Lingli Zeng; Jizhu Nan

Vector invariant ideals of abelian group algebras under the actions of the unitary groups and orthogonal groups

Czechoslovak Mathematical Journal, Vol. 66 (2016), No. 4, 1059–1078

Persistent URL: http://dml.cz/dmlcz/145919

Czechoslovak Mathematical Journal, 66 (141) (2016), 1059–1078

VECTOR INVARIANT IDEALS OF ABELIAN GROUP ALGEBRAS UNDER THE ACTIONS OF THE UNITARY GROUPS AND ORTHOGONAL GROUPS

Lingli Zeng, Xi'an, Jizhu Nan, Dalian

(Received March 6, 2015)

Abstract. Let F be a finite field of characteristic p and K a field which contains a primitive pth root of unity and $\operatorname{char} K \neq p$. Suppose that a classical group G acts on the F-vector space V. Then it can induce the actions on the vector space $V \oplus V$ and on the group algebra $K[V \oplus V]$, respectively. In this paper we determine the structure of G-invariant ideals of the group algebra $K[V \oplus V]$, and establish the relationship between the invariant ideals of $K[V]$ and the vector invariant ideals of $K[V \oplus V]$, if G is a unitary group or orthogonal group. Combining the results obtained by Nan and Zeng (2013), we solve the problem of vector invariant ideals for all classical groups over finite fields.

Keywords: vector invariant ideal; group algebra; unitary group; orthogonal group

MSC 2010: 16S34, 20G40

1. Introduction

Let V be an abelian group, G a group of automorphisms of V, and K a field. Then G can act on the group algebra $K[V]$. It is an interesting problem to determine all G-invariant ideals of $K[V]$. The motivation for this actually arises from the study of the lattice of ideals in group algebras of certain infinite locally finite groups. A natural special case of the problem occurs if V is a finite-dimensional vector space over a field F with $\operatorname{char} F \neq \operatorname{char} K$. In this case Brookes and Evans in [1] proved that if F is infinite then the $GL_n(F)$-invariant ideals of $K[V]$ are only 0, $K[V]$ and the augmentation ideal $\omega(V; V)$, while if F is finite then there is an

The research has been supported by the Scientific Research Program Funded by Shaanxi Provincial Education Department (Program No. 16JK1789), and the National Natural Science Foundation of China (Program No. 11371343).

additional $GL_n(F)$-invariant ideal generated by $\sum_{a\in V} a$. In [3], [8], the authors described the structure of G-invariant ideals by the augmentation ideals where G is the multiplicative group of a field or a division ring. They showed that the set of these G-invariant ideals is Noetherian. Passman in [6], [7] investigated the F^*-invariant ideals of the group algebra $K[V]$ under the torus action of F^* and proved that every F^*-invariant ideal of $K[V]$ is uniquely a finite irredundant intersection of augmentation ideals. In particular, $K[V]$ has exactly four F^*-invariant proper ideals of $K[V]$ if V is the direct sum of two 1-dimensional vector spaces. The articles [4], [5] were concerned with the simple linear groups and the finitary version of the classical linear groups. Also, these groups act on only a single vector space V and the G-invariant ideals are considered on the group algebra $K[V]$. Richman in [9], [11], [10] studied the ring of vector invariants $K[mV]^G$ with the diagonal action of G on the coordinate ring of $\bigoplus\limits^{m} V$, and determined the relationship between the ring of vector invariants $K[\bigoplus\limits^{m} V]^G$ and the ring of invariants $K[V]^G$. Thus, it is natural to ask what is the structure of the G-vector invariant ideals of the group algebra $K[\bigoplus\limits^{m} V]$, and what is the relationship between the vector invariant ideals of $K[\bigoplus\limits^{m} V]$ and the invariant ideals of $K[V]$.

Nan and Zeng in [2] studied the vector invariant ideals of the group algebra $K[V \oplus V]$ under the action of the symplectic group. The present paper determines the structure of the vector invariant ideals under the actions of unitary groups (or orthogonal groups) and establishes the relationship between the invariant ideals and the vector invariant ideals, which can be viewed as a continuation of the work [2]. Combining the results obtained in [2], we solve the problem of vector invariant ideals for all classical groups over finite fields.

Our paper is arranged as follows. Section 2 investigates the structure of the invariant ideals of the group algebra $K[V]$ and of the vector invariant ideals of the group algebra $K[V \oplus V]$ under the actions of unitary groups. In Section 3, we establish the relationship between the invariant ideals of $K[V]$ and the vector invariant ideals of $K[V \oplus V]$ under the actions of unitary groups using augmentation ideals. In Sections 4 and 5, we study the structure of the vector invariant ideals under the actions of orthogonal groups over finite fields of odd characteristic and characteristic 2, respectively.

2. Invariant ideals under the actions of unitary groups

In this section, we study the structure of G-invariant ideals under the actions of unitary groups. We first recall some relevant material about unitary groups in [12]. Throughout this paper, p will always be a prime number and q a power of p. Let us denote by F_{q^2} the finite field with q^2 elements. It is well known that F_{q^2} has an involutive automorphism $a \mapsto \overline{a} = a^q$, and the fixed field of this automorphism is F_q. Let $A = (a_{ik})_{1 \leqslant i \leqslant m, 1 \leqslant k \leqslant n}$ be an $m \times n$ matrix over F_{q^2}. Define the matrix $\overline{A} = (\overline{a}_{ik})_{1 \leqslant i \leqslant m, 1 \leqslant k \leqslant n}$ where $\overline{a}_{ik} = {a_{ik}}^q$ by the involutive automorphism. Two $n \times n$ matrices A and B over F_{q^2} are said to be cogredient, if there is an nonsingular matrix Q such that $QA\overline{Q}^{\mathrm{T}} = B$, where $\overline{Q}^{\mathrm{T}}$ denotes the transpose of $\overline{Q}$. It is easy to verify that cogredient matrices over F_{q^2} have the same rank. An $n \times n$ matrix H is called Hermitian if $\overline{H}^{\mathrm{T}} = H$.

If H is a nonsingular Hermitian matrix over F_{q^2}, then it is necessarily cogredient to the $n \times n$ identity matrix $I^{(n)}$, see [12], Theorem 5.2, Corollary 5.4, and so cogredient to

$$\begin{pmatrix} 0 & I^{(\nu)} \\ I^{(\nu)} & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & 1 \end{pmatrix}$$

where $n = 2\nu$ is even or $n = 2\nu + 1$ is odd, respectively. In this section, we suppose that $\nu \geqslant 2$.

