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Alireza Kamel Mirmostafaee; Zbigniew Piotrowski

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ON THE PRESERVATION OF BAIRE AND WEAKLY BAIRE CATEGORY

Alireza Kamel Mirmostafaee, Mashhad,
Zbigniew Piotrowski, Youngstown

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Abstract. We consider the question of preservation of Baire and weakly Baire category under images and preimages of certain kind of functions. It is known that Baire category is preserved under image of quasi-continuous feebly open surjections. In order to extend this result, we introduce a strictly larger class of quasi-continuous functions, i.e. the class of quasi-interior continuous functions. We show that Baire and weakly Baire categories are preserved under image of feebly open quasi-interior continuous surjections. We also give a new definition for countably fiber-completeness of a function. We prove that Baire category is preserved under inverse image of a countably fiber-complete function provided that it is feebly open and feebly continuous.

Keywords: feebly continuous mapping; quasi-interior continuity; Baire space; weakly Baire space; fiber-completeness

MSC 2010: 54E52, 54C10

1. Introduction

Following Bourbaki [2], page 75, a topological space is called *Baire* if its nonempty open subsets are of the second category. This notion plays a fundamental role in the proof of some important results in functional analysis such as the Uniform Boundedness Theorem, the Open Mapping Theorem, and the Closed Graph Theorem, see, e.g. [3], [4], [7], [10], [11], [15], [18].

Unfortunately, even continuous surjections do not preserve Baire category under images. For example, the space of rational numbers is the continuous image of any

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countable set with discrete topology. Moreover, the image of a Baire space under an open mapping may fail to be a Baire space [8].

Thus it is natural to ask under what condition(s) a function preserves Baire category under its image.

It is known that Baire spaces are invariant under continuous open surjections (see [8], Theorem 1.10). Frolík [9], Theorem 1, improved this result by showing that the result remains true if the function is only assumed to be quasi-continuous and feebly open. Neubrunn [12] proved that Baire category is preserved by feebly open, feebly continuous injections, and finally Dobos [5] showed that the injectivity condition in Neubrunn's result is necessary by exhibiting a Baire space X and a feebly continuous, feebly open function $f\colon X \to f(X)$ such that $f(X)$ is not a Baire space.

The question whether the Cartesian product (or a square) of Baire spaces is again a Baire space, was one of the most difficult questions in General Topology. Sikorski [19] asked this question in 1947. The Product Problem is closely related to the question of the preservation of Baire category under preimages. In fact, if the product of Baire spaces is not a Baire space, then the natural projection of, say, X^2 onto X, which is open and continuous, would show that Baire category is not preserved under preimages by open continuous functions. A completely satisfactory answer, i.e. in ZFC, came in [7], where a metric Baire space X was given whose square is of the first category. So, Baireness is not preserved under preimages of open continuous function, even if both the domain and the range of the function are metric.

In 1961, Oxtoby [14] proved that if a Baire space X has countable pseudo-base (= a collection of open sets such that every nonempty open subset contains a member of the collection), then X^2 is Baire. Frolík [8], Theorem 2, page 383, probably unaware of Oxtoby's result, showed:

Let $f\colon X \to Y$ be an open continuous function from a metrizable separable space X onto a space Y. If Y is a Baire space and the fibers are Baire, then X is Baire.

In 1989, Noll [13] proved that under certain conditions, Baire category is preserved under preimage of special functions, which we call quasi-interior continuous. In the next section, we show that every feebly open quasi-continuous mapping is quasi-interior continuous but that the converse is not true in general. Moreover, we prove that quasi-interior continuous functions preserve Baire category under images. This will improve Neubrunn's theorem mentioned above.

In 1988, Beer and Villar [1] defined the notion weakly Baire spaces. Rose et al. in [17] proved that the image of a weakly Baire space under quasi-continuous feebly open fiber countable function is weakly Baire. We improve this result by showing that this result is true if the function is fiber countable quasi-interior continuous.

2. Preservation of Baire category under images

Throughout this paper, we assume that X, Y and Z are Hausdorff topological spaces. We start by recalling some definitions.

Definition 2.1. A function f from X to Y is called

(a) *feebly open* if for every open subset U of X, $\operatorname{int}(f(U))$ is nonempty,
(b) *feebly continuous* if for every open subset W of Y, $\operatorname{int}(f^{-1}(W))$ is nonempty,
(c) a *feeble homeomorphism* if it is a feebly open and feebly continuous bijection.

Definition 2.2. A function $f\colon X \to Y$ is called *quasi-continuous* at $x_0 \in X$ if for every neighborhood U of x_0 and every neighborhood W of $f(x_0)$, there is a nonempty open subset U' of U such that $f(U') \subseteq W$. The function f is called quasi-continuous if it is quasi-continuous at each point of X.

The following class of functions was introduced by Noll in [13].

