Vladimir V. Bavula

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THE GROUPS OF AUTOMORPHISMS OF THE WITT W_n AND VIRASORO LIE ALGEBRAS

Vladimir V. Bavula, Sheffield

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Abstract. Let $L_n = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial algebra over a field K of characteristic zero, $W_n := \mathrm{Der}_K(L_n)$ the Lie algebra of K-derivations of the algebra L_n, the so-called Witt Lie algebra, and let Vir be the Virasoro Lie algebra which is a 1-dimensional central extension of the Witt Lie algebra. The Lie algebras W_n and Vir are infinite dimensional Lie algebras. We prove that the following isomorphisms of the groups of Lie algebra automorphisms hold: $\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vir}) \simeq \mathrm{Aut}_{\mathrm{Lie}}(W_1) \simeq \{\pm 1\} \ltimes K^*$, and give a short proof that $\mathrm{Aut}_{\mathrm{Lie}}(W_n) \simeq \mathrm{Aut}_{\mathrm{K-alg}}(L_n) \simeq \mathrm{GL}_n(\mathbb{Z}) \ltimes K^{*n}$.

Keywords: group of automorphisms; monomorphism; Lie algebra; Witt algebra; Virasoro algebra; automorphism; locally nilpotent derivation

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1. Introduction

In this paper, module means a left module, K is a field of characteristic zero and K^* is its group of units, and the following notation is fixed:

- ▷ $P_n := K[x_1, \dots, x_n] = \bigoplus_{\alpha \in \mathbb{N}^n} Kx^\alpha$ is a polynomial algebra over K where $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$,
- ▷ $G_n := \mathrm{Aut}_{\text{K-alg}}(P_n)$ is the group of automorphisms of the polynomial algebra P_n,
- ▷ $L_n := K[x_1^{\pm 1}, \dots, x_n^{\pm 1}] = \bigoplus_{\alpha \in \mathbb{Z}^n} Kx^\alpha$ is a Laurent polynomial algebra,
- ▷ $\mathbb{L}_n := \mathrm{Aut}_{\text{K-alg}}(L_n)$ is the group of K-algebra automorphisms of L_n,
- ▷ $\partial_1 := \partial/\partial x_1, \dots, \partial_n := \partial/\partial x_n$ are the partial derivatives (K-linear derivations) of P_n,

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▷ $D_n := \mathrm{Der}_K(P_n) = \bigoplus_{i=1}^{n} P_n\partial_i$ is the Lie algebra of K-derivations of P_n where $[\partial, \delta] := \partial\delta - \delta\partial$,

▷ $\mathbb{G}_n := \mathrm{Aut}_{\mathrm{Lie}}(D_n)$ is the group of automorphisms of the Lie algebra D_n,

▷ $W_n := \mathrm{Der}_K(L_n) = \bigoplus_{i=1}^{n} L_n\partial_i$ is the Witt Lie algebra where $[\partial, \delta] := \partial\delta - \delta\partial$,

▷ $\mathbb{W}_n := \mathrm{Aut}_{\mathrm{Lie}}(W_n)$ is the group of automorphisms of the Witt Lie algebra W_n,

▷ $\delta_1 := \mathrm{ad}(\partial_1), \dots, \delta_n := \mathrm{ad}(\partial_n)$ are the inner derivations of the Lie algebras D_n and W_n determined by $\partial_1, \dots, \partial_n$ where $\mathrm{ad}(a)(b) := [a, b]$,

▷ $\mathcal{D}_n := \bigoplus_{i=1}^{n} K\partial_i$,

▷ $\mathcal{H}_n := \bigoplus_{i=1}^{n} KH_i$ where $H_1 := x_1\partial_1, \dots, H_n := x_n\partial_n$.

The group of automorphisms of the Witt Lie algebra $\mathbb{W}_n$. The aim of the paper is to find the group of automorphisms of the Virasoro Lie algebra Vir (Theorem 1.2) and to give a short proof that $\mathrm{Aut}_{\mathrm{Lie}}(W_n) \simeq \mathrm{Aut}_{\text{K-alg}}(L_n) \simeq \mathrm{GL}_n(\mathbb{Z}) \ltimes K^{*n}$ (Theorem 1.1). In [10], it was proved that the Lie algebra of C^∞ vector fields on a smooth manifold is uniquely determined by the manifold. In [6], the same type of results are obtained for a larger class of Lie algebras including the Lie algebras of real analytic vector fields and the Lie algebras of complex analytic vector fields on Stein manifolds. Algebraic analogues of these results were given in [9], [8], [5], [7], and some other papers. In [9], a sketch of the theorem is given that states that for the Cartan-Lie algebras W_{A^n}, S_{A^n}, H_{A^n} and K_{A^n} their groups of automorphisms are determined by the automorphisms of the affine space A^n. In [4] a short algebraic proof is given for $W_{A^n} = D_n$. In [8], [5], similar results are proved for larger classes of Lie algebras introduced in these papers (like generalized Cartan type W Lie algebras).

The following lemma is an easy exercise.

▷ (Lemma 2.8) $\mathbb{L}_n \simeq \mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{T}^n$ *where* $\mathrm{GL}_n(\mathbb{Z})$ *is identified with a subgroup of* $\mathbb{L}_n$ *via the group monomorphism* $\mathrm{GL}_n(\mathbb{Z}) \to \mathbb{L}_n$, $A = (a_{ij}) \mapsto \sigma_a\colon x_i \mapsto \prod_{j=1}^{n} x_j^{a_{ji}}$, *and*

$$\mathbb{T}^n := \{t_l \in \mathbb{L}_n\colon\ t_l(x_1) = l_1x_1, \dots, t_l(x_n) = l_nx_n;\ l \in K^{*n}\} \simeq K^{*n}$$

is the algebraic n-dimensional torus.