Let H be an $n \times n$ nonsingular Hermitian matrix over F_{q^2}. A matrix A is called a unitary matrix with respect to H if $AH\overline{A}^{\mathrm{T}} = H$. The set of all such unitary matrices forms a group with respect to matrix multiplication, called the *unitary group* of degree n with respect to H and denoted by $U_n(F_{q^2}, H)$, i.e.,

$$U_n(F_{q^2}, H) = \{A \in GL_n(F_{q^2})\colon\ AH\overline{A'} = H\}.$$

Suppose that $U_n(F_{q^2}, H)$ acts on the F_{q^2}-vector space $F_{q^2}^{(n)}$ by

$$(x)^A = x{A^{-1}}^{\mathrm{T}}, \quad x \in F_{q^2}^{(n)},\ A \in U_n(F_{q^2}, H).$$

Then it can induce an action on the vector space $F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}$, defined by

$$(x, y)^A = (x{A^{-1}}^{\mathrm{T}}, y{A^{-1}}^{\mathrm{T}}), \quad x, y \in F_{q^2}^{(n)},\ A \in U_n(F_{q^2}, H).$$

For convenience, we define

$$G = \left\{(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\colon\ A \in U_n(F_{q^2}, H)\right\}.$$

The action of G on the vector space $F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}$ coincides with the action of $U_n(F_{q^2}, H)$ on $F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}$, i.e.,

$$(x, y)^{(A)} = (x{A^{-1}}^{\mathrm{T}}, y{A^{-1}}^{\mathrm{T}}), \quad x, y \in F_{q^2}^{(n)},\ (A) \in G.$$

And so G can act on the group algebra $K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}]$ by

$$\left(\sum_{i,j} a_{ij}(x_i, y_j)\right)^{(A)} = \sum_{i,j} a_{ij}(x_i, y_j)^{(A)},$$

where K is a field of characteristic different from p and K contains ε, a primitive pth root of unity.

To investigate the structure of invariant ideals, the following results given in [6] are necessary. Since F_{q^2} is a field with $\operatorname{char} F_{q^2} = p > 0$, it follows that any finite-dimensional F_{q^2}-vector space V is additively an elementary abelian p-group.

Lemma 2.1 ([6], Lemma 3.1)**.** *Let V be a finite elementary abelian p-group and let G be a group of automorphisms of V.*

(1) *The group algebra $K[V]$ is semisimple. Indeed, it is a direct sum of $|V|$ copies of K and every ideal is uniquely an intersection of maximal ideals.*
(2) *The maximal ideals of $K[V]$ are in a one-to-one correspondence with the linear characters $\chi\colon V \to K^*$. To be precise, the ideal corresponding to χ is the kernel of the natural algebra extension $\chi\colon K[V] \to K$.*
(3) *G permutes the linear characters of V by defining $\chi^g(x) = \chi(x^{g^{-1}})$ for all $g \in G$ and $x \in V$. This action corresponds to the permutation action of G on the maximal ideals of $K[V]$.*
(4) *Every G-invariant ideal of $K[V]$ is uniquely an intersection of maximal G-invariant ideals of $K[V]$. The latter are precisely the intersections of G-orbits of maximal ideals of $K[V]$.*
(5) *$\chi^g = \chi$ if and only if $x^{-1}x^{g^{-1}} \in \operatorname{Ker}\chi$ for all $x \in V$.*

Lemma 2.2 ([6], Lemma 2.4)**.** *Let $E \supseteq F$ be fields with $|E : F| < \infty$ and let $\lambda\colon E \to F$ be a nonzero F-linear function. Then every F-linear function from E to F is uniquely of the form λ_a for $a \in E$, where $\lambda_a(x) = \lambda(ax)$.*

We now briefly recall some facts and notation concerning linear characters and kernel ideals in [6]. Let $GF(p)$ denote the prime subfield of F_{q^2} and let $\mu\colon F_{q^2} \to GF(p)$ be a nonzero $GF(p)$-linear function. Then, by Lemma 2.2, all linear functions from F_{q^2} to $GF(p)$ are of the form $\mu_a\colon F_{q^2} \to GF(p)$, where $a \in F_{q^2}$ and $\mu_a(x) = \mu(ax)$. Hence all characters $\chi\colon F_{q^2} \to K^*$ are given by

$$\chi_a(x) = \varepsilon^{\mu_a(x)} = \varepsilon^{\mu(ax)}, \quad a, x \in F_{q^2}$$

where ε is a primitive pth root of unity in K. Furthermore, the characters from $F_{q^2}^{(n)}$ to K^* are necessarily products $\chi_1\chi_2\dots\chi_n$ where each χ_i is a character from F_{q^2}

to K^*, so they are all of the form

$$\chi_\alpha(x) = \prod_{i=1}^{n} \chi_{a_i}(x_i) = \varepsilon^{\sum_{i=1}^{n} \mu(a_i x_i)} = \varepsilon^{\mu(\sum_{i=1}^{n} a_i x_i)} = \varepsilon^{\mu(\alpha x^{\mathrm{T}})}, \quad \alpha, x \in F_{q^2}^{(n)}$$

where $\alpha x^{\mathrm{T}} = \sum_{i=1}^{n} a_i x_i$ denotes the usual dot product of two vectors $\alpha = (a_1, \ldots, a_n)$ and $x = (x_1, \ldots, x_n)$. These characters in turn extend naturally to the K-algebra homomorphisms $\chi_\alpha\colon K[F_{q^2}^{(n)}] \to K$ and their kernels $I_\alpha = \operatorname{Ker} \chi_\alpha$ are precisely the set of maximal ideals of the group algebra $K[F_{q^2}^{(n)}]$.

Note that the following result from [12] is very important for us to study the G-orbits.

Lemma 2.3 ([12], Lemma 5.12). *Let P_1 and P_2 be two $m \times 2\nu$ matrices of rank m. Then there exists an element $T \in U_n(F_{q^2}, H)$ such that $P_1 = P_2 T$ if and only if $P_1 H \overline{P_1}^{\mathrm{T}} = P_2 H \overline{P_2}^{\mathrm{T}}$.*

Lemma 2.4. *Let F_{q^2} be a field with q^2 elements, where q is a power of a prime p. Suppose that $U_n(F_{q^2}, H)$ acts on the F_{q^2}-vector space $F_{q^2}^{(n)}$. Then for any $0 \neq \alpha$, $\beta \in F_{q^2}^{(n)}$, the maximal ideals I_α and I_β of $K[F_{q^2}^{(n)}]$ are in the same orbit if and only if the unitary dot products are equal, i.e. $\alpha H \overline{\alpha}^{\mathrm{T}} = \beta H \overline{\beta}^{\mathrm{T}} = t$, $t \in F_q$.*

Proof. For any $A \in U_n(F_{q^2}, H)$ we have

$$\chi_\alpha^A(x) = \chi_\alpha(x^{A^{-1}}) = \chi_\alpha(x A^{\mathrm{T}}) = \varepsilon^{\mu(\alpha (x A^{\mathrm{T}})^{\mathrm{T}})} = \varepsilon^{\mu(\alpha A x^{\mathrm{T}})} = \chi_{\alpha A}(x).$$

