)) = \chi_m(P_4) = 2$, and the conclusion holds. Otherwise, suppose that c is a multiset coloring of $\chi_m(\mathrm{cor}^l(K_n))$. The graph $\mathrm{cor}^l(K_{n-1})$ can be obtained from $\mathrm{cor}^l(K_n)$ by deleting a vertex v of K_n and all its end-vertices. Obviously, the coloring c restricted to the graph $\mathrm{cor}^l(K_{n-1})$ is also a multiset coloring. □

Observation 2.3. For $n \geqslant 2$,

$$\chi_m(\mathrm{cor}^0(K_n)) \geqslant \chi_m(\mathrm{cor}^1(K_n)) \geqslant \ldots \geqslant \chi_m(\mathrm{cor}^{n-1}(K_n)).$$

Proof. It is a corollary of Observation 2.1. □

R e m a r k 2.1. What is the minimum l such that $\chi_m(\mathrm{cor}^l(K_n)) = 2$? We will answer this question in the next section.

O b s e r v a t i o n 2.4. For $l \geqslant 0$, if $H \subseteq G$, then the following inequality may be not right.

$$\chi_m(\mathrm{cor}^l(H)) \leqslant \chi_m(\mathrm{cor}^l(G)).$$

P r o o f. For example, $C_3 \subseteq C_6^2$, but from [5] we know that $\chi_m(C_3) = 3$, while $\chi_m(C_6^2) = 2$. $\square$

3. Generalized coronas of complete graphs

A heuristic question is: how small the l is such that $\chi_m(\mathrm{cor}^l(K_n)) = 2$? We answer this question by showing the following.

Theorem 3.1. *For $n \geqslant 2$,*

$$\min_{l \geqslant 0}\{l\colon\ \chi_m(\mathrm{cor}^l(K_n)) = 2\} = n - 2.$$

P r o o f. First, we show that $\chi_m(\mathrm{cor}^{n-2}(K_n)) = 2$. It is a consequence of Observation 1.1 that $\chi_m(\mathrm{cor}^{n-2}(K_n)) \geqslant 2$, thus we only need to give a multiset 2-coloring of $\mathrm{cor}^{n-2}(K_n)$. If $n = 2$, then the conclusion holds trivially. So we suppose that $n \geqslant 3$. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. The graph $\mathrm{cor}^{n-2}(K_n)$ can be obtained from K_n by adding, for each vertex v_i, $1 \leqslant i \leqslant n$, $n-2$ new vertices $v_{i,1}, v_{i,2}, \ldots, v_{i,n-2}$ and $n-2$ new edges $vv_{i,1}, vv_{i,2}, \ldots, vv_{i,n-2}$. Define a vertex 2-coloring c of $\mathrm{cor}^{n-2}(K_n)$ as follows.

$$c(v_i) = \begin{cases} 1, & 1 \leqslant i \leqslant n-1,\\ 2, & i = n; \end{cases}$$

$$c(v_{i,j}) = \begin{cases} 2, & 1 \leqslant i \leqslant n-2,\ 1 \leqslant j \leqslant n-i-1,\\ 1, & 2 \leqslant i \leqslant n-1,\ n-i \leqslant j \leqslant n-2,\\ 1, & i = n,\ 1 \leqslant j \leqslant n-2. \end{cases}$$

From the above coloring, it can be determined that for $1 \leqslant i \leqslant n$, $\mathrm{code}(v_i) = (n+i-3, n-i)$. It then follows immediately that c is a multiset 2-coloring of $\mathrm{cor}^{n-2}(K_n)$. Thus, $\chi_m(\mathrm{cor}^{n-2}(K_n)) = 2$.

Next, it only remains to show that $\chi_m(\mathrm{cor}^{n-3}(K_n)) > 2$ by Observation 2.3. Suppose that there exists a multiset 2-coloring of $\mathrm{cor}^{n-3}(K_n)$. Since the number of $(n-3)$-element multisets of $\{1, 2\}$ is $n-2$, there are at most $n-2$ vertices of K_n

which can be colored the same color. For $2 \leqslant t \leqslant n-2$, let t vertices of K_n be colored by 1 and the remaining $n-t$ vertices of K_n be colored by 2. It can be inferred that the multiset of each vertex of K_n contains at least $(t-1)$ 1's, and at most $(t+n-3)$ 1's. Therefore, the number of different multisets of the vertices of K_n is at most $(t+n-3)-(t-1)+1=n-1$. Since $n-1<n$, by the pigeonhole principle, there exist at least two vertices of K_n which have the same multisets. This is a contradiction, thus $\chi_m(\mathrm{cor}^{n-3}(K_n))>2$. $\square$

The multiset chromatic numbers of the generalized coronas of all complete graphs are characterized as follows.

Theorem 3.2. *For integers t and l with $t \geqslant 2$, $l \geqslant 0$, define*

$$n_t^l = \binom{t+l-2}{l} + (t-2)\binom{t+l-3}{l} + 1.$$

If $n \in [n_t^l, n_{t+1}^l)$, then

$$\chi_m(\mathrm{cor}^l(K_n)) = t.$$

Proof. First we show that $\chi_m(\mathrm{cor}^l(K_{n_t^l})) \geqslant t$. Assume, to the contrary, that there exists a multiset $(t-1)$-coloring of $\mathrm{cor}^l(K_{n_t^l})$. Let C be the set of $(t-1)$ colors. Since the number of l-element multisets of C is $\binom{t+l-2}{l}$, there are at most $\binom{t+l-2}{l}$ vertices of $K_{n_t^l}$ which can be colored the same color.

Suppose that there are exactly s, $0 \leqslant s \leqslant t-2$ colors, each of which is assigned to at most $\binom{t+l-3}{l}-1$ vertices of $K_{n_t^l}$. So each of the remaining $(t-1-s)$ colors is assigned to at least $\binom{t+l-3}{l}$ vertices of $K_{n_t^l}$. Without loss of generality, let these $(t-1-s)$ colors be $1, 2, \ldots, t-1-s$. The simple calculation yields

$$\begin{aligned} n_t^l &- s\left(\binom{t+l-3}{l}-1\right) - (t-1-s)\binom{t+l-3}{l} \\ &= \binom{t+l-2}{l} + (t-2)\binom{t+l-3}{l} + 1 - s\binom{t+l-3}{l} + s \\ &\quad - (t-1)\binom{t+l-3}{l} + s\binom{t+l-3}{l} \\ &= \binom{t+l-3}{l-1} + s + 1. \end{aligned}$$

Let S be the set of these $\binom{t+l-3}{l-1}+s+1$ vertices of $K_{n_t^l}$. Suppose that n_1 vertices of S are colored by 1, n_2 vertices of S are colored by $2, \ldots, n_p$ vertices of S are colored by p, where

$$\begin{cases} n_1 \geqslant 1,\ n_2 \geqslant 1, \ldots, n_p \geqslant 1, \\ n_1 + n_2 + \ldots + n_p = \binom{t+l-3}{l-1} + s + 1, \\ 1 \leqslant p \leqslant t - 1 - s. \end{cases}$$

For $1 \leqslant r \leqslant p$, there are $\binom{t+l-3}{l} + n_r$ vertices of $K_{n_t^l}$ which are colored by r. Observe that the number of l-element multisets of the $(t-2)$ colors is $\binom{t+l-3}{l}$, therefore, the color r appears in the leaves of each of the remaining n_r vertices which are colored by r. Realize the fact that $n_1 + n_2 + \ldots + n_p = \binom{t+l-3}{l-1} + s + 1$, and the number of $(l-1)$-element multisets of the $(t-1)$ colors is $\binom{t+l-3}{l-1}$. Since $\binom{t+l-3}{l-1} + s + 1 > \binom{t+l-3}{l-1}$, there exist two vertices of $K_{n_t^l}$, each of them being colored as illustrated in Figure 2. It is clear that $M(u) = M(v)$, which is a contradiction, thus $\chi_m(\mathrm{cor}^l(K_{n_t^l})) \geqslant t$.