Theorem 1.1. $\mathbb{W}_n = \mathbb{L}_n$.

Structure of the proof.

(i) $\mathbb{L}_n$ is a subgroup of $\mathbb{W}_n$ (Lemma 2.2) via the group monomorphism

$$\mathbb{L}_n \to \mathbb{W}_n, \quad \sigma \mapsto \sigma\colon\ \partial \mapsto \sigma(\partial) := \sigma\partial\sigma^{-1}.$$

Let $\sigma \in \mathbb{W}_n$. We have to show that $\sigma \in \mathbb{L}_n$.

(ii) (crux) $\sigma(\mathcal{H}_n) = \mathcal{H}_n$ (Lemma 2.5), i.e.,

$$\sigma(H) = A_\sigma H \text{ for some } A_\sigma \in \mathrm{GL}_n(K)$$

where $H := (H_1, \ldots, H_n)^{\mathrm{T}}$.

(iii) $A_\sigma \in \mathrm{GL}_n(\mathbb{Z})$ (Corollary 2.7).

(iv) There exists an automorphism $\tau \in \mathbb{L}_n$ such that $\tau\sigma \in \mathrm{Fix}_{\mathbb{W}_n}(H_1, \ldots, H_n)$ (Lemma 2.10).

(v) $\mathrm{Fix}_{\mathbb{W}_n}(H_1, \ldots, H_n) = \mathbb{T}^n \subseteq \mathbb{L}_n$ (Lemma 2.12) and so $\sigma \in \mathbb{L}_n$. □

The group of automorphisms of the Virasoro Lie algebra. The *Virasoro Lie algebra* $\mathrm{Vir} = W_1 \oplus Kc$ is a 1-dimensional central extension of the Witt Lie algebra W_1 where $Z(\mathrm{Vir}) = Kc$ is the centre of Vir and for all $i, j \in \mathbb{Z}$,

$$[x^i H, x^j H] = (j-i)x^{i+j} H + \delta_{i,-j} \frac{i^3 - i}{12} c \tag{1}$$

where $x = x_1$ and $H = H_1$.

Theorem 1.2. $\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vir}) \simeq \mathbb{W}_1 \simeq \mathbb{L}_1 \simeq \mathrm{GL}_1(\mathbb{Z}) \ltimes \mathbb{T}^1$.

The key point in the proof of Theorem 1.2 is to use Theorem 1.3 of which Theorem 1.2 is a special case (where $\mathcal{G} = \mathrm{Vir}$, $W = W_1$ and $Z = Kc$, see Section 3).

Theorem 1.3. *Let $\mathcal{G}$ be a Lie algebra, Z a subspace of the centre of $\mathcal{G}$ and $W = \mathcal{G}/Z$. Suppose that*

1. *every automorphism σ of the Lie algebra W can be extended to an automorphism $\widehat{\sigma}$ of the Lie algebra $\mathcal{G}$,*
2. *$Z \subseteq [G, G]$, and*
3. *$W = [W, W]$.*

Then, for each σ, the extension $\widehat{\sigma}$ is unique and the map $\mathrm{Aut}_{\mathrm{Lie}}(W) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathcal{G})$, $\sigma \mapsto \widehat{\sigma}$, is a group isomorphism.

The groups $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{u}_n)$ and $\mathrm{Aut}_{\mathrm{Lie}}(D_n)$ where found in [3] and [4], respectively. The Lie algebras $\mathfrak{u}_n$ have been studied in great detail in [1] and [2]. In particular, in [1] it was proved that every monomorphism of the Lie algebra $\mathfrak{u}_n$ is an automorphism but this is not true for epimorphisms.

2. Proof of Theorem 1.1

This section can be seen as a proof of Theorem 1.1. The proof is split into several statements that reflect 'Structure of the proof of Theorem 1.1' given in Introduction.

By the very definition, $\mathcal{H}_n = \bigoplus_{i=1}^n KH_i$ is an abelian Lie subalgebra of W_n of dimension n. Each element H of $\mathcal{H}_n$ is a unique sum $H = \sum_{i=1}^n l_i H_i$ where $l_i \in K$. Let us define the bilinear map

$$\mathcal{H}_n \times \mathbb{Z}^n \to K, \quad (H, \alpha) \mapsto (H, \alpha) := \sum_{i=1}^n l_i \alpha_i.$$

The Witt algebra W_n is a $\mathbb{Z}^n$-graded Lie algebra. The Witt algebra

$$W_n = \bigoplus_{\alpha \in \mathbb{Z}^n} \bigoplus_{i=1}^n Kx^\alpha \partial_i = \bigoplus_{\alpha \in \mathbb{Z}^n} x^\alpha \mathcal{H}_n \tag{2}$$

is a $\mathbb{Z}^n$-graded Lie algebra, that is $[x^\alpha \mathcal{H}_n, x^\beta \mathcal{H}_n] \subseteq x^{\alpha+\beta}\mathcal{H}_n$ for all $\alpha, \beta \in \mathbb{Z}^n$. This follows from the identity

$$[x^\alpha H, x^\beta H'] = x^{\alpha+\beta}((H, \beta)H' - (H', \alpha)H). \tag{3}$$

In particular,

$$[H, x^\alpha H'] = (H, \alpha) x^\alpha H'. \tag{4}$$

So, $x^\alpha \mathcal{H}_n$ is the *weight subspace* $W_{n,\alpha} := \{w \in W_n \colon [H, w] = (H, \alpha)w\}$ of W_n with respect to the adjoint action of the abelian Lie algebra $\mathcal{H}_n$ on W_n. The direct sum (2) is the weight decomposition of W_n and $\mathbb{Z}^n$ is the *set of weights* of $\mathcal{H}_n$.