Definition 2.3. A feebly continuous function $f\colon X \to Y$ is called *quasi-interior continuous* if for every nonempty open set U in X and every nonempty open set $W \subseteq \operatorname{int} f(U)$, the set $U \cap \operatorname{int} f^{-1}(W)$ is nonempty.

Theorem 2.1. *If $f\colon X \to Y$ is quasi-continuous and feebly open, then it is quasi-interior continuous.*

P r o o f. Let U be a nonempty open subset of X and W a nonempty open subset of $\operatorname{int} f(U)$. Then there is an $x_0 \in U$ such that $f(x_0) \in W$. Thanks to quasi-continuity of f at x_0, there is a nonempty open set $U' \subseteq U$ such that $f(U') \subseteq W$. It follows that $U' \subseteq U \cap f^{-1}(W)$. Thus, the set $U \cap f^{-1}(W)$ is not empty. □

The following example shows that the converse of the above result is not true in general.

E x a m p l e 2.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x-1, & x<0,\\ 0, & x=0,\\ x+1, & x>0. \end{cases}$$

Then f is quasi-interior continuous and feebly open, but it is not quasi-continuous at zero.

In order to state the main result of this section, we need the following auxiliary result.

Lemma 2.1. *If $f\colon X \to Y$ is quasi-interior continuous and feebly open and V is a dense open subset of Y, then $\operatorname{int} f^{-1}(V)$ is a dense subset of X.*

Proof. Let U be an open subset of X. By feeble openness of f, $\operatorname{int} f(U)$ is nonempty. Since V is dense in Y, the set $G = \operatorname{int} f(U) \cap V$ is nonempty. The inclusion $G \subseteq \operatorname{int} f(U)$ together with quasi-interior continuity of f implies that $U \cap \operatorname{int} f^{-1}(G) \neq \emptyset$. It follows that $U \cap \operatorname{int} f^{-1}(V) \supset U \cap \operatorname{int} f^{-1}(G) \neq \emptyset$. Hence, $\operatorname{int} f^{-1}(V)$ is dense in X. □

Now, we are ready to state the main result of this section.

Theorem 2.2. *Let $f\colon X \to Y$ be a quasi-interior continuous feebly open surjection. Then f preserves Baire category under image, i.e. if X is Baire, then so is Y.*

Proof. Let $\{W_n\}$ be a sequence of dense open subsets of Y and G an arbitrary nonempty open subset of Y. Thanks to Lemma 2.1, $\{\operatorname{int} f^{-1}(W_n)\}$ is a sequence of dense subsets of X. Since $G \subseteq \operatorname{int} f(X) = Y$, there is a nonempty open subset U of X such that $f(U) \subset G$. By the Baireness of X, $U \cap \Big(\bigcap\limits_{i\geqslant 1} \operatorname{int} f^{-1}(W_n)\Big) \neq \emptyset$. Therefore, we have

$$\emptyset \neq f\Big(U \cap \Big(\bigcap_{i\geqslant 1} \operatorname{int} f^{-1}(W_n)\Big)\Big) \subseteq f(U) \cap f\Big(\bigcap_{i\geqslant 1} \operatorname{int} f^{-1}(W_n)\Big) \subseteq G \cap \Big(\bigcap_{i\geqslant 1} W_n\Big).$$

This completes our proof. □

The following result follows immediately from Theorems 2.1 and 2.2.

Corollary 2.1 ([8]). *Let $f\colon X \to Y$ be a quasi-continuous feebly open surjection. If X is Baire, then so is Y.*

3. Preservation of Baire category under preimage

In 1989, Noll [12] introduced a notion of fiber-completeness of a function to show that Baire category is preserved under preimage of such a function provided that f is quasi-interior continous. He asked if a fiber-complete feebly open and feebly continuous function preserves Baire category under preimage. Piotrowski and Reilly in [16] provided an example to show that this conjecture is not true in general. In this section, we give another type of fiber-completeness. We show that this kind of fiber-completeness preserves Baire category under inverse image provided that the function is feebly open and feebly continuous.

Definition 3.1. Let X and Y be topological spaces. A function $f\colon X \to Y$ is called *countably fiber-complete* if for every centered sequence $\{U_i\}_{i\geqslant 1}$ of open subsets of X, $\bigcap\limits_{i=1}^{\infty} U_i \neq \emptyset$, provided that there is $y \in Y$ such that $f^{-1}(y) \cap U_i \neq \emptyset$ for each $i \geqslant 1$.