Hence

$$I_\alpha^A = I_{\alpha A}. \tag{2.1}$$

Suppose now that the maximal ideals I_α and I_β are in the same orbit. Then there exists an $A \in U_n(F_{q^2}, H)$ such that $\beta = \alpha A$ due to the equality (2.1). Further, we have

$$\beta H \overline{\beta}^{\mathrm{T}} = (\alpha A) H \overline{(\alpha A)}^{\mathrm{T}} = \alpha (A H \overline{A}^{\mathrm{T}}) \overline{\alpha}^{\mathrm{T}} = \alpha H \overline{\alpha}^{\mathrm{T}}.$$

Let $t = \alpha H \overline{\alpha}^{\mathrm{T}}$. Then $\overline{t}^{\mathrm{T}} = \overline{\alpha H \overline{\alpha}^{\mathrm{T}}}^{\mathrm{T}} = \alpha \overline{H}^{\mathrm{T}} \overline{\alpha}^{\mathrm{T}} = \alpha H \overline{\alpha}^{\mathrm{T}} = t$. Since $\overline{t} \in F_{q^2}$, it follows that $\overline{t}^{\mathrm{T}} = \overline{t}$ and so $\overline{t} = t$. Hence $t \in F_q$.

Conversely, if $\alpha H \overline{\alpha}^{\mathrm{T}} = \beta H \overline{\beta}^{\mathrm{T}}$, then by Lemma 2.3 there exists an $A \in U_n(F_{q^2}, H)$ such that $\beta = \alpha A$. This implies that $I_\alpha^A = I_{\alpha A} = I_\beta$. Thus I_α and I_β are in the same orbit. $\square$

Let

$$M(m,r) = \begin{pmatrix} I^{(r)} & \\ & 0^{(m-r)} \end{pmatrix}.$$

We denote by $\mathfrak{n}(m,r;n)$ the number of $m \times n$ matrices P of rank m which satisfy

$$PH\overline{P}^{\mathrm{T}} = M(m,r).$$

Lemma 2.5 (Anzahl theorem, [12], Lemma 5.18)**.** *Let $2r \leqslant 2m \leqslant n+r$. Then*

$$\mathfrak{n}(m,r;n) = q^{rn+(m-r)(m-r-1)-r(r+1)/2} \prod_{i=n+r-2m+1}^{n} (q^i - (-1)^i).$$

Proposition 2.6. *Let F_{q^2} be a field with q^2 elements, where q is a power of a prime p. Suppose that $U_n(F_{q^2},H)$ acts on the F_{q^2}-vector space $F_{q^2}^{(n)}$. Then there are exactly $(q+1)$ of $U_n(F_{q^2},H)$-orbits of maximal ideals of $K[F_{q^2}^{(n)}]$, i.e.,*

$$\Omega_{\overline{0}} = \{I_0\}, \quad \Omega_t = \{I_\alpha\colon\ 0 \neq \alpha \in F_{q^2}^{(n)},\ \alpha H \overline{\alpha}^{\mathrm{T}} = t\}, \quad t \in F_q.$$

P r o o f. First, it is easy to see that $\Omega_{\overline{0}}$ and $\Omega_t, t \in F_q$ are distinct $U_n(F_{q^2},H)$ -orbits of maximal ideals by Lemma 2.4.

We next show that every orbit Ω_t is nonempty. By Lemma 2.5, if $t \neq 0$, then the number of α is $\mathfrak{n}(1,1;n) = q^{n-1}(q^n - (-1)^n)$. If $t = 0$, then the number of α is $\mathfrak{n}(1,0;n) = (q^{n-1} - (-1)^{n-1})(q^n - (-1)^n)$. Thus $\Omega_t \neq \emptyset$, and this implies that there are exactly $(q+1)$ of $U_n(F_{q^2},H)$ -orbits of maximal ideals of $K[F_{q^2}^{(n)}]$. $\square$

Remark 1. We define $I_{\overline{0}} \triangleq I_0$, $I_t \triangleq \bigcap\limits_{I_\alpha \in \Omega_t} I_\alpha$, $t \in F_q$ and $Q_1 \triangleq \{I_{\overline{0}}\} \cup \{I_t\colon\ t \in F_q\}$.

We shall now prove the first main result of this paper.

Theorem 2.7. *Let F_{q^2} be a field with q^2 elements, where q is a power of a prime p. Suppose that $U_n(F_{q^2},H)$ acts on the F_{q^2}-vector space $F_{q^2}^{(n)}$. Then every $U_n(F_{q^2},H)$-invariant ideal I of $K[F_{q^2}^{(n)}]$ is an intersection of finitely many ideals in the set Q_1 (the same notation as in Remark 1), i.e. $I = \bigcap\limits_{I' \in Q_1'} I'$ where Q_1' is a subset of Q_1.*

P r o o f. By Lemma 2.1 (4), we know that the maximal $U_n(F_{q^2},H)$-invariant ideals of $K[F_{q^2}^{(n)}]$ are precisely the intersections of $U_n(F_{q^2},H)$-orbits of maximal ideals. Then it follows by Proposition 2.6 that Q_1 is the set of all maximal $U_n(F_{q^2},H)$-invariant ideals of $K[F_{q^2}^{(n)}]$. Again by Lemma 2.1 (4), we deduce that each $U_n(F_{q^2},H)$-invariant ideal is the intersection of finitely many maximal $U_n(F_{q^2},H)$-invariant ideals in Q_1, i.e., that $I = \bigcap\limits_{I' \in Q_1'} I'$ where Q_1' $(\leqslant Q_1)$ is a subset of Q_1. $\square$

Next, we shall study the structure of G-vector invariant ideals of the group algebra $K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}]$. By a similar argument (on page 4) we can show that the characters from $F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}$ to K^* are all of the form

$$\chi_{(\alpha,\beta)}(x,y) = \varepsilon^{\mu(\alpha x^{\mathrm{T}} + \beta y^{\mathrm{T}})}, \quad \alpha, \beta, x, y \in F_{q^2}^{(n)}.$$

These characters can be extended to the K-algebra homomorphisms

$$\chi_{(\alpha,\beta)}\colon\ K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}] \to K$$

and their kernels $I_{(\alpha,\beta)} = \operatorname{Ker}\chi_{(\alpha,\beta)}$ are precisely the set of maximal ideals of $K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}]$.