Let $\mathcal{G}$ be a Lie algebra and $\mathcal{H}$ its Lie subalgebra. The *centralizer* $C_\mathcal{G}(\mathcal{H}) := \{x \in \mathcal{G} \colon [x, \mathcal{H}] = 0\}$ of $\mathcal{H}$ in $\mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$. In particular, $Z(\mathcal{G}) := C_\mathcal{G}(\mathcal{G})$ is the *centre* of the Lie algebra $\mathcal{G}$. The *normalizer* $N_\mathcal{G}(\mathcal{H}) := \{x \in \mathcal{G} \colon [x, \mathcal{H}] \subseteq \mathcal{H}\}$ of $\mathcal{H}$ in $\mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$, it is the largest Lie subalgebra of $\mathcal{G}$ that contains $\mathcal{H}$ as an ideal. Each element $a \in \mathcal{G}$ determines the derivation of the Lie algebra $\mathcal{G}$ by the rule $\mathrm{ad}(a) \colon \mathcal{G} \to \mathcal{G}$, $b \mapsto [a, b]$, which is called the *inner derivation* associated with a. An element $a \in \mathcal{G}$ is called a *locally finite element* if so is the inner derivation $\mathrm{ad}(a)$ of the Lie algebra $\mathcal{G}$, that is $\dim_K \Big(\sum_{i \in \mathbb{N}} K\mathrm{ad}(a)^i(b) \Big) < \infty$ for all $b \in \mathcal{G}$. Let $\mathrm{LF}(\mathcal{G})$ be the set of locally finite elements of $\mathcal{G}$.

The Cartan subalgebra $\mathcal{H}_n$ of W_n. A nilpotent Lie subalgebra C of a Lie algebra $\mathcal{G}$ such that $C = N_\mathcal{G}(C)$ is called a *Cartan subalgebra* of $\mathcal{G}$. We often use the following obvious observation: *An abelian Lie subalgebra that coincides with its centralizer is a maximal abelian Lie subalgebra.*

Lemma 2.1.

1. $\mathcal{H}_n = C_{W_n}(\mathcal{H}_n)$ *is a maximal abelian Lie subalgebra of* W_n.
2. $\mathcal{H}_n$ *is a Cartan subalgebra of* W_n.

Proof. Both the statements follow from (2) and (4). $\square$

The next lemma is very useful and can be applied in many different situations. It allows one to see the group of automorphisms of a ring as a subgroup of the group of automorphisms of its Lie algebra of derivations.

Lemma 2.2. *Let R be a commutative ring such that there exists a derivation $\partial \in \mathrm{Der}(R)$ such that $r\partial \neq 0$ for all nonzero elements $r \in R$ (e.g., $R = P_n, L_n$ and $\delta = \partial_1$). Then the group homomorphism*

$$\mathrm{Aut}(R) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Der}(R)), \quad \sigma \mapsto \sigma\colon\ \delta \mapsto \sigma(\delta) := \sigma\delta\sigma^{-1},$$

is a monomorphism.

Proof. If an automorphism $\sigma \in \mathrm{Aut}(R)$ belongs to the kernel of the group homomorphism $\sigma \mapsto \sigma$ then for all $r \in R$, $r\partial = \sigma(r\partial)\sigma^{-1} = \sigma(r)\sigma\partial\sigma^{-1} = \sigma(r)\partial$, i.e. $\sigma(r) = r$ for all $r \in R$. This means that σ is the identity automorphism. Therefore, the homomorphism $\sigma \mapsto \sigma$ is a monomorphism. $\square$

The $(\mathbb{Z}, l)$-grading and the filtration $\mathcal{F}_l$ on W_n. Each vector $l = (l_1, \dots, l_n) \in \mathbb{Z}^n$ determines the $\mathbb{Z}$-grading on the Lie algebra W_n by the rule

$$W_n = \bigoplus_{i\in\mathbb{Z}} W_{n,i}(l), \quad W_{n,i}(l) := \bigoplus_{(l,\alpha)=i} x^\alpha \mathcal{H}_n, \quad (l,\alpha) := \sum_{i=1}^n l_i\alpha_i,$$

$[W_{n,i}(l), W_{n,j}(l)] \subseteq W_{n,i+j}(l)$ for all $i, j \in \mathbb{Z}$, as follows from (3) and (4). The $\mathbb{Z}$-grading above is called the $(\mathbb{Z}, l)$-*grading* on W_n. Every element $a \in W_n$ is the unique sum of homogeneous elements with respect to the $(\mathbb{Z}, l)$-grading on W_n,

$$a = a_{i_1} + a_{i_2} + \ldots + a_{i_s}, \quad a_{i_\nu} \in W_{n,i_\nu}(l),$$

and $i_1 < i_2 < \ldots < i_s$. The elements $l_l^+(a) := a_{i_s}$ and $l_l^-(a) := a_{i_1}$ are called the *leading term* and the *least term* of a, respectively. So,

$$a = l_l^+(a) + \ldots,$$
$$a = l_l^-(a) + \ldots,$$

where the three dots denote smaller and larger terms, respectively. For all $a, b \in W_n$,

$$(5) \qquad [a,b] = [l_l^+(a), l_l^+(b)] + \dots,$$

$$(6) \qquad [a,b] = [l_l^-(a), l_l^-(b)] + \dots,$$

where the three dots denote smaller and larger terms, respectively (the brackets on the right hand side can be zero).