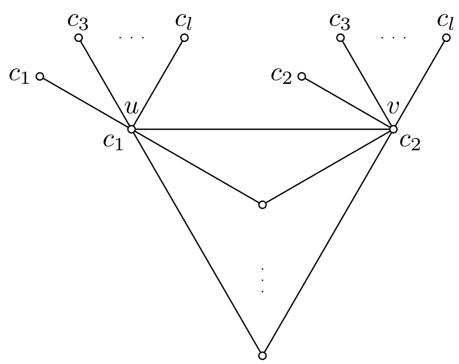

Figure 2. $M(u) = M(v)$.

Now, by Observation 2.2, we only need to give a multiset t-coloring c of $\mathrm{cor}^l(K_{n_{t+1}^l - 1})$. Obviously,

$$n_{t+1}^l - 1 = (t-1)\binom{t+l-2}{l} + \binom{t+l-1}{l} = (t-1)a + b.$$

Let

$$V(K_{n_{t+1}^l - 1}) = \{v_{1,1}, \ldots, v_{1,a}\} \cup \ldots \cup \{v_{t-1,1}, \ldots, v_{t-1,a}\} \cup \{v_{t,1}, \ldots, v_{t,b}\} := V.$$

For each i, $1 \leqslant i \leqslant t-1$, color $v_{i,1}, \ldots, v_{i,a}$ by i. Color $v_{t,1}, \ldots, v_{t,b}$ by t.

In order to color the remaining leaves of $\mathrm{cor}^l(K_{n_{t+1}^l - 1})$, we introduce some notation.

For each i, $1 \leqslant i \leqslant t-1$, let

$$R_i = \{1, \dots, t\} - \{i\}.$$

The number of l-multisets of R_i is $\binom{t+l-2}{l}$, let these a multisets be $S_{i,1}, \dots, S_{i,a}$. The number of l-multisets of $\{1, \dots, t\}$ is $\binom{t+l-1}{l}$, let these b multisets be $S_{t,1}, \dots, S_{t,b}$.

Now we are ready to color the leaves. For $1 \leqslant i \leqslant t-1$ and $1 \leqslant j \leqslant a$, color the leaves corresponding to $v_{i,j}$ so that $M(J^l(v_{i,j})) = S_{i,j}$. For $1 \leqslant k \leqslant b$, color the leaves corresponding to $v_{t,k}$ so that $M(J^l(v_{t,k})) = S_{t,k}$.

We show that such coloring c is a multiset t-coloring of $\operatorname{cor}^l(K_{n^l_{t+1}-1})$.

(1) For $1 \leqslant i \leqslant t-1$, $1 \leqslant j_1 < j_2 \leqslant a$, observe that

$$\begin{aligned} M(v_{i,j_1}) &= M(V) - \{c(v_{i,j_1})\} + S_{i,j_1}, \\ M(v_{i,j_2}) &= M(V) - \{c(v_{i,j_2})\} + S_{i,j_2}. \end{aligned}$$

Since $c(v_{i,j_1}) = c(v_{i,j_2}) = i$ and $S_{i,j_1} \neq S_{i,j_2}$, thus $M(v_{i,j_1}) \neq M(v_{i,j_2})$.

(2) For $1 \leqslant i_1 < i_2 \leqslant t-1$, $1 \leqslant j_1 \leqslant j_2 \leqslant a$, it is clear that

$$\begin{aligned} M(v_{i_1,j_1}) &= M(V) - \{c(v_{i_1,j_1})\} + S_{i_1,j_1}, \\ M(v_{i_2,j_2}) &= M(V) - \{c(v_{i_2,j_2})\} + S_{i_2,j_2}. \end{aligned}$$

Observe that $c(v_{i_1,j_1}) = i_1$, $c(v_{i_2,j_2}) = i_2$ and $i_1 \notin S_{i_1,j_1}$ hence $M(v_{i_1,j_1})$ contains less i_1's than $M(v_{i_2,j_2})$. Thus $M(v_{i_1,j_1}) \neq M(v_{i_2,j_2})$.

(3) For $1 \leqslant i \leqslant t-1$, $1 \leqslant j_1 \leqslant a$, $1 \leqslant j_2 \leqslant b$, it is clear that

$$\begin{aligned} M(v_{i,j_1}) &= M(V) - \{c(v_{i,j_1})\} + S_{i,j_1}, \\ M(v_{t,j_2}) &= M(V) - \{c(v_{t,j_2})\} + S_{t,j_2}. \end{aligned}$$

Observe that $c(v_{i,j_1}) = i$, $c(v_{t,j_2}) = t$ and $i \notin S_{i,j_1}$ hence $M(v_{i,j_1})$ contains less i_1's than $M(v_{t,j_2})$. Thus $M(v_{i,j_1}) \neq M(v_{t,j_2})$. $\square$

For example, let $t = 3$, $l = 2$, then

$$n^l_{t+1} - 1 = \binom{4}{2} + 2\binom{3}{2} = 6 + 2 \times 3 = 12.$$

A multiset 3-coloring of $\operatorname{cor}^2(K_{12})$ is depicted in Figure 3.

Remark 3.1. The above theorem implies Theorem 1.1 which also can be found in [8]. Let $l = 1$, then $n^1_t = t^2 - 3t + 4$, $n^1_{t+1} = t^2 - t + 2$. By the above theorem, if $n \in [t^2 - 3t + 4, t^2 - t + 2)$, then $\chi_m(\operatorname{cor}(K_n)) = t$. Since $(t^2 - t + 1) \in [t^2 - 3t + 4, t^2 - t + 2)$, $t = \lceil (1 + \sqrt{4n-3})/2 \rceil$ when taking $n = t^2 - t + 1$. Thus $\chi_m(\operatorname{cor}(K_n)) = \lceil (1 + \sqrt{4n-3})/2 \rceil$. Given $l = 0$, the above theorem also implies the well-known result that $\chi_m(K_n) = n$.

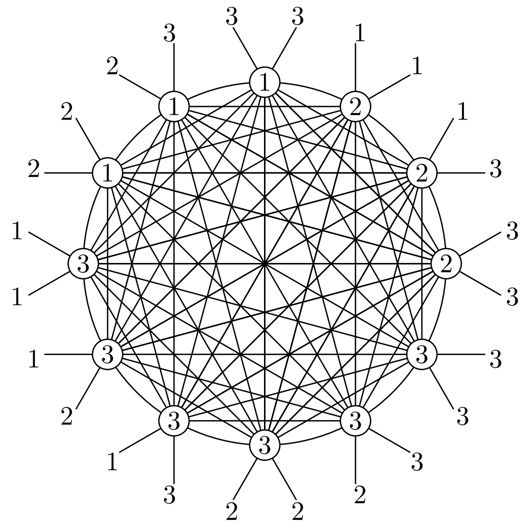

Figure 3. A multiset 3-coloring of $\operatorname{cor}^2(K_{12})$.

4. Generalized coronas of complete multipartite graphs

The multiset chromatic numbers of generalized coronas of $K_{k(n)}$ are obtained in this section. To this end, we first present a formula. For $p \geqslant 0$, $l \geqslant 0$, $n \geqslant 1$, define

$$g(p,l,n) = \binom{l+n-p-1}{n-p} + \sum_{i=0}^{p-1}\sum_{j=1}^{p-i}\binom{l}{j}\binom{p-i-1}{p-i-j}\binom{n+l-i-j-1}{n-i}.$$

Obviously, $g(p,l,n)$ is strictly increasing with respect to l. Thus, for $k \geqslant 1$, there exists a unique integer $l \geqslant 1$ such that

$$g(p,l-1,n) < k \leqslant g(p,l,n).$$

Theorem 4.1. *For integers $k \geqslant 2$, $n \geqslant 1$, $p \geqslant 0$, the multiset chromatic number of $\operatorname{cor}^p(K_{k(n)})$ is the unique positive integer l such that*

$$g(p,l-1,n) < k \leqslant g(p,l,n).$$

Proof. Obviously, such an integer $l \geqslant 2$ exists. Let $G = K_{k(n)}$ with $V(G) = U = U_1 \cup U_2 \cup \ldots \cup U_k$, where $U_i = \{u_{i,1}, u_{i,2}, \ldots, u_{i,n}\}$ is a partite set for $1 \leqslant i \leqslant k$. We obtain the p-corona of G by adding, for each vertex $u_{i,j}$, p end-vertices $W_{i,j} = \{w_{i,j,1}, w_{i,j,2}, \ldots, w_{i,j,p}\}$, where $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$.