Lemma 2.8. *Let F_{q^2} be a field with q^2 elements, where q is a power of a prime p. Suppose that*

$$G = \left\{ (A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} :\ A \in U_n(F_{q^2}, H) \right\}$$

acts on the vector space $F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}$. If two maximal ideals $I_{(\alpha,\beta)}$ and $I_{(\gamma,\delta)}$ of $K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}]$ are in the same orbit, then the unitary dot products are equal, i.e. that $\alpha H \overline{\beta}^{\mathrm{T}} = \gamma H \overline{\delta}^{\mathrm{T}}$.

Proof. For any $(A) \in G$ we have

$$\begin{aligned}
\chi_{(\alpha,\beta)}^{(A)}(x,y) &= \chi_{(\alpha,\beta)}((x,y)^{(A)^{-1}}) = \chi_{(\alpha,\beta)}(xA^{\mathrm{T}}, yA^{\mathrm{T}}) \\
&= \varepsilon^{\mu(\alpha(xA^{\mathrm{T}})^{\mathrm{T}} + \beta(yA^{\mathrm{T}})^{\mathrm{T}})} = \varepsilon^{\mu(\alpha A x^{\mathrm{T}} + \beta A y^{\mathrm{T}})} = \chi_{(\alpha A, \beta A)}(x,y).
\end{aligned}$$

Hence

$$I_{(\alpha,\beta)}^{(A)} = I_{(\alpha A, \beta A)}. \tag{2.2}$$

Since the unitary dot product of αA and βA is

$$\alpha A H \overline{(\beta A)}^{\mathrm{T}} = \alpha A H \overline{A}^{\mathrm{T}} \overline{\beta}^{\mathrm{T}} = \alpha H \overline{\beta}^{\mathrm{T}};$$

by combining the equality (2.2), the required result follows. □

Lemma 2.9. *Let F_{q^2} be a field with q^2 elements, where q is a power of a prime p. Let*

$$G = \left\{ (A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} :\ A \in U_n(F_{q^2}, H) \right\}$$

act on the vector space $F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}$. Suppose that $0 \neq \alpha, \beta, \gamma, \delta \in F_{q^2}^{(n)}$. If the vectors α and β are linearly dependent, i.e., $\beta = k\alpha$ for some $k \in F_{q^2}$, then the maximal ideals $I_{(\alpha,\beta)}$ and $I_{(\gamma,\delta)}$ of $K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}]$ are in the same orbit if and only if $\delta = k\gamma$ and $\alpha H \overline{\alpha}^{\mathrm{T}} = \gamma H \overline{\gamma}^{\mathrm{T}}$.

Proof. Suppose that the maximal ideals $I_{(\alpha,\beta)}$ and $I_{(\gamma,\delta)}$ are in the same orbit. By the equality (2.2), there exists an $(A) \in G$ such that $\gamma = \alpha A, \delta = \beta A$. Since $\beta = k\alpha$, we have

$$\delta = \beta A = k\alpha A = k\gamma \quad \text{and} \quad \gamma H \overline{\gamma}^{\mathrm{T}} = \alpha A H \overline{A}^{\mathrm{T}} \overline{\alpha}^{\mathrm{T}} = \alpha H \overline{\alpha}^{\mathrm{T}}.$$

Conversely, if $\alpha H \overline{\alpha}^{\mathrm{T}} = \gamma H \overline{\gamma}^{\mathrm{T}}$ then by Lemma 2.3 there exists an $A \in U_n(F_{q^2}, H)$ such that $\gamma = \alpha A$, and so $\delta = k\gamma = k\alpha A = \beta A$. It follows that

$$I_{(\alpha,\beta)}^{(A)} = I_{(\alpha A, \beta A)} = I_{(\gamma,\delta)}.$$

□

Lemma 2.10. *Let F_{q^2} be a field with q^2 elements, where q is a power of a prime p. Let*

$$G = \left\{ (A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} :\ A \in U_n(F_{q^2}, H) \right\}$$

act on the vector space $F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}$. Suppose that $0 \neq \alpha, \beta, \gamma, \delta \in F_{q^2}^{(n)}$. If the vectors α and β are linearly independent, then the maximal ideals $I_{(\alpha,\beta)}$ and $I_{(\gamma,\delta)}$ of $K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}]$ are in the same orbit if and only if γ and δ are linearly independent and $\alpha H \overline{\alpha}^{\mathrm{T}} = \gamma H \overline{\gamma}^{\mathrm{T}}$, $\beta H \overline{\beta}^{\mathrm{T}} = \delta H \overline{\delta}^{\mathrm{T}}$, $\alpha H \overline{\beta}^{\mathrm{T}} = \gamma H \overline{\delta}^{\mathrm{T}}$.

Proof. Suppose first that the maximal ideals $I_{(\alpha,\beta)}$ and $I_{(\gamma,\delta)}$ are in the same orbit. Then, by the equality (2.2), there exists an $A \in U_n(F_{q^2}, H)$ such that $\gamma = \alpha A$, $\delta = \beta A$. It is easy to verify that

$$\alpha H \overline{\alpha}^{\mathrm{T}} = \gamma H \overline{\gamma}^{\mathrm{T}}, \quad \beta H \overline{\beta}^{\mathrm{T}} = \delta H \overline{\delta}^{\mathrm{T}} \quad \text{and} \quad \alpha H \overline{\beta}^{\mathrm{T}} = \gamma H \overline{\delta}^{\mathrm{T}}.$$

If γ and δ are linearly dependent, then α and β are also linearly dependent by Lemma 2.9, a contradiction.

Conversely, suppose that γ and δ are linearly independent and $\alpha H\overline{\alpha}^{\mathrm{T}} = \gamma H\overline{\gamma}^{\mathrm{T}}$, $\beta H\overline{\beta}^{\mathrm{T}} = \delta H\overline{\delta}^{\mathrm{T}}$, $\alpha H\overline{\beta}^{\mathrm{T}} = \gamma H\overline{\delta}^{\mathrm{T}}$. Then $\binom{\alpha}{\beta}$ and $\binom{\gamma}{\delta}$ are two $2\times n$ matrices of rank 2. Since

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} H \overline{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}}^{\mathrm{T}} = \begin{pmatrix}\alpha H\overline{\alpha}^{\mathrm{T}} & \alpha H\overline{\beta}^{\mathrm{T}}\\ \beta H\overline{\alpha}^{\mathrm{T}} & \beta H\overline{\beta}^{\mathrm{T}}\end{pmatrix}$$

and

$$\begin{pmatrix}\gamma\\ \delta\end{pmatrix} H \overline{\begin{pmatrix}\gamma\\ \delta\end{pmatrix}}^{\mathrm{T}} = \begin{pmatrix}\gamma H\overline{\gamma}^{\mathrm{T}} & \gamma H\overline{\delta}^{\mathrm{T}}\\ \delta H\overline{\gamma}^{\mathrm{T}} & \delta H\overline{\delta}^{\mathrm{T}}\end{pmatrix},$$

we obtain that

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} H \overline{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}}^{\mathrm{T}} = \begin{pmatrix}\gamma\\ \delta\end{pmatrix} H \overline{\begin{pmatrix}\gamma\\ \delta\end{pmatrix}}^{\mathrm{T}}.$$

Hence, by Lemma 2.3, there exists an $A \in U_n(F_{q^2}, H)$ such that

$$\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} A$$

and so

$$I_{(\alpha,\beta)}^{(A)} = I_{(\alpha A,\beta A)} = I_{(\gamma,\delta)}.$$

$\square$

By the previous lemmas, we can describe a classification of the G-orbits of maximal ideals of $K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}]$.