The Newton polygon of an element of W_n. Each element $a \in W_n$ is the unique finite sum $a = \sum_{\alpha \in \mathbb{Z}^n} l_\alpha x^\alpha H_\alpha$ where $l_\alpha \in K$ and $H_\alpha \in \mathcal{H}_n$. The set $\operatorname{Supp}(a) := \{\alpha \in \mathbb{Z}^n \colon l_\alpha \neq 0\}$ is called the *support* of a and its convex hull in $\mathbb{R}^n$ is called the *Newton polygon* of a, denoted by $\operatorname{NP}(a)$.

Lemma 2.3. *Let a be a locally finite element of W_n. Then the elements $l_l^+(a)$ and $l_l^-(a)$ are locally finite for all $l \in \mathbb{Z}^n$.*

Proof. The statement follows from (5) and (6). $\square$

Let $\operatorname{LF}(W_n)_h$ be the set of *homogeneous* (with respect to the $\mathbb{Z}^n$-grading on W_n) locally finite elements of the Lie algebra W_n.

Lemma 2.4. $\operatorname{LF}(W_n)_h = \mathcal{H}_n$.

Proof. $\mathcal{H}_n \subseteq \operatorname{LF}(W_n)_h$ since every element of $\mathcal{H}_n$ is a semi-simple element of W_n for all $H = \sum_{i=1}^n l_i H_i$ where $l_i \in K$,

$$(7) \qquad [H, x^\alpha H'] = (l, \alpha) x^\alpha H' \quad \alpha \in \mathbb{Z}^n, \ H' \in \mathcal{H}_n.$$

It suffices to show that no homogeneous element $x^\alpha H'$ that does not belong to $\mathcal{H}_n$, i.e. $\alpha \neq 0$, is locally finite. Fix i such that $\alpha_i \neq 0$. Let $\delta = \operatorname{ad}(x^\alpha H')$.

Suppose that $(H', \alpha) \neq 0$. This is the case for $n = 1$. Then

$$\delta^m(x^{2\alpha} H') = (m-1)!\, 2^{m-1} (H', \alpha)^m x^{(1+2m)\alpha} H' \quad \text{for } m \geqslant 1.$$

Therefore, the element $x^\alpha H'$ is not locally finite.

Suppose that $(H', \alpha) = 0$. Then necessarily $n \geqslant 2$. Fix $\beta \in \mathbb{Z}^n$ such that $(H', \beta) = 1$. Then

$$\delta^m(x^\beta H') = x^{\beta + m\alpha} H' \quad \text{for } m \geqslant 1.$$

Therefore, the element $x^\alpha H'$ is not locally finite. $\square$

Lemma 2.5. $\sigma(\mathcal{H}_n) = \mathcal{H}_n$ *for all $\sigma \in \mathbb{W}_n$.*

Proof. Let $\sigma \in \mathbb{W}_n$ and $H \in \mathcal{H}_n$. We have to show that $H' := \sigma(H) \in \mathcal{H}_n$. The element H is a locally finite element, hence so is H'. By Lemma 2.3 and Lemma 2.4, the Newton polygon $\mathrm{NP}(H')$ has the single vertex 0, i.e. $H' \in \mathcal{H}_n$. $\square$

Let $H = (H_1, \dots, H_n)^{\mathrm{T}}$ where T stands for the transposition. By Lemma 2.5,

$$\sigma(H) = A_\sigma H, \quad \sigma \in \mathbb{W}_n \tag{8}$$

where $A_\sigma = (a_{ij}) \in \mathrm{GL}_n(K)$ and $\sigma(H_i) = \sum_{j=1}^{n} a_{ij} H_j$. Let ${}^\sigma W_n$ be the W_n-module of W_n twisted by the automorphism $\sigma \in \mathbb{W}_n$. As a vector space, ${}^\sigma W_n = W_n$, but the adjoint action is twisted by σ:

$$w x^\alpha H'' = [\sigma(w), x^\alpha H'']$$

for all $w \in W_n$ and $\alpha \in \mathbb{Z}^n$. The map $\sigma\colon W_n \to {}^\sigma W_n$, $w \mapsto \sigma(w)$, is a W_n-module isomorphism. By Lemma 2.5, every weight subspace $x^\alpha \mathcal{H}_n$ of the $\mathcal{H}_n$-module $W_n = \bigoplus_{\alpha \in \mathbb{Z}^n} x^\alpha \mathcal{H}_n$ is also a weight subspace for the $\mathcal{H}_n$-module ${}^\sigma W_n$, and vice versa. Moreover,

$$W_{n,\alpha} = x^\alpha \mathcal{H}_n = ({}^\sigma W_n)_{A_\sigma \alpha}, \quad \alpha \in \mathbb{Z}^n \tag{9}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)^{\mathrm{T}} \in \mathbb{Z}^n$ is a column: for all $H' = \sum_{i=1}^{n} l_i H_i \in \mathcal{H}_n$,

$$[\sigma(H'), x^\alpha H''] = \sum_{i,j=1}^{n} l_i a_{ij} \alpha_j x^\alpha H'' = (H', A_\sigma \alpha) x^\alpha H''. \tag{10}$$

Since $\sigma(\mathcal{H}_n) = \mathcal{H}_n$ and $\sigma\colon W_n \to {}^\sigma W_n$ is a W_n-module isomorphism, the automorphism σ permutes the weight components $\{W_{n,\alpha} = x^\alpha \mathcal{H}_n\}_{\alpha \in \mathbb{Z}^n}$. There is a bijection $\sigma'\colon \mathbb{Z}^n \to \mathbb{Z}^n$, $\alpha \mapsto \sigma'(\alpha)$, such that $\sigma(W_{n,\alpha}) = W_{n,\sigma'(\alpha)}$ for all $\alpha \in \mathbb{Z}^n$.