We first show that $\chi_m(\mathrm{cor}^p(G)) \geqslant l$. Suppose to the contrary that there exists a multiset $(l-1)$-coloring of $\mathrm{cor}^p(G)$. Let these $(l-1)$ colors be $1, 2, \ldots, l-1$. For $1 \leqslant q \leqslant l-1$, let t_q be the number of vertices in U that are colored by q. Then $\sum\limits_{q=1}^{l-1} t_q = nk$.

Consider an arbitrary vertex in U, say $u_{1,1} \in U_1$. For $1 \leqslant q \leqslant l-1$, let a_q be the number of vertices in U_1 that are colored by q, b_q the number of vertices in $W_{1,1}$ that are colored by q. Then

$$\sum_{q=1}^{l-1} a_q = n, \quad \sum_{q=1}^{l-1} b_q = p,$$

and

$$\mathrm{code}(u_{1,1}) = (t_1, t_2, \ldots, t_{l-1}) - (a_1, a_2, \ldots, a_{l-1}) + (b_1, b_2, \ldots, b_{l-1}).$$

We now determine all possible color codes for $u_{1,1}$.

(1) $b_q \leqslant a_q$ for $1 \leqslant q \leqslant l-1$. Let $c_q = a_q - b_q$ for $1 \leqslant q \leqslant l-1$, then $c_q \geqslant 0$ and $\sum\limits_{q=1}^{l-1} c_q = n-p$. Therefore, the number of possible color codes for $u_{1,1}$ is

$$\binom{(l-1)+(n-p)-1}{n-p} = \binom{l+n-p-2}{n-p}.$$

(2) There exists an integer q, $1 \leqslant q \leqslant l-1$ such that $b_q > a_q$. To this end, the coloring for the vertices in U_1 and $W_{1,1}$ should be like this: for $0 \leqslant i \leqslant p-1$, U_1 and $W_{1,1}$ can be divided into two subsets, say U_1^i, U_1^{n-i} and $W_{1,1}^i$, $W_{1,1}^{p-i}$, respectively, such that

$$|U_1^i| = |W_{1,1}^i| = i, \quad M(U_1^i) = M(W_{1,1}^i), \tag{4.1}$$

and

$$|U_1^{n-i}| = n-i, \quad |W_{1,1}^{p-i}| = p-i, \quad M(U_1^{n-i}) \cap M(W_{1,1}^{p-i}) = \emptyset. \tag{4.2}$$

In this case, we claim that the number of possible color codes for $u_{1,1}$ is

$$\sum_{i=0}^{p-1}\sum_{j=1}^{p-i} \binom{l-1}{j}\binom{p-i-1}{p-i-j}\binom{n+l-i-j-2}{n-i}.$$

In fact, the color code for $u_{1,1}$ is only dependent on the different coloring ways for U_1^{n-i} and $W_{1,1}^{p-i}$ satisfying equation (4.2). Moreover, it can be realized as follows.

Step 1, for $0 \leqslant i \leqslant p-1$, $1 \leqslant j \leqslant p-i$, choose j colors from the set of $l-1$ colors. There are $\binom{l-1}{j}$ different ways to do that.

Step 2, use these j colors to color $W_{1,1}^{p-i}$ so that each of these j colors must be assigned to the vertices of $W_{1,1}^{p-i}$. This can be done by coloring the j vertices of $W_{1,1}^{p-i}$ pairwise different using the given j colors, while the remaining $p-i-j$ vertices of $W_{1,1}^{p-i}$ are colored using these colors arbitrarily. The number of different ways to do such job is $\binom{p-i-1}{p-i-j}$.

Step 3, color the $n-i$ vertices of U_i^{n-i} using the remaining $l-1-j$ colors. The number of different ways to do so is $\binom{n+l-i-j-2}{n-i}$.

Therefore, by (1) and (2), the number of distinct color codes for the vertices in U is at most

$$\binom{l+n-p-2}{n-p} + \sum_{i=0}^{p-1}\sum_{j=1}^{p-i}\binom{l-1}{j}\binom{p-i-1}{p-i-j}\binom{n+l-i-j-2}{n-i} = g(p,l-1,n).$$

Since $\operatorname{cor}^p(G)$ contains K_k as a subgraph, the number of distinct color codes for the vertices in U is at least k. Thus $k \leqslant g(p,l-1,n)$, which is a contradiction.

We now show that $\chi_m(\operatorname{cor}^p(G)) \leqslant l$ by providing a multiset l-coloring for $\operatorname{cor}^p(G)$. For this purpose, we introduce some notation first.

Since the number of $(n-p)$-element multisets of the set of l colors is $\binom{l+n-p-1}{n-p} := a$, let these a multisets be $A_1, A_2, \ldots, A_a$.

For $r \in \{1,2,\ldots,k\}$, color U_r and $W_{r,1}, W_{r,2}, \ldots, W_{r,n}$ using l colors as follows.

For $0 \leqslant i \leqslant p-1$, color $u_{r,1}, \ldots, u_{r,i}$ by 1; for $1 \leqslant s \leqslant n$, color $w_{r,s,1}, \ldots, w_{r,s,i}$ by 1. Now color the rest $n-i$ vertices in U_r and $p-i$ vertices in each $W_{r,s}$ as described below.

Step 1, for $0 \leqslant i \leqslant p-1$, $1 \leqslant j \leqslant p-i$, choose j colors from the set of l colors. There are $\binom{l}{j}$ different ways to do that.

Step 2, use these j colors to color the remaining $p-i$ vertices in each $W_{r,s}$ so that each of these j colors must be assigned to these $p-i$ vertices. This can be done by coloring the j vertices of these $p-i$ vertices pairwise different using the given j colors, while the remaining $p-i-j$ vertices of these $p-i$ vertices are colored using these colors arbitrarily. The number of different ways to do such job is $\binom{p-i-1}{p-i-j}$.

Step 3, color the remaining $n-i$ vertices in U_r using the remaining $l-j$ colors. The number of different ways to do so is $\binom{n+l-i-j-1}{n-i}$.

The Addition and Multiplication Principle tells us that the number of different such ways to color U_r and $W_{r,1}, W_{r,2}, \ldots, W_{r,n}$ is

$$\sum_{i=0}^{p-1}\sum_{j=1}^{p-i}\binom{l}{j}\binom{p-i-1}{p-i-j}\binom{n+l-i-j-1}{n-i} := b.$$

Let $B_1, B_2, \ldots, B_b$ be these b different coloring ways for U_r and $W_{r,1}, W_{r,2}, \ldots, W_{r,n}$. Please note that

$$a + b = g(p, l, n).$$

We are now ready to give a multiset l-coloring for $\mathrm{cor}^p(G)$.

(1) $2 \leqslant k \leqslant a$. For $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$, $1 \leqslant h \leqslant p$, color $w_{i,j,h}$ by 1; color $u_{i,1}, \ldots, u_{i,p}$ by 1; and color $u_{i,p+1}, \ldots, u_{i,n}$ so that $M(\{u_{i,p+1}, \ldots, u_{i,n}\}) = A_i$.

Let $x \in U_{i_1}$, $y \in U_{i_2}$, where $1 \leqslant i_1 < i_2 \leqslant k$. Then

$$M(x) = M(U) - \{\underbrace{1, \ldots, 1}_{p}\} - A_{i_1} + \{\underbrace{1, \ldots, 1}_{p}\} = M(U) - A_{i_1},$$
$$M(y) = M(U) - \{\underbrace{1, \ldots, 1}_{p}\} - A_{i_2} + \{\underbrace{1, \ldots, 1}_{p}\} = M(U) - A_{i_2}.$$

Since $A_{i_1} \neq A_{i_2}$, $M(x) \neq M(y)$.