Lemma 3.1. *Let f be a feebly open and feebly continuous function from a topological space X to a topological space Y and let U be an open dense subset of X. Then* $\operatorname{int} f(U)$ *is dense in Y.*

P r o o f. Let G be a nonempty open subset of Y. Since f is feebly continuous, $\operatorname{int} f^{-1}(G)$ is a nonempty open subset of X. Therefore, $U \cap \operatorname{int} f^{-1}(G)$ is a nonempty open subset of X. Since f is feebly open, $\operatorname{int} f(U \cap \operatorname{int} f^{-1}(G)) \neq \emptyset$. We have

$$\emptyset \neq \operatorname{int} f(U \cap \operatorname{int} f^{-1}(G)) \subseteq \operatorname{int} f(U \cap f^{-1}(G)) \subseteq \operatorname{int} f(U) \cap G.$$

This completes our proof. □

Theorem 3.1. *Let f be a feebly open and feebly continuous countably fiber-complete function from a topological space X onto a Baire space Y. Then X is a Baire space.*

P r o o f. Let $\{U_i\}_{i\geqslant 1}$ be a sequence of dense open subsets of X. We will show that $\bigcap\limits_{i=1}^{\infty} U_i$ is dense in X. Let V be a nonempty open subset of X. Let $W = \operatorname{int} f(V)$. Thanks to feeble openness of f, W is a nonempty open subset of Y. By the above lemma, $\{\operatorname{int} f(U_i)\}_{i\geqslant 1}$ is a sequence of dense open subsets of Y. Since Y is Baire, $\bigcap\limits_{i=1}^{\infty} \operatorname{int} f(U_i)$ is dense in Y. Hence $\bigcap\limits_{i=1}^{\infty} \operatorname{int} f(U_i) \cap W$ is nonempty. Let y belong to this set. It follows that $f^{-1}(y) \cap (U_i \cap V) \neq \emptyset$ for each $i \geqslant 1$. Since f is countably fiber-complete, $\bigcap\limits_{i=1}^{\infty} U_i \cap V \neq \emptyset$. This completes our proof. □

Since every one to one mapping satisfies the properties of Theorem 3.1, we have the following result.

Corollary 3.1 ([6], [12]). *If $f\colon X \to Y$ is a feeble homeomorphism, then X is a Baire space if and only if Y is a Baire space.*

4. Preservation of weakly Baire spaces under image

The notion of a weakly Baire space was introduced by Beer and Villar in [1]. Rose et al. in [17] gave the following equivalent condition of a weakly Baire space.

Definition 4.1. Let (X,τ) be a topological T_1-space. Let $M(\tau)$ and $C(X)$ denote meager and countable subsets of X, respectively. Then (X,τ) is called *weakly Baire* if $M(\tau)\cap C(X)\cap\tau=\emptyset$.

Definition 4.2. Let (X,τ) be a topological space. Define $X_C=\bigcup_{U\in C(X)\cap\tau}U$ with the relative topology.

Rose et al. [17] proved that if $f\colon X\to Y$ is a quasi-continuous feebly open surjection such that $f^{-1}(y)$ is countable for each $y\in X_C$, then Y is weakly Baire if X is weakly Baire. In order to give a generalization of this result, we need the following lemma.

Lemma 4.1. *Let $f\colon X\to Y$ be a quasi-interior continuous feebly open function and E a nowhere dense subset of Y. Then $f^{-1}(E)$ is a nowhere dense subset of X.*

P r o o f. By Lemma 2.1, $f^{-1}(Y\setminus\overline{E})$ is a dense subset of X. Therefore $f^{-1}(E)$ is nowhere dense in X. □

Theorem 4.1. *Let (X,τ) be a weakly Baire space and $f\colon(X,\tau)\to(Y,\sigma)$ a quasi-interior continuous feebly open surjection. Then (Y,σ) is a weakly Baire space provided that $f^{-1}(y)$ is countable for each $y\in Y_C$.*

P r o o f. Let $V\in M(\sigma)\cap C(Y)\cap\sigma$. By Lemma 4.1, $f^{-1}(V)\in M(\tau)$. Since V is countable and, by our assumption, $f^{-1}(y)$ is countable for each $y\in V$, we have $\operatorname{int} f^{-1}(V)\in M(\tau)\cap C(X)\cap\tau$. The last set is empty, since Y is weakly Baire. Therefore, by the weak openness of f, $V=\emptyset$. This completes our proof. □

Corollary 4.1. *Let $f\colon X\to Y$ be a feeble homeomorphism. Then X is weakly Baire if and only if Y is weakly Baire.*

P r o o f. The result follows from the fact that if f is a feeble homeomorphism, both f and f^{-1} are quasi-interior continuous feebly open. □

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Authors' addresses: *Alireza Kamel Mirmostafaee*, Center of Excellence in Analysis on Algebraic Structures, Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Azadi Sq., Mashhad 91775, Iran, e-mail: `mirmostafaei@um.ac.ir`; *Zbigniew Piotrowski*, Department of Mathematics and Statistics, Youngstown State University, Lincoln Building Office 518, One University Plaza, Youngstown, Ohio, 44555, USA, e-mail: `zpiotr@as.ysu.edu`.