Proposition 2.11. *Let F_{q^2} be a field with q^2 elements, where q is a power of a prime p. Suppose that*

$$G = \left\{ (A) = \begin{pmatrix} A & 0\\ 0 & A\end{pmatrix} :\ A \in U_n(F_{q^2}, H) \right\}$$

acts on the vector space $F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}$. Then there are exactly (q^4+q^3+q+1) G-orbits of maximal ideals of $K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}]$, i.e.,

$$\begin{aligned}
\Psi^0 &= \{I_{(0,0)}\};\\
\Psi_t^{0_r} &= \{I_{(\alpha,0)}\colon\ 0 \neq \alpha \in F_{q^2}^{(n)},\ \alpha H\overline{\alpha}^{\mathrm{T}} = t\}, \quad t \in F_q;\\
\Psi_t^{0_l} &= \{I_{(0,\beta)}\colon\ 0 \neq \beta \in F_{q^2}^{(n)},\ \beta H\overline{\beta}^{\mathrm{T}} = t\}, \quad t \in F_q;\\
\Psi_{t,k} &= \{I_{(\alpha,\beta)}\colon\ 0 \neq \alpha \in F_{q^2}^{(n)},\ \beta = k\alpha,\ \alpha H\overline{\alpha}^{\mathrm{T}} = t\}, \quad t \in F_q,\ k \in F_{q^2}^*;\\
\Psi_{t,t',t''} &= \{I_{(\alpha,\beta)}\colon\ 0 \neq \alpha \in F_{q^2}^{(n)},\ \beta \neq k\alpha,\ k \in F_{q^2},\ \alpha H\overline{\beta}^{\mathrm{T}} = t,\ \alpha H\overline{\alpha}^{\mathrm{T}} = t',\\
&\qquad \beta H\overline{\beta}^{\mathrm{T}} = t''\}, \quad t \in F_{q^2},\ t', t'' \in F_q.
\end{aligned}$$

P r o o f. For any $\alpha, \beta \in F_{q^2}^{(n)}$, we now distinguish four cases:

Case 1. $\alpha = \beta = 0$. It is obvious that Ψ^0 is a G-orbit.

Case 2. $\alpha \neq 0$, $\beta = 0$. $\Psi_t^{0_r}$ is a G-orbit by the equality (2.1) and Proposition 2.6.

Case 3. $\alpha = 0$, $\beta \neq 0$. It is similar to Case 2.

Case 4. $\alpha \neq 0$, $\beta \neq 0$. Consider the following two subcases:

Subcase 4.1. $\beta = k\alpha$ for some $k \in F_{q^2}^*$. $\Psi_{t,k}$ is a G-orbit by Lemma 2.9.

Subcase 4.2. $\beta \neq k\alpha$ for all $k \in F_{q^2}$. $\Psi_{t,t',t''}$ is a G-orbit by Lemma 2.10.

Then it follows that the above sets Ψ are all the G-orbits.

In the sequel we shall prove that the above orbits are all nonempty. We first consider the G-orbits $\Psi_{t,t',t''}$, $t \in F_{q^2}$, $t', t'' \in F_q$. If $I_{(\alpha,\beta)} \in \Psi_{t,t',t''}$, then

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} H \overline{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}}^{\mathrm{T}} = \begin{pmatrix} \alpha H \overline{\alpha}^{\mathrm{T}} & \alpha H \overline{\beta}^{\mathrm{T}} \\ \beta H \overline{\alpha}^{\mathrm{T}} & \beta H \overline{\beta}^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} t' & t \\ \overline{t} & t'' \end{pmatrix}, \tag{2.3}$$

and so $\binom{\alpha}{\beta} H \overline{\binom{\alpha}{\beta}}^{\mathrm{T}}$ is a 2×2 Hermitian matrix. Thus it is cogredient to one of

$$H_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; H_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In other words, there exists a nonsingular matrix C such that $C\binom{\alpha}{\beta} H \overline{\binom{\alpha}{\beta}}^{\mathrm{T}} \overline{C}^{\mathrm{T}}$ is equal to one of H_0, H_1 and H_2. By Lemma 2.5, it follows that the numbers n_i, $i = 0, 1, 2$, of matrices D_i which satisfy

$$D_i H \overline{D_i}^{\mathrm{T}} = H_i$$

are $n_0 = \mathfrak{n}(2,0;n) = \prod\limits_{i=n-3}^{n} (q^i - (-1)^i) q^2$, $n_1 = \mathfrak{n}(2,1;n) = \prod\limits_{i=n-2}^{n} (q^i - (-1)^i) q^{n-1}$ and $n_2 = \mathfrak{n}(2,2;n) = \prod\limits_{i=n-1}^{n} (q^i - (-1)^i) q^{2n-3}$, respectively. Let $\binom{\alpha}{\beta} = C^{-1} D_i$. Then there is a one-one correspondence between $\binom{\alpha}{\beta}$ and D_i, and so the numbers of (α, β) satisfying the equality (2.3) are n_0, n_1 and n_2, respectively. Since $\nu \geqslant 2$, $n \geqslant 4$, the numbers n_i, $i = 0, 1, 2$, are all larger than q^2. Hence there exists at least one (α, β) satisfying the equality (2.3) and $\beta \neq k\alpha$ for all $k \in F_{q^2}$. Consequently, the orbits $\Psi_{t,t',t''}$, $t \in F_{q^2}$, $t', t'' \in F_q$ are nonempty.

Similarly, we can show that the remaining orbits are all nonempty.