Lemma 2.6. *For all $\sigma \in \mathbb{W}_n$ and $\alpha \in \mathbb{Z}^n$, $\sigma'(\alpha) = A_{\sigma^{-1}} \alpha$.*

Proof. By (10),

$$\begin{aligned}(H', \sigma'(\alpha)) \sigma(x^\alpha H'') &= [H', \sigma(x^\alpha H'')] = \sigma([\sigma^{-1}(H'), x^\alpha H'']) = \sigma((H', A_{\sigma^{-1}} \alpha) x^\alpha H'') \\ &= (H'', A_{\sigma^{-1}} \alpha) \sigma(x^\alpha H'').\end{aligned}$$

Therefore, $\sigma'(\alpha) = A_{\sigma^{-1}} \alpha$. $\square$

Corollary 2.7. *For all $\sigma \in \mathbb{W}_n$, $A_\sigma \in \mathrm{GL}_n(\mathbb{Z})$.*

Proof. This follows from Lemma 2.6. $\square$

The group of automorphisms $\mathbb{L}_n = \mathrm{Aut}_{\mathrm{Lie}}(L_n)$**.** The group $\mathbb{L}_n$ contains two obvious subgroups: the *algebraic n-dimensional torus* $\mathbb{T}^n = \{t_l\colon\, l \in K^{*n}\} \simeq K^{*n}$ where $t_l(x_i) = l_i x_i$ for $i = 1, \dots, n$, and $\mathrm{GL}_n(\mathbb{Z})$ which can be seen as a subgroup of $\mathbb{L}_n$ via the group monomorphism

$$\mathrm{GL}_n(\mathbb{Z}) \to \mathbb{L}_n, \quad A \mapsto \sigma_A\colon\, x_i \mapsto \prod_{j=1}^{n} x_j^{a_{ji}}. \tag{11}$$

For all $\alpha \in \mathbb{Z}^n$, $\sigma_A(x^\alpha) = x^{A\alpha}$. Hence $\sigma_{AB} = \sigma_A \sigma_B$ and $\sigma_A^{-1} = \sigma_{A^{-1}}$.

Lemma 2.8. $\mathbb{L}_n = \mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{T}^n$.

Proof. The group of units L_n^* of the algebra L_n is equal to the direct product of its two subgroups $K^* \times \mathbb{X}$ where $\mathbb{X} = \{x^\alpha\colon\, \alpha \in \mathbb{Z}^n\} \simeq \mathbb{Z}^n$ via $x^\alpha \mapsto \alpha$. Since $\sigma(K^*) = K^*$ for all $\sigma \in \mathbb{L}_n$, there is a group homomorphism (where $\mathrm{Aut}_{\mathrm{gr}}(G)$ is the group of automorphisms of a *group* G)

$$\theta\colon\, \mathbb{L}_n \to \mathrm{Aut}_{\mathrm{gr}}(L_n/K^*), \quad \sigma \mapsto \overline{\sigma}\colon\, K^* x^\alpha \mapsto K^* \sigma(x^\alpha).$$

Notice that $\mathrm{Aut}_{\mathrm{gr}}(L_n/K^*) \simeq \mathrm{Aut}_{\mathrm{gr}}(\mathbb{Z}^n) \simeq \mathrm{GL}_n(\mathbb{Z})$ and $\theta|_{\mathrm{GL}_n(\mathbb{Z})}\colon\, \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{Aut}_{\mathrm{gr}}(L_n/K^*)$, $A \mapsto A$. Then $\mathbb{L}_n \simeq \mathrm{GL}_n(\mathbb{Z}) \ltimes \mathrm{Ker}(\theta)$ but $\mathrm{Ker}(\theta) = \mathbb{T}^n$. Clearly, $\mathbb{L}_n = \mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{T}^n$. $\square$

Lemma 2.9. *Let $\sigma_A \in \mathbb{L}_n$ be as in* (11) *where $A \in \mathrm{GL}_n(\mathbb{Z})$, $\partial = (\partial_1, \dots, \partial_n)^{\mathrm{T}}$, $H = (H_1, \dots, H_n)^{\mathrm{T}}$ and let $\mathrm{diag}(l_{11}, \dots, l_{nn})$ be the diagonal matrix with the diagonal elements $l_{11}, \dots, l_{nn}$. Then*

1. *$\sigma(\partial) = C_\sigma \partial$ where $C_\sigma = \mathrm{diag}(\sigma(x_1)^{-1}, \dots, \sigma(x_n)^{-1}) A^{-1} \mathrm{diag}(x_1, \dots, x_n)$,*
2. *$\sigma(H) = A^{-1} H$.*

Proof. 1. Let $\partial_i' = \sigma(\partial_i)$ and $x_j' = \sigma(x_j)$. Clearly, $\sigma(\partial) = C_\sigma \partial$ for some matrix $C_\sigma = (c_{ij}) \in M_n(L_n)$. Applying the automorphism σ to the equalities $\delta_{ij} = \partial_i * x_j$ where $i, j = 1, \dots, n$, we obtain the equalities

$$\delta_{ij} = \sigma \partial_i \sigma^{-1} \sigma(x_j) = \partial_i' * x_j' = \left(\sum_{k=1}^{n} c_{ik} \partial_k \right) * \prod_{l=1}^{n} x_l^{a_{lj}} = \left(\sum_{k,l=1}^{n} c_{ik} x_k^{-1} a_{kj} \right) x_j'$$

where $i, j = 1, \dots, n$. Equivalently, $C_\sigma \,\mathrm{diag}(x_1^{-1}, \dots, x_n^{-1}) A = \mathrm{diag}(x_1'^{-1}, \dots, x_n'^{-1})$, and statement 1 follows.