(2) $a < k \leqslant a + b$. For $1 \leqslant i \leqslant a$, $1 \leqslant j \leqslant n$, $1 \leqslant h \leqslant p$, color $w_{i,j,h}$ by 1; color $u_{i,1}, \ldots, u_{i,p}$ by 1; and color $u_{i,p+1}, \ldots, u_{i,n}$ so that $M(\{u_{i,p+1}, \ldots, u_{i,n}\}) = A_i$. For $a + 1 \leqslant i \leqslant k$, color U_i and $W_{i,1}, \ldots, W_{i,n}$ by B_{i-a}.

Let $u_{i_1,j_1} \in U_{i_1}$, $u_{i_2,j_2} \in U_{i_2}$, where $1 \leqslant i_1 < i_2 \leqslant k$, $1 \leqslant j_1 \leqslant j_2 \leqslant n$.

(I): $1 \leqslant i_1 < i_2 \leqslant a$. Similarly to (1), it is obvious that $M(u_{i_1,j_1}) \neq M(u_{i_2,j_2})$.

(II): $a < i_1 < i_2 \leqslant k$. It is clear that

$$M(u_{i_1,j_1}) = M(U) - M(U_{i_1}) + M(W_{i_1,j_1}),$$
$$M(u_{i_2,j_2}) = M(U) - M(U_{i_2}) + M(W_{i_2,j_2}).$$

Let $0 \leqslant t_1, t_2 \leqslant p - 1$, denote

$$M(U_{i_1}) = \{\underbrace{1, \ldots, 1}_{t_1}\} \cup M(U'_{i_1}),$$
$$M(W_{i_1,j_1}) = \{\underbrace{1, \ldots, 1}_{t_1}\} \cup M(W'_{i_1,j_1}),$$
$$M(U_{i_2}) = \{\underbrace{1, \ldots, 1}_{t_2}\} \cup M(U'_{i_2}),$$
$$M(W_{i_2,j_2}) = \{\underbrace{1, \ldots, 1}_{t_2}\} \cup M(W'_{i_2,j_2}).$$

In addition, $M(U'_{i_1}) \cap M(W'_{i_1,j_1}) = \emptyset$, $M(U'_{i_2}) \cap M(W'_{i_2,j_2}) = \emptyset$. Therefore,

$$M(u_{i_1,j_1}) = M(U) - M(U'_{i_1}) + M(W'_{i_1,j_1}),$$
$$M(u_{i_2,j_2}) = M(U) - M(U'_{i_2}) + M(W'_{i_2,j_2}).$$

If $M(U'_{i_1}) = M(U'_{i_2})$, then $M(W'_{i_1,j_1}) \neq M(W'_{i_2,j_2})$, so $M(u_{i_1,j_1}) \neq M(u_{i_2,j_2})$. If $M(U'_{i_1}) \neq M(U'_{i_2})$ and $M(W'_{i_1,j_1}) = M(W'_{i_2,j_2})$, then obviously $M(u_{i_1,j_1}) \neq M(u_{i_2,j_2})$. Suppose $M(U'_{i_1}) \neq M(U'_{i_2})$ and $M(W'_{i_1,j_1}) \neq M(W'_{i_2,j_2})$. Then there exists a color α, $1 \leqslant \alpha \leqslant l$ such that the number of the color α in $M(W'_{i_1,j_1})$ is different from that in $M(W'_{i_2,j_2})$. Observe that $M(U'_{i_1}) \cap M(W'_{i_1,j_1}) = \emptyset$ and $M(U'_{i_2}) \cap M(W'_{i_2,j_2}) = \emptyset$, thus the number of the color α in $M(u_{i_1,j_1})$ is different from that in $M(u_{i_2,j_2})$, so $M(u_{i_1,j_1}) \neq M(u_{i_2,j_2})$.

(III): $1 \leqslant i_1 \leqslant a < i_2 \leqslant k$.

$$\begin{aligned} M(u_{i_1,j_1}) &= M(U) - M(U_{i_1}) + M(W_{i_1,j_1}), \\ M(u_{i_2,j_2}) &= M(U) - M(U_{i_2}) + M(W_{i_2,j_2}). \end{aligned}$$

Observe that there exists a color α, $1 \leqslant \alpha \leqslant l$ such that $M(W_{i_2,j_2})$ contains more α's than $M(U_{i_2})$. Besides, $M(W_{i_1,j_1}) \subseteq M(U_{i_1})$. Therefore, $M(u_{i_2,j_2})$ contains more α's than $M(u_{i_1,j_1})$, so $M(u_{i_1,j_1}) \neq M(u_{i_2,j_2})$. □

Remark 4.1. If $p = 1$, then the above theorem implies Theorem 1.2 which was earlier presented in [8]. The verification only requires noticing that $g(0, l, n) = \binom{l+n-2}{n-1} + l\binom{l+n-2}{n}$. If $n = 1$, then we get the multiset of the generalized corona of all the complete graph K_k. To verify that the previous Theorem 3.2 is coincident with that of the above theorem when taking $n = 1$, one only needs to justify the equality

$$g(p, l, 1) = \binom{l+p-1}{p} + (l-1)\binom{l+p-2}{p}. \tag{4.3}$$

In fact, since

$$g(p, l, 1) = \binom{l-p}{1-p} + \sum_{i=0}^{p-1}\sum_{j=1}^{p-i} \binom{l}{j}\binom{p-i-1}{p-i-j}\binom{l-i-j}{1-i}, \tag{4.4}$$

one only needs to justify the equality

$$\begin{aligned} &\binom{l-p}{1-p} + \sum_{i=0}^{p-1}\sum_{j=1}^{p-i} \binom{l}{j}\binom{p-i-1}{p-i-j}\binom{l-i-j}{1-i} \\ &\qquad = \binom{l+p-1}{p} + (l-1)\binom{l+p-2}{p}. \end{aligned} \tag{4.5}$$

If $p = 0$, then (4.5) holds since both sides are equal to l. If $p = 1$, then both sides of (4.5) are equal to $l^2 - l + 1$. Now we suppose $p \geqslant 2$. Then (4.5) follows from the calculations

$$\begin{aligned}
\binom{l-p}{1-p} &+ \sum_{i=0}^{p-1}\sum_{j=1}^{p-i}\binom{l}{j}\binom{p-i-1}{p-i-j}\binom{l-i-j}{1-i} \\
&= \sum_{j=1}^{p}\binom{l}{j}\binom{p-1}{p-j}\binom{l-j}{1} + \sum_{j=1}^{p-1}\binom{l}{j}\binom{p-2}{p-1-j}\binom{l-1-j}{0} \\
&= l\binom{l+p-2}{p} + \binom{l+p-2}{p-1} \\
&= \binom{l+p-2}{p} + \binom{l+p-2}{p-1} + (l-1)\binom{l+p-2}{p} \\
&= \binom{l+p-1}{p} + (l-1)\binom{l+p-2}{p}.
\end{aligned}$$

5. Generalized coronas of $K_r \,\square\, K_2$

We introduce some product graphs as follows.

Definition 5.1. The *Cartesian product* $G \,\square\, H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and edge set $\{(a,x)(b,y)\colon\ ab \in E(G), x = y \text{ or } xy \in E(H), a = b\}$. The *tensor product* $G \times H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and edge set $\{(a,x)(b,y)\colon\ ab \in E(G) \text{ and } xy \in E(H)\}$. The *strong product* $G \boxtimes H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and edge set $E(G \,\square\, H) \cup E(G \times H)$.

The following result can be found in [8].

Theorem 5.1. *For each positive integer r,*

$$\chi_m(K_r \,\square\, K_2) = \left\lceil \frac{1+\sqrt{4r+1}}{2} \right\rceil.$$

Since $K_r \times K_2$ is a regular bipartite graph, by Proposition 1.1, $\chi_m(K_r \times K_2) = 2$. In addition, $K_r \boxtimes K_2$ is a complete graph of order $2r$, thus $\chi_m(K_r \boxtimes K_2) = 2r$. We now focus our attention on the discussion of the multiset chromatic number of generalized corona of $K_r \,\square\, K_2$.