Then it follows that there are exactly $(q^4 + q^3 + q + 1)$ G-orbits of maximal ideals of $K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}]$. $\square$

Remark 2. For convenience, we define

$$
\begin{aligned}
&I_{0,0} \triangleq I_{(0,0)};\\
&I_t^{0_r} \triangleq \bigcap_{I_{(\alpha,\beta)}\in\Psi_t^{0_r}} I_{(\alpha,\beta)},\ t\in F_q;\quad I_{t,k} \triangleq \bigcap_{I_{(\alpha,\beta)}\in\Psi_{t,k}} I_{(\alpha,\beta)},\ t\in F_q,\ k\in F_{q^2}^*;\\
&I_t^{0_l} \triangleq \bigcap_{I_{(\alpha,\beta)}\in\Psi_t^{0_l}} I_{(\alpha,\beta)},\ t\in F_q;\quad I_{t,t',t''} \triangleq \bigcap_{I_{(\alpha,\beta)}\in\Psi_{t,t',t''}} I_{(\alpha,\beta)},\ t\in F_{q^2},\ t',t''\in F_q;\\
&Q_2 \triangleq \{I_{0,0},\, I_t^{0_r},\, t\in F_q,\, I_t^{0_l},\, t\in F_q,\, I_{t,k},\, t\in F_q,\, k\in F_{q^2}^*,\, I_{t,t',t''},\, t\in F_{q^2},\, t',t''\in F_q\}.
\end{aligned}
$$

By an argument similar to that in Theorem 2.7, from Lemma 2.1 (4) and Proposition 2.11 we deduce one of the main results of this paper.

Theorem 2.12. *Let F_{q^2} be a field with q^2 elements, where q is a power of a prime p. Suppose that*

$$G=\left\{(A)=\begin{pmatrix} A & 0\\ 0 & A\end{pmatrix}:\ A\in U_n(F_{q^2},H)\right\}$$

acts on the vector space $F_{q^2}^{(n)}\oplus F_{q^2}^{(n)}$. Then every G-invariant ideal I of $K[F_{q^2}^{(n)}\oplus F_{q^2}^{(n)}]$ is an intersection of finitely many ideals in the set Q_2 (the same notation as in Remark 2), i.e., $I=\bigcap_{I'\in Q_2'} I'$ where Q_2' is a subset of Q_2.

3. Relationship

So far, we have investigated the structure of invariant ideals of the group algebra $K[F_{q^2}^{(n)}]$ and of vector invariant ideals of the group algebra $K[F_{q^2}^{(n)}\oplus F_{q^2}^{(n)}]$ in Section 2. In this section we shall establish the relationship between invariant ideals of $K[F_{q^2}^{(n)}]$ and vector invariant ideals of $K[F_{q^2}^{(n)}\oplus F_{q^2}^{(n)}]$ using augmentation ideals.

Let V be an abelian group, viewed multiplicatively, and let $K[V]$ denote its group algebra over the field K. If U is a subgroup of V, then there exists a natural epimorphism φ: $K[V]\to K[V/U]$. Let us denote the kernel of φ by $\omega(U;V)$ and call it the augmentation ideal of U in V.

We first compute the maximal ideals by characters, and then describe the corresponding augmentation ideals. Finally, by an argument similar to that in Section 3 of [2] we may show the following two lemmas.

Lemma 3.1. *Let F_{q^2} be a field with q^2 elements, where q is a power of a prime p. Let $U_n(F_{q^2}, H)$ act on the F_{q^2}-vector space $F_{q^2}^{(n)}$. Suppose that K is a field with* $\operatorname{char} K \neq p$ *and K contains a primitive pth root of unity ε. Then $I_{\overline{0}} = \omega(F_{q^2}^{(n)}; F_{q^2}^{(n)})$ and $I_t = \bigl\{\sum_i a_i x_i \colon \sum_i a_i \varepsilon^{\mu(\alpha x_i{}^{\mathrm{T}})} = 0,$ for all $0 \neq \alpha \in F_{q^2}^{(n)}$ and $\alpha H \overline{\alpha}^{\mathrm{T}} = t\bigr\}$, $t \in F_q$.*

Lemma 3.2. *Let F_{q^2} be a field with q^2 elements, where q is a power of a prime p. Let*

$$G = \left\{(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \colon A \in U_n(F_{q^2}, H)\right\}$$

act on the vector space $F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}$. Suppose that K is a field with $\operatorname{char} K \neq p$ *and K contains a primitive pth root of unity ε. Then*

$$I_{0,0} = \omega(F_{q^2}^{(n)} \oplus F_{q^2}^{(n)};\ F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}),$$

$$I_t^{0_r} = \left\{\sum_{i,j} a_{ij}(x_i, y_j) \colon \sum_{i,j} a_{ij} \varepsilon^{\mu(\alpha x_i{}^{\mathrm{T}})} = 0,\ \forall\, 0 \neq \alpha \in F_{q^2}^{(n)} \text{ and } \alpha H \overline{\alpha}^{\mathrm{T}} = t\right\}, \quad t \in F_q,$$

$$I_t^{0_l} = \left\{\sum_{i,j} a_{ij}(x_i, y_j) \colon \sum_{i,j} a_{ij} \varepsilon^{\mu(\beta y_j{}^{\mathrm{T}})} = 0,\ \forall\, 0 \neq \beta \in F_{q^2}^{(n)} \text{ and } \beta H \overline{\beta}^{\mathrm{T}} = t\right\}, \quad t \in F_q,$$

$$\bigcap_{t \in F_q} I_t^{0_r} \bigcap I_{0,0} = \omega(0 \oplus F_{q^2}^{(n)};\ F_{q^2}^{(n)} \oplus F_{q^2}^{(n)})$$

and

$$\bigcap_{t \in F_q} I_t^{0_l} \bigcap I_{0,0} = \omega(F_{q^2}^{(n)} \oplus 0;\ F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}).$$

Let

$$\pi \colon\ K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}] \to K[F_{q^2}^{(n)}],$$
$$\sum_{i,j} a_{ij}(x_i, y_j) \mapsto \sum_{i,j} a_{ij} x_i$$

be the natural projection. The next theorem establishes the relationship between invariant ideals of $K[F_{q^2}^{(n)}]$ and vector invariant ideals of $K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}]$ by the natural projection π.

Theorem 3.3. *Let F_{q^2} be a finite field of characteristic p. Let $U_n(F_{q^2}, H)$ act on the vector space $F_{q^2}^{(n)}$ and*

$$G = \left\{(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \colon A \in U_n(F_{q^2}, H)\right\}$$

act on the vector space $F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}$. Then the invariant ideals of $K[F_{q^2}^{(n)}]$ are precisely the intersections of the natural projections of some vector invariant ideals of $K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}]$.