2. Statement 2 follows from statement 1:

$$\begin{aligned}\sigma(H) &= \sigma(\mathrm{diag}(x_1,\dots,x_n)\partial) = \sigma(\mathrm{diag}(x_1,\dots,x_n))\sigma(\partial)\\ &= \mathrm{diag}(\sigma(x_1),\dots,\sigma(x_n))C_\sigma\partial\\ &= \mathrm{diag}(\sigma(x_1),\dots,\sigma(x_n))(\mathrm{diag}(\sigma(x_1)^{-1},\dots,\sigma(x_n)^{-1})A^{-1}\,\mathrm{diag}(x_1,\dots,x_n)\\ &= A^{-1}H.\end{aligned}$$

□

Let a group G act on a set S and $T \subseteq S$. Then $\mathrm{Fix}_G(T) := \{g \in G\colon gt = t$ for all $t \in T\}$ is the *fixator* of the the set T. $\mathrm{Fix}_G(T)$ is a subgroup of G.

Lemma 2.10. *Let $\sigma \in \mathbb{W}_n$. Then $\sigma(H) = A_{\sigma^{-1}}H$ for some $A_{\sigma^{-1}} \in \mathrm{GL}_n(\mathbb{Z})$* (see (8) and Lemma 2.7) *and $\sigma_{A_{\sigma^{-1}}}\sigma \in \mathrm{Fix}_{\mathbb{W}_n}(H_1,\dots,H_n)$ where $\sigma_{A_{\sigma^{-1}}} \in \mathrm{GL}_n(\mathbb{Z}) \subseteq \mathbb{L}_n$, see* (11).

P r o o f. The statement follows from Lemma 2.9 (2). □

$\mathrm{Sh}_n := \{s_l \in G_n\colon s_l(x_1) = x_1 + l_1,\dots,s_l(x_n) = x_n + l_n\}$ is the *shift group* of automorphisms of the polynomial algebra P_n where $l = (l_1,\dots,l_n) \in K^n$; $\mathrm{Sh}_n \subset \mathrm{Aut}_{\text{K-alg}}(P_n) \subseteq \mathrm{Aut}_{\mathrm{Lie}}(D_n)$.

Proposition 2.11. $\mathrm{Fix}_{\mathbb{W}_n}(\partial_1,\dots,\partial_n) = \{e\}$.

P r o o f. Let $\sigma \in F := \mathrm{Fix}_{\mathbb{W}_n}(\partial_1,\dots,\partial_n)$. We have to show that $\sigma = e$. Let $N := \mathrm{Nil}_{W_n}(\partial_1,\dots,\partial_n) := \{w \in W_n\colon \delta_i^s(w) = 0$ for some $s = s(w)$ and all $i = 1,\dots,n\}$. Clearly, $N = D_n$. The automorphisms σ and σ^{-1} preserve the space $N = D_n$, that is $\sigma^{\pm1}(D_n) \subseteq D_n$. Hence $\sigma(D_n) = D_n$ and $\sigma|_{D_n} \in \mathrm{Fix}_{G_n}(\partial_1,\dots,\partial_n) = \mathrm{Sh}_n$, [4]. The only element s_l of Sh_n that can be extended to an automorphism of W_n is e (since $s_l(x_i^{-1}\partial_i) = (x_i + l_1)^{-1}\partial_i$). Therefore, $\sigma = e$. In more detail, suppose that s_l can be extended to an automorphism of the Witt algebra W_n and $l_i \neq 0$; we seek a contradiction. Applying s_l to the relation $[x_i^{-1}\partial_i, x_i^2\partial_i] = 3\partial_i$ we obtain the relation $[s_l(x_i^{-1}\partial_i), (x_i+l_i)^2\partial_i] = 3\partial_i$. On the other hand, $[(x_i+l_i)^{-1}\partial_i, (x_i+l_i)^2\partial_i] = 3\partial_i$ in the Lie algebra $K(x_i)\partial_i$. Hence, $s_l(x_i^{-1}\partial_i) - (x_i+l_i)^{-1}\partial_i \in C := C_{K(x_i)\partial_i}((x_i+l_i)^2\partial_i)$. Since $C = K(x+l_i)^2\partial_i$, we see that $s_l(x_i^{-1}\partial_i) \notin W_n$, a contradiction. In more detail, let $\alpha = (x_i+l_i)^2$. Then $\beta\partial_i \in C$ where $\beta \in K(x_i)$ if and only if (where $\alpha' := \mathrm{d}\alpha/\mathrm{d}x_i$, etc.) $0 = [\alpha\partial_i, \beta\partial_i] = (\alpha\beta' - \alpha'\beta)\partial_i = \alpha^2(\beta/\alpha)'\partial_i$ if and only if $(\beta/\alpha)' = 0$ if and only if $\beta/\alpha \in \mathrm{Ker}_{K(x_i)}(\partial_i) = K$. Hence, $\beta \in K\alpha$, as required. □

Lemma 2.12. $\mathrm{Fix}_{\mathbb{W}_n}(H_1,\dots,H_n) = \mathbb{T}^n$.