For $l \geqslant 1$, $p \geqslant 1$, define

$$h(l,p) = \binom{l+p-1}{p} + l\binom{l+p-1}{p+1}.$$

Theorem 5.2. *For $r \geqslant 2$, $p \geqslant 1$, the multiset chromatic number of* $\operatorname{cor}^p(K_r \,\square\, K_2)$ *is the unique positive integer l such that*

$$h(l-1,p) < r \leqslant h(l,p).$$

P r o o f. Since $h(l,p)$ is strictly increasing with respect to l, such an integer $l \geqslant 2$ exists. Let H_1 and H_2 be two disjoint copies of K_r, where

$$V(H_1) = \{u_1, u_2, \ldots, u_r\}, \quad V(H_2) = \{v_1, v_2, \ldots, v_r\}.$$

Then

$$\begin{aligned} V(K_r \,\square\, K_2) &= V(H_1) \cup V(H_2), \\ E(K_r \,\square\, K_2) &= E(H_1) \cup E(H_2) \cup \{u_i v_i \colon 1 \leqslant i \leqslant r\}. \end{aligned}$$

Construct $\operatorname{cor}^p(K_r \,\square\, K_2)$ from $K_r \,\square\, K_2$ by adding, for each vertex u_i, $1 \leqslant i \leqslant r$, p end-vertices $X_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,p}\}$, and for each vertex v_i, $1 \leqslant i \leqslant r$, p end-vertices $Y_i = \{y_{i,1}, y_{i,2}, \ldots, y_{i,p}\}$. Let

$$\begin{aligned} X &= X_1 \cup X_2 \cup \ldots \cup X_r, \\ Y &= Y_1 \cup Y_2 \cup \ldots \cup Y_r. \end{aligned}$$

Then

$$\begin{aligned} V(\operatorname{cor}^p(K_r \,\square\, K_2)) &= V(H_1) \cup V(H_2) \cup X \cup Y, \\ E(\operatorname{cor}^p(K_r \,\square\, K_2)) &= E(H_1) \cup E(H_2) \cup \{u_i v_i \colon 1 \leqslant i \leqslant r\} \\ &\quad \cup \{u_i x_{i,j} \colon 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant p\} \\ &\quad \cup \{v_i y_{i,j} \colon 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant p\}. \end{aligned}$$

We first show that $\chi_m(\operatorname{cor}^p(K_r \,\square\, K_2)) > l-1$. Suppose that there exists a multiset $(l-1)$-coloring f of $\operatorname{cor}^p(K_r \,\square\, K_2)$. For each vertex $u_i \in V(H_1)$, $1 \leqslant i \leqslant r$, it is clear that

$$M(u_i) = M(V(H_1)) - \{f(u_i)\} + \{f(v_i)\} + M(X_i).$$

We now determine the number of possible multisets for each vertex $u_i \in V(H_1)$.

(1) $f(u_i) \in M(X_i) \cup \{f(v_i)\}$. Let one vertex in $X_i \cup \{v_i\}$ be colored the same as u_i. And then color the remaining p vertices in $X_i \cup \{v_i\}$ using the $(l-1)$ colors arbitrarily. Thus the number of possible multisets for u_i is

$$\binom{l+p-2}{p}.$$

(2) $f(u_i) \notin M(X_i) \cup \{f(v_i)\}$. Choose a color for u_i from the set of the $(l-1)$ colors; there are just $(l-1)$ different ways to do that. Then color the vertices in $X_i \cup \{v_i\}$ using the remaining $(l-2)$ colors; there are $\binom{l+p-2}{p+1}$ different ways to do that. So in this case, the number of possible multisets for u_i is

$$(l-1)\binom{l+p-2}{p+1}.$$

Therefore, the number of distinct multisets for the vertices in $V(H_1)$ is at most

$$\binom{l+p-2}{p} + (l-1)\binom{l+p-2}{p+1} = h(l-1,p).$$

Since $G[V(H_1)]$ is a complete graph of order r which served as a subgraph of $\mathrm{cor}^p(K_r \,\square\, K_2)$, the r vertices in $V(H_1)$ should have multisets pairwise distinct. Thus $h(l-1,p) \geqslant r$, which is a contradiction, so $\chi_m(\mathrm{cor}^p(K_r \,\square\, K_2)) \geqslant l$.

Next we show that $\chi_m(\mathrm{cor}^p(K_r \,\square\, K_2)) \leqslant l$ by providing a multiset l-coloring for $\mathrm{cor}^p(K_r \,\square\, K_2)$. To this end, we introduce some notation first.

Let the set of l colors be $\{0,1,\dots,l-1\}$.

Let $\mathcal{A}$ be the set of p-element multisets of $\{0,1,\dots,l-1\}$; obviously,

$$|\mathcal{A}| = \binom{l+p-1}{p} := a \geqslant 2.$$

Let

$$\mathcal{A} = \{A_0, A_1, \dots, A_{a-1}\}.$$

For $0 \leqslant i \leqslant l-1$, $0 \leqslant j \leqslant l-2$, let

$$R_{i,j} = \{0,1,\dots,l-1\} - \{i, [i+1]_l, \dots, [i+j]_l\}.$$

Let $\mathcal{B}_{i,j}$ be the set of p-element multisets of $R_{i,j}$. It is clear that

$$|\mathcal{B}_{i,j}| = \binom{l-j-1+p-1}{p} = \binom{l+p-j-2}{p} := b_j.$$

Let

$$\mathcal{B}_{i,j} = \{B_{i,j,k}\colon\ 0 \leqslant k \leqslant b_j - 1\}.$$

For $0 \leqslant t \leqslant l-2$, let

$$s_t = \sum_{j=0}^{t} b_j.$$

In particular,

$$s_{-1} = 0.$$

By the well-known Pascal's formula

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

we obtain

$$s_{l-2} = \sum_{j=0}^{l-2} b_j = \sum_{j=0}^{l-2} \binom{l+p-j-2}{p} = \binom{l+p-1}{p+1} := b.$$

Thus

$$\sum_{i=0}^{l-1} \sum_{j=0}^{l-2} |\mathcal{B}_{i,j}| = \sum_{i=0}^{l-1} \sum_{j=0}^{l-2} b_j = l\binom{l+p-1}{p+1} = lb.$$

Then

$$a + lb = h(l,p).$$

Let S_1 and S_2 be two disjoint copies of $K_{h(l,p)}$, where

$$V(S_1) = \{u_0, u_1, \ldots, u_{lb-1}, w_0, w_1, \ldots, w_{a-1}\} := U,$$
$$V(S_2) = \{v_0, v_1, \ldots, v_{lb-1}, z_0, z_1, \ldots, z_{a-1}\} := V.$$

Then

$$\begin{aligned} V(K_{h(l,p)} \,\Box\, K_2) &= U \cup V, \\ E(K_{h(l,p)} \,\Box\, K_2) &= E(S_1) \cup E(S_2) \\ &\quad \cup \{u_i v_i \colon\ 0 \leqslant i \leqslant lb-1\} \\ &\quad \cup \{w_j z_j \colon\ 0 \leqslant j \leqslant a-1\}. \end{aligned}$$

Construct $\mathrm{cor}^p(K_{h(l,p)} \,\Box\, K_2)$ from $K_{h(l,p)} \,\Box\, K_2$ by adding,

(1) For each vertex u_i, $0 \leqslant i \leqslant lb-1$, p end-vertices $X_i = \{x_{i,0}, x_{i,1}, \ldots, x_{i,p-1}\}$, and for each vertex v_i, $0 \leqslant i \leqslant lb-1$, p end-vertices $Y_i = \{y_{i,0}, y_{i,1}, \ldots, y_{i,p-1}\}$.

(2) For w_j, $0 \leqslant j \leqslant a-1$, p end-vertices $E_j = \{e_{j,0}, e_{j,1}, \ldots, e_{j,p-1}\}$, and for z_j, $0 \leqslant j \leqslant a-1$, p end-vertices $F_j = \{f_{j,0}, f_{j,1}, \ldots, f_{j,p-1}\}$.