P r o o f. By Lemma 3.1 and Lemma 3.2, we have

$$\pi(I_{0,0}) = I_{\overline{0}}, \quad \pi(I_t^{0_r}) = I_t.$$

Then, by Theorem 2.7, every $U_n(F_{q^2}, H)$-invariant ideal I must be of the form

$$I = \bigcap_{I_{t'} \in Q_1', \, Q_1' \leqslant Q_1} I_{t'} = \bigcap_{I_{t'}^{0_r} \in Q_2', \, Q_2' \leqslant Q_2} \pi(I_{t'}^{0_r})$$

where Q_2' corresponds to Q_1' $(I_{t'}^{0_r} \to I_{t'})$. If there exists $I_{\overline{0}}$ in the above equation, we need only add $\pi(I_{0,0})$. Therefore the invariant ideals of $K[F_{q^2}^{(n)}]$ are precisely the intersections of the natural projections of some vector invariant ideals of $K[F_{q^2}^{(n)} \oplus F_{q^2}^{(n)}]$. □

4. Orthogonal groups over finite fields of odd characteristic

This section is concerned with the orthogonal groups over finite fields of odd characteristic. We first recall some relevant material in [12]. Suppose that F_q is a finite field of odd characteristic p. If S is an $n \times n$ nonsingular symmetric matrix such that $S^{\mathrm{T}} = S$ over F_q, then S is necessarily cogredient to one of the following four forms:

$$S_{2\nu} = \begin{pmatrix} 0 & I^{(\nu)} \\ I^{(\nu)} & 0 \end{pmatrix}, \quad S_{2\nu+1,1} = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & 1 \end{pmatrix},$$

$$S_{2\nu+1,z} = \begin{pmatrix} 0 & I^{(\nu)} & \\ I^{(\nu)} & 0 & \\ & & z \end{pmatrix}, \quad S_{2\nu+2} = \begin{pmatrix} 0 & I^{(\nu)} & & \\ I^{(\nu)} & 0 & & \\ & & 1 & \\ & & & -z \end{pmatrix},$$

where z is a fixed non-square element of $F_q \setminus \{0\}$ and $n = 2\nu, 2\nu+1, 2\nu+1$ and $2\nu+2$, respectively. In this section, we suppose that $\nu \geqslant 2$.

Let S be an $n \times n$ nonsingular symmetric matrix over F_q. A matrix A is called an orthogonal matrix with respect to S if $ASA^{\mathrm{T}} = S$. The set of all such orthogonal matrices forms a group with respect to matrix multiplication, called the *orthogonal group* of degree n with respect to S over F_q and is denoted by $O_n(F_q, S)$, i.e.,

$$O_n(F_q, S) = \{A \in GL_n(F_q) \colon \ ASA^{\mathrm{T}} = S\}.$$

Suppose that $O_n(F_q, S)$ acts on the F_q-vector space $F_q^{(n)}$ by

$$(x)^A = xA^{-1^{\mathrm{T}}}, \quad x \in F_q^{(n)},\ A \in O_n(F_q, S).$$

Then it can induce an action on the vector space $F_q^{(n)} \oplus F_q^{(n)}$, defined by

$$(x, y)^A = (xA^{-1^{\mathrm{T}}}, yA^{-1^{\mathrm{T}}}), \quad x, y \in F_q^{(n)},\ A \in O_n(F_q, S).$$

For convenience, we define

$$\widehat{G} = \left\{ (A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} :\ A \in O_n(F_q, S) \right\}.$$

The action of $\widehat{G}$ on the vector space $F_q^{(n)} \oplus F_q^{(n)}$ coincides with the action of $O_n(F_q, S)$ on $F_q^{(n)} \oplus F_q^{(n)}$, i.e.,

$$(x, y)^{(A)} = (xA^{-1^{\mathrm{T}}}, yA^{-1^{\mathrm{T}}}), \quad x, y \in F_q^{(n)},\ (A) \in \widehat{G}.$$

And so $\widehat{G}$ can act on the group algebra $K[F_q^{(n)} \oplus F_q^{(n)}]$ by

$$\left(\sum_{i,j} a_{ij}(x_i, y_j) \right)^{(A)} = \sum_{i,j} a_{ij}(x_i, y_j)^{(A)},$$

where K is a field of characteristic different from p and K contains ε, a primitive pth root of unity.

The following propositions and theorems will describe the structure of $O_n(F_q, S)$-invariant ideals of $K[F_q^{(n)}]$ and $\widehat{G}$-vector invariant ideals of $K[F_q^{(n)} \oplus F_q^{(n)}]$, and also establish the relationship between them. The proofs of these results are similar to those of the corresponding results in Section 2 and Section 3, and thus omitted.

Proposition 4.1. *Let F_q be a field with q elements, where q is a power of an odd prime p. Suppose that $O_n(F_q, S)$ acts on the F_q-vector space $F_q^{(n)}$. Then there are $(q+1)$ of $O_n(F_q, S)$-orbits of maximal ideals of $K[F_q^{(n)}]$, i.e.,*

$$\widehat{\Omega}_{\overline{0}} = \{I_0\}, \quad \widehat{\Omega}_t = \{I_\alpha :\ 0 \neq \alpha \in F_q^{(n)},\ \alpha S \alpha^{\mathrm{T}} = t\}, \quad t \in F_q.$$

Remark 3. We define $I_{\overline{0}} \triangleq I_0$, $I_t \triangleq \bigcap_{I_\alpha \in \widehat{\Omega}_t} I_\alpha$, $t \in F_q$ and $\widehat{Q}_1 \triangleq \{I_{\overline{0}}\} \cup \{I_t :\ t \in F_q\}$.

Theorem 4.2. *Let F_q be a field with q elements, where q is a power of an odd prime p. Suppose that $O_n(F_q, S)$ acts on the F_q-vector space $F_q^{(n)}$. Then every $O_n(F_q, S)$-invariant ideal I of $K[F_q^{(n)}]$ is an intersection of finitely many ideals in the set $\widehat{Q}_1$ (the same notation as in Remark 3), i.e., $I = \bigcap_{I' \in \widehat{Q}'_1} I'$ where $\widehat{Q}'_1$ is a subset of $\widehat{Q}_1$.*

Proposition 4.3. *Let F_q be a field with q elements, where q is a power of an odd prime p. Suppose that*

$$\widehat{G} = \left\{ (A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : \ A \in O_n(F_q, S) \right\}$$

acts on the vector space $F_q^{(n)} \oplus F_q^{(n)}$. Then there are $(q^3 + q^2 + q + 1)$ $\widehat{G}$ -orbits of the maximal ideals of $K[F_q^{(n)} \oplus F_q^{(n)}]$, i.e.,

$$\begin{aligned}
\widehat{\Psi}^0 &= \{I_{(0,0)}\}; \\
\widehat{\Psi}_t^{0_r} &= \{I_{(\alpha,0)} \colon \ 0 \neq \alpha \in F_q^{(n)}, \ \alpha S \alpha^{\mathrm{T}} = t\}, \quad t \in F_q; \\
\widehat{\Psi}_t^{0_l} &= \{I_{(0,\beta)} \colon \ 0 \neq \beta \in F_q^{(n)}, \ \beta S \beta^{\mathrm{T}} = t\}, \quad t \in F_q; \\
\widehat{\Psi}_{t,k} &= \{I_{(\alpha,\beta)} \colon \ 0 \neq \alpha \in F_q^{(n)}, \ \beta = k\alpha, \ \alpha S \alpha^{\mathrm{T}} = t\}, \quad t \in F_q, \ k \in F_q^*; \\
\widehat{\Psi}_{t,t',t''} &= \{I_{(\alpha,\beta)} \colon \ 0 \neq \alpha \in F_q^{(n)}, \ \beta \neq k\alpha, \ k \in F_q, \ \alpha S \beta^{\mathrm{T}} = t, \ \alpha S \alpha^{\mathrm{T}} = t', \\
&\qquad \beta S \beta^{\mathrm{T}} = t''\}, \quad t, t', t'' \in F_q.
\end{aligned}$$