P r o o f. The inclusion $\mathbb{T}^n \subseteq F := \mathrm{Fix}_{\mathbb{W}_n}(H_1, \dots, H_n)$ is obvious. Let $\sigma \in F$. We have to show that $\sigma \in \mathbb{T}^n$. In view of Proposition 2.11, it suffices to show that $\sigma(\partial_1) = l_1\partial_1, \dots, \sigma(\partial_n) = l_n\partial_n$ for some $l = (l_1, \dots, l_n) \in K^{*n}$ since then $t_l\sigma \in \mathrm{Fix}_{\mathbb{W}_n}(\partial_1, \dots, \partial_n) = \{e\}$ (Proposition 2.11), and so $\sigma = t_l^{-1} \in \mathbb{T}^n$. Since $\sigma \in F$, the automorphism respects the weight components of the Lie algebra W_n, that is $\sigma(x^\alpha \mathcal{H}_n) = x^\alpha \mathcal{H}_n$ for all $\alpha \in \mathbb{Z}^n$. In particular, for $i = 1, \dots, n$,

$$\partial'_i = \sigma(\partial_i) = \sigma(x_i^{-1} H_i) = x_i^{-1} \sum_{j=1}^{n} l_{ij} H_j = -x_i^{-1} \sum_{j=1}^{n} l_{ij} x_j \partial_j, \tag{12}$$

$\partial' = D^{-1}\Lambda D\partial$ where $D = \mathrm{diag}(x_1, \dots, x_n)$ and $D^{-1}\Lambda D \in \mathrm{GL}(L_n)$, and so $\Lambda = (l_{ij}) \in \mathrm{GL}_n(K)$. In view of (12), we have to show that Λ is a diagonal matrix. The elements $\partial_1, \dots, \partial_n$ commute, so do $\partial'_1, \dots, \partial'_n$: for all $i, j = 1, \dots, n$,

$$0 = [\partial'_i, \partial'_j] = \left[x_i^{-1} \sum_{k=1}^{n} l_{ik} H_k, x_j^{-1} \sum_{l=1}^{n} l_{jl} H_l \right].$$

Therefore, for all $i, j, l = 1, \dots, n$, we have $l_{ij} l_{jl} = l_{ji} l_{il}$. For each $i = 1, \dots, n$, let $c_i := \sum\limits_{j=1}^{n} l_{ji}$. The above equalities yield the equalities

$$\sum_{j=1}^{n} l_{ij} l_{jl} = c_i l_{il} \quad \text{for } i, l = 1, \dots, n.$$

Equivalently, $\Lambda^2 = \mathrm{diag}(c_1, \dots, c_n)\Lambda$. Therefore, $\Lambda = \mathrm{diag}(c_1, \dots, c_n)$ since $\Lambda \in \mathrm{GL}_n(K)$, as required. $\square$

P r o o f of Theorem 1.1. Let $\sigma \in \mathbb{W}_n$. We have to show that $\sigma \in \mathbb{L}_n$. By Lemma 2.10 and Lemma 2.12, $\tau\sigma \in \mathrm{Fix}_{\mathbb{W}_n}(H_1, \dots, H_n) = \mathbb{T}^n$ for some $\tau \in \mathbb{L}_n$, hence $\sigma \in \mathbb{L}_n$. $\square$

3. The group of automorphisms of the Virasoro algebra

The aim of this section is to find the group of automorphisms of the Virasoro algebra (Theorem 1.3). The key idea is to use Theorem 1.3.

P r o o f of Theorem 1.3. Let a K-linear map $s\colon W \to \mathcal{G}$ be a section to the surjection $\pi\colon \mathcal{G} \to W$, $a \mapsto a + Z$, i.e. $\pi s = \mathrm{id}_W$. The map σ is an injection and we identify the vector space W with its image in $\mathcal{G}$ via s. Then $\mathcal{G} = W \oplus Z$, a direct sum of vector spaces.

(i) $\widehat{\sigma}$ is unique: Suppose we have another extension, say $\widehat{\sigma}_1$. Then $\tau := \widehat{\sigma}_1^{-1}\widehat{\sigma} \in G := \mathrm{Aut}_{\mathrm{Lie}}(\mathcal{G})$ and

$$\varphi(w) := \tau(w) - w \in Z, \quad w \in W,$$

where $\varphi \in \mathrm{Hom}_K(W, Z)$. By condition 2, the inclusion $Z \subseteq [\mathcal{G}, \mathcal{G}] = [W+Z, W+Z] = [W, W]$ implies that $\tau(z) = z$ for all $z \in Z$. For all $w_1, w_2 \in W$,

$$[w_1, w_2] = [w_1, w_2]_W + z(w_1, w_2) \tag{13}$$

where $[\cdot,\cdot]$ and $[\cdot,\cdot]_W$ are the Lie brackets in $\mathcal{G}$ and W, respectively, and $z(w_1, w_2) \in Z$. Moreover, $[w_1, w_2]_W$ means $s([w_1, w_2]_W)$. Applying the automorphism τ to the above equality we have

$$\begin{aligned}[w_1, w_2] &= [\tau(w_1), \tau(w_2)] = \tau([w_1, w_2]) = \tau([w_1, w_2]_W + z(w_1, w_2)) \\ &= [w_1, w_2]_W + \varphi([w_1, w_2]_W) + z(w_1, w_2) = [w_1, w_2] + \varphi([w_1, w_2]_W).\end{aligned}$$

Hence, $\varphi([w_1, w_2]_W) = 0$ for all $w_1, w_2 \in W$. By condition 3, $\varphi = 0$, that is $\tau(w) = w$ for all $w \in W$. Together with the condition $\tau(z) = z$ for all $z \in Z$, this gives $\tau = e$. So, $\widehat{\sigma} = \widehat{\sigma}_1$.