Let

$$\begin{aligned} X &= X_0 \cup X_1 \cup \ldots \cup X_{lb-1}, \\ Y &= Y_0 \cup Y_1 \cup \ldots \cup Y_{lb-1}, \\ E &= E_0 \cup E_1 \cup \ldots \cup E_{a-1}, \\ F &= F_0 \cup F_1 \cup \ldots \cup F_{a-1}. \end{aligned}$$

Then

$$
\begin{aligned}
V(\operatorname{cor}^p(K_{h(l,p)} \,\square\, K_2)) &= U \cup V \cup X \cup Y \cup E \cup F,\\
E(\operatorname{cor}^p(K_{h(l,p)} \,\square\, K_2)) &= E(S_1) \cup E(S_2)\\
&\quad \cup \{u_i v_i\colon\ 0 \leqslant i \leqslant lb-1\}\\
&\quad \cup \{w_j z_j\colon\ 0 \leqslant j \leqslant a-1\}\\
&\quad \cup \{u_i x_{i,q}\colon\ 0 \leqslant i \leqslant lb-1,\ 0 \leqslant q \leqslant p-1\}\\
&\quad \cup \{v_i y_{i,q}\colon\ 0 \leqslant i \leqslant lb-1,\ 0 \leqslant q \leqslant p-1\}\\
&\quad \cup \{w_j e_{j,q}\colon\ 0 \leqslant j \leqslant a-1,\ 0 \leqslant q \leqslant p-1\}\\
&\quad \cup \{z_j f_{j,q}\colon\ 0 \leqslant j \leqslant a-1,\ 0 \leqslant q \leqslant p-1\}.
\end{aligned}
$$

Find a multiset l-coloring c of $\operatorname{cor}^p(K_{h(l,p)} \,\square\, K_2)$ such that for $0 \leqslant i \leqslant l-1$, $0 \leqslant j \leqslant l-2$, $0 \leqslant k \leqslant b_j - 1$,

$$
\begin{aligned}
c(u_{ib+s_{(j-1)}+k}) &= i,\\
c(v_{ib+s_{(j-1)}+k}) &= [i+j+1]_l,\\
M(X_{ib+s_{(j-1)}+k}) &= B_{i,j,k},\\
M(Y_{ib+s_{(j-1)}+k}) &= B_{[i+1]_l,j,k}.
\end{aligned}
$$

In addition, for $0 \leqslant i \leqslant a-1$,

$$
\begin{aligned}
c(w_i) = c(z_i) &= 0,\\
M(E_i) &= A_i,\\
M(F_i) &= A_{[i+1]_a}.
\end{aligned}
$$

The above coloring implies that $M(U) = M(V) = M$.

We verify that c is a multiset coloring of $\operatorname{cor}^p(K_{h(l,p)} \,\square\, K_2)$.

(1) For $0 \leqslant i_1,\, i_2 \leqslant l-1$, $0 \leqslant j_1,\, j_2 \leqslant l-2$, $0 \leqslant k_1 \leqslant b_{j_1} - 1$, $0 \leqslant k_2 \leqslant b_{j_2} - 1$,

$$
\begin{aligned}
M(u_{i_1 b+s_{(j_1-1)}+k_1}) &= M(U) - \{c(u_{i_1 b+s_{(j_1-1)}+k_1})\} + \{c(v_{i_1 b+s_{(j_1-1)}+k_1})\}\\
&\quad + M(X_{i_1 b+s_{(j_1-1)}+k_1})\\
&= M - \{i_1\} + \{[i_1+j_1+1]_l\} + B_{i_1,j_1,k_1},\\
M(u_{i_2 b+s_{(j_2-1)}+k_2}) &= M(U) - \{c(u_{i_2 b+s_{(j_2-1)}+k_2})\} + \{c(v_{i_2 b+s_{(j_2-1)}+k_2})\}\\
&\quad + M(X_{i_2 b+s_{(j_2-1)}+k_2})\\
&= M - \{i_2\} + \{[i_2+j_2+1]_l\} + B_{i_2,j_2,k_2}.
\end{aligned}
$$

If $i_1 \neq i_2$, since $i_1 \notin \{[i_1+j_1+1]_l\} \cup B_{i_1,j_1,k_1}$, $M(u_{i_1b+s_{(j_1-1)}+k_1})$ contains less i_1's than $M(u_{i_2b+s_{(j_2-1)}+k_2})$, so $M(u_{i_1b+s_{(j_1-1)}+k_1}) \neq M(u_{i_2b+s_{(j_2-1)}+k_2})$. If $i_1 = i_2$ and $j_1 \neq j_2$, without loss of generality, let $j_1 < j_2$. Since

$$\{i_2, [i_2+1]_l, \ldots, [i_2+j_2]_l\} \cap B_{i_2,j_2,k_2} = \emptyset$$

and

$$[i_1+j_1+1]_l \in \{i_2, [i_2+1]_l, \ldots, [i_2+j_2]_l\},$$

we have $[i_1+j_1+1]_l \notin B_{i_2,j_2,k_2}$. Observe that $[i_1+j_1+1]_l \neq [i_2+j_2+1]_l$, thus $M(u_{i_1b+s_{(j_1-1)}+k_1})$ contains more $[i_1+j_1+1]_l$'s than $M(u_{i_2b+s_{(j_2-1)}+k_1})$. Therefore, $M(u_{i_1b+s_{(j_1-1)}+k_1}) \neq M(u_{i_2b+s_{(j_2-1)}+k_2})$. If $i_1 = i_2$, $j_1 = j_2$ and $k_1 \neq k_2$, then obviously, $B_{i_1,j_1,k_1} \neq B_{i_2,j_2,k_2}$. Thus $M(u_{i_1b+s_{(j_1-1)}+k_1}) \neq M(u_{i_2b+s_{(j_2-1)}+k_2})$.

(2) For $0 \leqslant i \leqslant l-1$, $0 \leqslant j \leqslant l-2$, $0 \leqslant k \leqslant b_j - 1$, $0 \leqslant r \leqslant a-1$,

$$\begin{aligned} M(u_{ib+s_{(j-1)}+k}) &= M(U) - \{c(u_{ib+s_{(j-1)}+k})\} + \{c(v_{ib+s_{(j-1)}+k})\} \\ &\quad + M(X_{ib+s_{(j-1)}+k}) \\ &= M - \{i\} + \{[i+j+1]_l\} + B_{i,j,k}, \\ M(w_r) &= M(U) - \{c(w_r)\} + \{c(z_r)\} + M(E_r) \\ &= M - \{0\} + \{0\} + A_r = M + A_r. \end{aligned}$$

Since $i \notin \{[i+j+1]_l\} \cup B_{i,j,k}$, $M(u_{ib+s_{(j-1)}+k})$ contains less i's than $M(w_r)$. Thus $M(u_{ib+s_{(j-1)}+k}) \neq M(w_r)$.

(3) For $0 \leqslant r_1 < r_2 \leqslant a-1$,

$$\begin{aligned} M(w_{r_1}) &= M(U) - \{c(w_{r_1})\} + \{c(z_{r_1})\} + M(E_{r_1}) \\ &= M - \{0\} + \{0\} + A_{r_1} = M + A_{r_1}, \\ M(w_{r_2}) &= M(U) - \{c(w_{r_2})\} + \{c(z_{r_2})\} + M(E_{r_2}) \\ &= M - \{0\} + \{0\} + A_{r_2} = M + A_{r_2}. \end{aligned}$$

Since $A_{r_1} \neq A_{r_2}$, $M(w_{r_1}) \neq M(w_{r_2})$.

(4) For $0 \leqslant i_1, i_2 \leqslant l-1$, $0 \leqslant j_1, j_2 \leqslant l-2$, $0 \leqslant k_1 \leqslant b_{j_1} - 1$, $0 \leqslant k_2 \leqslant b_{j_2} - 1$,

$$\begin{aligned} M(v_{i_1b+s_{(j_1-1)}+k_1}) &= M(V) - \{c(v_{i_1b+s_{(j_1-1)}+k_1})\} + \{c(u_{i_1b+s_{(j_1-1)}+k_1})\} \\ &\quad + M(Y_{i_1b+s_{(j_1-1)}+k_1}) \\ &= M - \{[i_1+j_1+1]_l\} + \{i_1\} + B_{[i_1+1]_l,j_1,k_1}, \\ M(v_{i_2b+s_{(j_2-1)}+k_2}) &= M(V) - \{c(v_{i_2b+s_{(j_2-1)}+k_2})\} + \{c(u_{i_2b+s_{(j_2-1)}+k_2})\} \\ &\quad + M(Y_{i_2b+s_{(j_2-1)}+k_2}) \\ &= M - \{[i_2+j_2+1]_l\} + \{i_2\} + B_{[i_2+1]_l,j_2,k_2}. \end{aligned}$$

If $[i_1+j_1+1]_l \neq [i_2+j_2+1]_l$, then $M(v_{i_1b+s_{(j_1-1)}+k_1})$ contains less $[i_1+j_1+1]_l$'s than $M(v_{i_2b+s_{(j_2-1)}+k_2})$ since $[i_1+j_1+1]_l \notin \{i_1\} \cup B_{[i_1+1]_l,j_1,k_1}$. Therefore, $M(v_{i_1b+s_{(j_1-1)}+k_1}) \neq M(v_{i_2b+s_{(j_2-1)}+k_2})$.

If $[i_1+j_1+1]_l = [i_2+j_2+1]_l$ and $i_1 = i_2$, then $j_1 = j_2$ and $k_1 \neq k_2$. Since $B_{[i_1+1]_l,j_1,k_1} \neq B_{[i_2+1]_l,j_2,k_2}$, we have $M(v_{i_1b+s_{(j_1-1)}+k_1}) \neq M(v_{i_2b+s_{(j_2-1)}+k_2})$.

If $[i_1+j_1+1]_l = [i_2+j_2+1]_l$ and $i_1 \neq i_2$, say $i_1 < i_2$, then

$$i_2 \in \{[i_1+1]_l, [i_1+2]_l, \dots, [i_1+j_1+1]_l\}.$$

Observe that

$$\{[i_1+1]_l, [i_1+2]_l, \dots, [i_1+j_1+1]_l\} \cap B_{[i_1+1]_l,j_1,k_1} = \emptyset.$$

Therefore,

$$i_2 \notin B_{[i_1+1]_l,j_1,k_1}.$$

Thus $M(v_{i_1b+s_{(j_1-1)}+k_1})$ contains less i_2's than $M(v_{i_2b+s_{(j_2-1)}+k_2})$, so

$$M(v_{i_1b+s_{(j_1-1)}+k_1}) \neq M(v_{i_2b+s_{(j_2-1)}+k_2}).$$

(5) For $0 \leqslant i \leqslant l-1$, $0 \leqslant j \leqslant l-2$, $0 \leqslant k \leqslant b_j-1$, $0 \leqslant r \leqslant a-1$,

$$\begin{aligned}
M(v_{ib+s_{(j-1)}+k}) &= M(V) - \{c(v_{ib+s_{(j-1)}+k})\} + \{c(u_{ib+s_{(j-1)}+k})\} \\
&\quad + M(Y_{ib+s_{(j-1)}+k}) \\
&= M - \{[i+j+1]_l\} + \{i\} + B_{[i+1]_l,j,k}, \\
M(z_r) &= M(V) - \{c(z_r)\} + \{c(w_r)\} + M(F_r) \\
&= M - \{0\} + \{0\} + A_{[r+1]_a} = M + A_{[r+1]_a}.
\end{aligned}$$

Since $[i+j+1]_l \notin \{i\} \cup B_{[i+1]_l,j,k}$, $M(v_{ib+s_{(j-1)}+k})$ contains less $[i+j+1]_l$'s than $M(z_r)$. Thus $M(v_{ib+s_{(j-1)}+k}) \neq M(z_r)$.

(6) For $0 \leqslant r_1 < r_2 \leqslant a-1$,

$$\begin{aligned}
M(z_{r_1}) &= M(V) - \{c(z_{r_1})\} + \{c(w_{r_1})\} + M(F_{r_1}) \\
&= M - \{0\} + \{0\} + A_{[r_1+1]_l} = M + A_{[r_1+1]_l}, \\
M(z_{r_2}) &= M(V) - \{c(z_{r_2})\} + \{c(w_{r_2})\} + M(F_{r_2}) \\
&= M - \{0\} + \{0\} + A_{[r_2+1]_l} = M + A_{[r_2+1]_l}.
\end{aligned}$$

Since $A_{[r_1+1]_l} \neq A_{[r_2+1]_l}$, $M(z_{r_1}) \neq M(z_{r_2})$.

(7) For $0 \leqslant i \leqslant l-1$, $0 \leqslant j \leqslant l-2$, $0 \leqslant k \leqslant b_j - 1$,

$$\begin{aligned}
M(u_{ib+s_{(j-1)}+k}) &= M(U) - \{c(u_{ib+s_{(j-1)}+k})\} + \{c(v_{ib+s_{(j-1)}+k})\} \\
&\quad + M(X_{ib+s_{(j-1)}+k}) \\
&= M - \{i\} + \{[i+j+1]_l\} + B_{i,j,k}, \\
M(v_{ib+s_{(j-1)}+k}) &= M(V) - \{c(v_{ib+s_{(j-1)}+k})\} + \{c(u_{ib+s_{(j-1)}+k})\} \\
&\quad + M(Y_{ib+s_{(j-1)}+k}) \\
&= M - \{[i+j+1]_l\} + \{i\} + B_{[i+1]_l,j,k}.
\end{aligned}$$

Since $i \notin \{[i+j+1]_l\} \cup B_{i,j,k}$, $M(u_{ib+s_{(j-1)}+k})$ contains less i's than $M(v_{ib+s_{(j-1)}+k})$, we have $M(u_{ib+s_{(j-1)}+k}) \neq M(v_{ib+s_{(j-1)}+k})$.

(8) For $0 \leqslant r \leqslant a-1$,

$$\begin{aligned}
M(w_r) &= M(U) - \{c(w_r)\} + \{c(z_r)\} + M(E_r) \\
&= M - \{0\} + \{0\} + A_r = M + A_r, \\
M(z_r) &= M(V) - \{c(z_r)\} + \{c(w_r)\} + M(F_r) \\
&= M - \{0\} + \{0\} + A_{[r+1]_a} = M + A_{[r+1]_a}.
\end{aligned}$$

Since $A_r \neq A_{[r+1]_a}$, $M(w_r) \neq M(z_r)$.

To obtain $\mathrm{cor}^p(K_r \square K_2)$, we need to delete some vertices of $\mathrm{cor}^p(K_{h(l,p)} \square K_2)$. The following algorithm tells us how to do that.

Deletion Algorithm:

▷ If $lb \leqslant r < a + lb$, then delete w_i, z_i, E_i, F_i $(i = 0, 1, \ldots, a + lb - r - 1)$;

▷ If $2 \leqslant r < lb$, let $I = \emptyset$.

Step 1: delete w_i, z_i, E_i, F_i $(i = 0, 1, \ldots, a-1)$;

Step 2:

(A): let $h \in \{0, 1, \ldots, lb-1\} \setminus I$;

(B): find unique i, j, k such that $h = ib + s_{j-1} + k$, where $0 \leqslant i \leqslant l-1$, $0 \leqslant j \leqslant l-2$, $0 \leqslant k \leqslant b_j - 1$. Delete u_h, v_h, E_h, F_h, let $I := I \cup h$.

(C): if $|I| = lb - r$, then stop;
else $h := [h + (j+1)b]_{lb}$;
if $h \notin I$, then go to (B);
else go to (A).

Let c^* be the restriction of c to $\mathrm{cor}^p(K_r \square K_2)$. Denote by $U_{a+lb-r} \subseteq U$ and $V_{a+lb-r} \subseteq V$ two sets of vertices in $K_{h(l,p)} \square K_2$ which are deleted by the above algorithm. Suppose

$$M(U_{a+lb-r}) = \{c_0, c_1, \ldots, c_{a+lb-r-1}\}.$$

Then

$$M(V_{a+lb-r}) = \{c_1, \ldots, c_{a+lb-r-1}, c_{a+lb-r}\}.$$

We now verify that c^* is a multiset l-coloring of $\operatorname{cor}^p(K_r \,\square\, K_2)$.