(ii) The map $\sigma \mapsto \widehat{\sigma}$ is a monomorphism: Let $\widehat{\sigma}$ and $\widehat{\tau}$ be the extensions of σ and τ, respectively. By the uniqueness, $\widehat{\sigma}\widehat{\tau}$ is the extension of $\sigma\tau$, that is $\widehat{\sigma\tau} = \widehat{\sigma}\widehat{\tau}$, and so the map $\sigma \mapsto \widehat{\sigma}$ is a homomorphism. Again, by the uniqueness, $\sigma \mapsto \widehat{\sigma}$ is a monomorphism.

(iii) The map $\sigma \mapsto \widehat{\sigma}$ is an isomorphism: By condition 1, the map $\sigma \mapsto \widehat{\sigma}$ is a surjection, hence an isomorphism, by (ii). $\square$

P r o o f of Theorem 1.2. The conditions of Theorem 1.3 are satisfied for the Virasoro algebra: $Z = Z(\mathrm{Vir}) = Kc$, $\mathrm{Vir}/Z \simeq W_1$, $[W_1, W_1] = W_1$ (since W_1 is a simple Lie algebra), $Z \subseteq [\mathrm{Vir}, \mathrm{Vir}]$ and each automorphism $\sigma \in \mathbb{W}_1 = \mathrm{Aut}_{\mathrm{Lie}}(W_1) = \mathrm{Aut}_{\text{K-alg}}(L_1) = \mathrm{GL}_1(\mathbb{Z}) \ltimes \mathbb{T}^1$ is extended to an automorphism $\widehat{\sigma} \in \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Vir})$ by the rule $\widehat{\sigma}(c) = c$. The last condition is obvious for $\sigma \in \mathbb{T}^1$ but for $e \neq \sigma \in \mathrm{GL}_1(\mathbb{Z}) = \{\pm 1\}$, i.e. $\sigma\colon L_1 \to L_1$, $x \mapsto x^{-1}$, i.e. $\sigma\colon W_1 \to W_1$, $x^i H \mapsto -x^{-1} H$ for all $i \in \mathbb{Z}$, it follows from the relations (1). $\square$

Corollary 3.1.

1. *Each automorphism σ of the Witt algebra W_1 is uniquely extended to an automorphism $\widehat{\sigma}$ of the Virasoro algebra* Vir. *Moreover, $\widehat{\sigma}(c) = c$.*
2. *All automorphisms of the Virasoro algebra* Vir *act trivially on its centre.*

When we drop condition 3 of Theorem 1.3, we obtain a more general result.

Corollary 3.2. *Let $\mathcal{G}$ be a Lie algebra, Z a subspace of the centre of $\mathcal{G}$ and $W = \mathcal{G}/Z$. Fix a K-linear map $s\colon W \to \mathcal{G}$ which is a section to the surjection $\pi\colon \mathcal{G} \to W$, $a \mapsto a + Z$, and identify W with* $\mathrm{im}(s)$*, and so $\mathcal{G} = W \oplus Z$ (a direct sum of vector spaces). Let $\mathcal{K} := \{\tau = \tau_\varphi \in \mathrm{End}_K(\mathcal{G})\colon \tau(w) = w + \varphi(w)$ and $\tau(z) = z$ for all $w \in W$ and $z \in Z$, and $\varphi \in \mathrm{Hom}_K(W, Z)$ is such that $\varphi([W, W]) = 0\}$. Suppose that*

1. *every automorphism σ of the Lie algebra W can be extended to an automorphism $\widehat{\sigma}$ of the Lie algebra $\mathcal{G}$, and*
2. *$Z \subseteq [G, G]$.*

Then the short exact sequence of groups

$$1 \to \mathcal{K} \to \mathrm{Aut}_{\mathrm{Lie}}(\mathcal{G}) \xrightarrow{\psi} \mathrm{Aut}_{\mathrm{Lie}}(W) \to 1$$

is exact where $\psi(\sigma)\colon\ a + Z \mapsto \sigma(a) + Z$ for all $a \in \mathcal{G}$.

P r o o f. By condition 1, ψ is a group epimorphism. It remains to show that $\mathrm{Ker}(\psi) = \mathcal{K}$. Let $\tau \in \mathrm{Ker}(\psi)$. Each element $g \in \mathcal{G} = W \oplus Z$ is a unique sum $g = w + z$ where $w \in W$ and $z \in Z$. Then $\tau(w) = w + \varphi(w)$ for some $\varphi \in \mathrm{Hom}_K(W, Z)$. We keep the notation of the proof of Theorem 1.3. By condition 2, $Z \subseteq [\mathcal{G}, \mathcal{G}] = [W, W]$, hence $\tau(z) = z$ for all elements $z \in Z$. Applying the automorphism τ to the equality (13) yields $\varphi([w_1, w_2]) = 0$ (see the proof of Theorem 1.3). It follows that $\mathrm{Ker}(\psi) = \mathcal{K}$. $\square$

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Author's address: Vladimir V. Bavula, Department of Pure Mathematics, University of Sheffield, Hounsfield Rd, Hicks Building, Sheffield, S3 7RH, United Kingdom, e-mail: `v.bavula@sheffield.ac.uk`.