Let

$$M(U) - \{c_1, \ldots, c_{a+lb-r-1}\} = M(V) - \{c_1, \ldots, c_{a+lb-r-1}\} = M'.$$

Since

$$\begin{aligned} M_{c^*}(u_{ib+s_{(j-1)}+k}) &= M(U) - M(U_{a+lb-r}) - \{c(u_{ib+s_{(j-1)}+k})\} + \{c(v_{ib+s_{(j-1)}+k})\} \\ &\quad + M(X_{ib+s_{(j-1)}+k}) \\ &= M' - \{i\} + \{[i+j+1]_l\} + B_{i,j,k} - \{c_0\}, \\ M_{c^*}(v_{ib+s_{(j-1)}+k}) &= M(V) - M(V_{a+lb-r}) - \{c(v_{ib+s_{(j-1)}+k})\} + \{c(u_{ib+s_{(j-1)}+k})\} \\ &\quad + M(Y_{ib+s_{(j-1)}+k}) \\ &= M' - \{[i+j+1]_l\} + \{i\} + B_{[i+1]_l,j,k} - \{c_{a+lb-r}\}. \end{aligned}$$

As $i \notin \{[i+j+1]_l\} \cup B_{i,j,k}$, $M_{c^*}(u_{ib+s_{(j-1)}+k})$ contains less i's than $M_{c^*}(v_{ib+s_{(j-1)}+k})$, we have $M_{c^*}(u_{ib+s_{(j-1)}+k}) \neq M_{c^*}(v_{ib+s_{(j-1)}+k})$. $\square$

Remark 5.1. Please note that $a = \binom{l+p-1}{p} \geqslant 2$ when $p \geqslant 1$. If $p = 0$, then $a = \binom{l+p-1}{p} = 1$, and we cannot distinguish w_0 and z_0. Therefore, if $p = 0$ is allowed in the above theorem, then we must drop the first term of $\binom{l+0-1}{0} + l\binom{l+0-1}{0+1}$, i.e., let $h(l,0) = l(l-1)$. Hence, if $p = 0$ and $h(l,0) = l(l-1)$, then Theorem 5.3 and Theorem 5.2 are identical.

The *Deletion Algorithm* in the above proof implies the following corollary.

Corollary 5.1. *For $r \geqslant 2$, $p \geqslant 1$, $r_1 \leqslant r_2$,*

$$\chi_m(\operatorname{cor}^p(K_{r_1} \,\square\, K_2)) \leqslant \chi_m(\operatorname{cor}^p(K_{r_2} \,\square\, K_2)).$$

6. Generalized coronas of regular graphs

We conclude the paper by discussing the multiset chromatic number of generalized coronas of regular graphs. We obtain

Theorem 6.1. *Let G be a regular graph. Then*

$$\min_{p\geqslant 0}\{p\colon\ \chi_m(\mathrm{cor}^p(G))=2\}\leqslant \chi(G)-2.$$

P r o o f. Let $\chi(G)=k$, then $k\geqslant 2$. We only need to show that $\chi_m(\mathrm{cor}^{k-2}(G))=2$. $V(G)$ can be partitioned into k independent sets, say $V_0,V_1,\ldots,V_{k-1}$, such that each vertex in $V_0\cup V_1\cup\ldots\cup V_{k-2}$ has neighbors in V_{k-1}. For $0\leqslant i\leqslant k-2$, $1\leqslant j\leqslant r$, denote $U_{i,j}=\{v\colon\ v\in V_i, N(v)\cap V_{k-1}=j\}$. Then

$$U_{i,1}\cup U_{i,2}\cup\ldots\cup U_{i,r}=V_i.$$

Define a vertex 2-coloring of $\mathrm{cor}^{k-2}(G)$ as follows.

(1) For $0\leqslant i\leqslant k-2$, color each vertex in V_i by 1; and color each vertex in V_{k-1} by 2.

(2) Color all end-vertices of each vertex in V_{k-1} by 1. For $1\leqslant j\leqslant r$, $1-j\leqslant i'\leqslant k-1-j$, color all the $(k-2)$ end-vertices of each vertex in $U_{[i']_{k-1},j}$ so that exactly $i'+j-1$ end-vertices are colored by 1.

We now verify that the above coloring is a multiset 2-coloring of $\mathrm{cor}^{k-2}(G)$.

Let $xy\in G$, where $x\in V_i$, $0\leqslant i\leqslant k-2$, $y\in V_{k-1}$. Since $2\in M(x)$, $2\notin M(y)$, we have $M(x)\neq M(y)$.

For $1\leqslant j\leqslant r$, $1-j\leqslant i'\leqslant k-1-j$, the multiset of each vertex in $U_{[i']_{k-1},j}$ contains $j+(k-1-i'-j)=(k-1-i')$ 2's. Let $xy\in G$ and $x\in U_{[i_1]_{k-1},j_1}$, $y\in U_{[i_2]_{k-1},j_2}$, where $1\leqslant j_1$, $j_2\leqslant r$, $1-j_1\leqslant i_1\leqslant k-1-j_1$, $1-j_2\leqslant i_2\leqslant k-1-j_2$. From the above coloring, $M(x)$ contains $(k-1-i_1)$ 2's, $M(y)$ contains $(k-1-i_2)$ 2's. Since $xy\in E(G)$, $i_1\neq i_2$, hence $M(x)\neq M(y)$. $\square$

R e m a r k 6.1. The upper bound $\chi(G)-2$ in the above theorem is sharp since $G=K_n$, $n\geqslant 2$ attains it by Theorem 3.1. However, it is not clear if there exist other graphs which attain this upper bound. In Theorem 5.3, let $l=2$, then $h(l-1,p)=h(1,p)=1$, $h(l,p)=h(2,p)=p+3$. Therefore, if $1<r\leqslant p+3$, then we have $\chi_m(\mathrm{cor}^p(K_r\,\square\,K_2))=2$. Observe that $\chi(\mathrm{cor}^p(K_r\,\square\,K_2))=r$, so we get: if $p\geqslant\chi(\mathrm{cor}^p(K_r\,\square\,K_2))-3$, then $\chi_m(\mathrm{cor}^p(K_r\,\square\,K_2))=2$. Hence, if complete graphs are excluded, then the upper bound $\chi(G)-2$ is almost sharp. One might also be tempted to continue the research on establishing the sharpness of $\chi(G)-2$ when G is a regular graph but not a complete graph.

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References

[1] *L. Addario-Berry, R. E. L. Aldred, K. Dalal, B. A. Reed*: Vertex colouring edge partitions. J. Comb. Theory, Ser. B *94* (2005), 237–244.

[2] *J.-L. Baril, O. Togni*: Neighbor-distinguishing k-tuple edge-colorings of graphs. Discrete Math. *309* (2009), 5147–5157.

[3] *A. C. Burris, R. H. Schelp*: Vertex-distinguishing proper edge-colorings. J. Graph Theory *26* (1997), 73–82.

[4] *G. Chartrand, L. Lesniak, D. W. VanderJagt, P. Zhang*: Recognizable colorings of graphs. Discuss. Math., Graph Theory *28* (2008), 35–57.

[5] *G. Chartrand, F. Okamoto, E. Salehi, P. Zhang*: The multiset chromatic number of a graph. Math. Bohem. *134* (2009), 191–209.

[6] *Y. Feng, W. Lin*: A proof of a conjecture on multiset coloring the powers of cycles. Inf. Process. Lett. *112* (2012), 678–682.

[7] *M. Karoński, T. Łuczak, A. Thomason*: Edge weights and vertex colours. J. Comb. Theory, Ser. B *91* (2004), 151–157.

[8] *F. Okamoto, E. Salehi, P. Zhang*: On multiset colorings of graphs. Discuss. Math., Graph Theory *30* (2010), 137–153.

[9] *M. Radcliffe, P. Zhang*: Irregular colorings of graphs. Bull. Inst. Comb. Appl. *49* (2007), 41–59.

[10] *Z. Zhang, X. Chen, J. Li, B. Yao, X. Lu, J. Wang*: On adjacent-vertex-distinguishing total coloring of graphs. Sci. China, Ser. A *48* (2005), 289–299.

[11] *Z. Zhang, L. Liu, J. Wang*: Adjacent strong edge coloring of graphs. Appl. Math. Lett. *15* (2002), 623–626.

[12] *Z. Zhang, P. Qiu, B. Xu, J. Li, X. Chen, B. Yao*: Vertex-distinguishing total coloring of graphs. Ars Comb. *87* (2008), 33–45.

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