Remark 4. For convenience, we define

$$\begin{aligned}
I^0 &\triangleq I_{(0,0)}; \\
I_t^{0_r} &\triangleq \bigcap_{I_{(\alpha,\beta)} \in \widehat{\Psi}_t^{0_r}} I_{(\alpha,\beta)}, \quad t \in F_q; \\
I_{t,k} &\triangleq \bigcap_{I_{(\alpha,\beta)} \in \widehat{\Psi}_{t,k}} I_{(\alpha,\beta)}, \quad t \in F_q, \ k \in F_q^*; \\
I_t^{0_l} &\triangleq \bigcap_{I_{(\alpha,\beta)} \in \widehat{\Psi}_t^{0_l}} I_{(\alpha,\beta)}, \quad t \in F_q; \\
I_{t,t',t''} &\triangleq \bigcap_{I_{(\alpha,\beta)} \in \widehat{\Psi}_{t,t',t''}} I_{(\alpha,\beta)}, \quad t, t', t'' \in F_q; \\
\widehat{Q}_2 &\triangleq \{I^0, \ I_t^{0_r}, \ t \in F_q, \ I_t^{0_l}, \ t \in F_q, \ I_{t,k}, \ t \in F_q, \ k \in F_q^*, \ I_{t,t',t''}, \ t, t', t'' \in F_q\}.
\end{aligned}$$

Theorem 4.4. *Let F_q be a field with q elements, where q is a power of an odd prime p. Suppose that*

$$\widehat{G} = \left\{ (A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in O_n(F_q, S) \right\}$$

acts on the vector space $F_q^{(n)} \oplus F_q^{(n)}$. Then every $\widehat{G}$-invariant ideal I of $K[F_q^{(n)} \oplus F_q^{(n)}]$ is an intersection of finitely many ideals in the set $\widehat{Q}_2$ (the same notation as in Remark 4), i.e., $I = \bigcap\limits_{I' \in \widehat{Q}'_2} I'$ where $\widehat{Q}'_2$ is a subset of $\widehat{Q}_2$.

Theorem 4.5. *Let F_q be a field of odd characteristic. Suppose that $O_n(F_q, S)$ acts on the F_q -vector space $F_q^{(n)}$ and*

$$\widehat{G} = \left\{ (A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in O_n(F_q, S) \right\}$$

acts on the vector space $F_q^{(n)} \oplus F_q^{(n)}$. Then the invariant ideals of $K[F_q^{(n)}]$ are precisely the intersections of the natural projections of some vector invariant ideals of $K[F_q^{(n)} \oplus F_q^{(n)}]$.

5. Orthogonal groups over finite fields of characteristic 2

This section is concerned with orthogonal groups over finite fields of characteristic 2. Recall some relevant material in [12]. Suppose that F_q is a finite field of characteristic 2. Then we have $F_q^2 = F_q$. An $n \times n$ matrix $D = (d_{ij})$ is called alternate if $d_{ij} = d_{ji}$ for all $i \neq j$, $1 \leqslant i, j \leqslant n$ and $d_{ii} = 0$ for all $i = 1, 2, \ldots, n$. Denote the set of all $n \times n$ alternate matrices over F_q by $\mathcal{K}_n$. Two $n \times n$ matrices A and B over F_q are said to be congruent mod $\mathcal{K}_n$ if $A + B \in \mathcal{K}_n$, which is usually denoted by

$$A \equiv B \pmod{\mathcal{K}_n}$$

or simply, $A \equiv B$. Two $n \times n$ matrices A and B over F_q are said to be 'cogredient', if there is a nonsingular matrix Q such that $QAQ^{\mathrm{T}} \equiv B$. A matrix A is called definite if $xAx^{\mathrm{T}} = 0$, where $x \in F_q^{(n)}$ implies $x = 0$.

By [12], Theorem 1.30, we know that any $n \times n$ matrix R over F_q is 'cogredient' to a matrix of the form

$$M = \begin{pmatrix} A & I^{(p)} & & \\ & B & & \\ & & C & \\ & & & 0 \end{pmatrix},$$

where A and B are $p\times p$ diagonal matrices, and C is a $d\times d$ definite matrix. Moreover, p and d are uniquely determined by R. An $n\times n$ matrix R is said to be regular if $n=2p+d$.

Let R be an $n\times n$ regular matrix over F_q. A matrix A is called an orthogonal matrix with respect to R if $ARA^{\mathrm{T}} \equiv R$. The set of all such orthogonal matrices forms a group with respect to matrix multiplication, called the *orthogonal group* of degree n with respect to R over F_q, and denoted by $O_n(F_{q,}R)$, i.e.,

$$O_n(F_{q,}R)=\{A\in GL_n(F_q):\ ARA^{\mathrm{T}}\equiv R\}.$$

Suppose that $O_n(F_q,R)$ acts on the F_q -vector space $F_q^{(n)}$ by

$$(x)^A = xA^{-1^{\mathrm{T}}},\quad x\in F_q^{(n)},\ A\in O_n(F_q,R).$$

Then it can induce an action on the vector space $F_q^{(n)}\oplus F_q^{(n)}$, defined by

$$(x,y)^A=(xA^{-1^{\mathrm{T}}},yA^{-1^{\mathrm{T}}}),\quad x,y\in F_q^{(n)},\ A\in O_n(F_q,R).$$

For convenience, we define

$$\widetilde{G}=\left\{(A)=\begin{pmatrix} A & 0\\ 0 & A\end{pmatrix}:\ A\in O_n(F_q,R)\right\}.$$

The action of $\widetilde{G}$ on the vector space $F_q^{(n)}\oplus F_q^{(n)}$ coincides with the action of $O_n(F_q,R)$ on $F_q^{(n)}\oplus F_q^{(n)}$, i.e.,

$$(x,y)^{(A)}=(xA^{-1^{\mathrm{T}}},yA^{-1^{\mathrm{T}}}),\quad x,y\in F_q^{(n)},\ (A)\in\widetilde{G}.$$

And so $\widetilde{G}$ can act on the group algebra $K[F_q^{(n)}\oplus F_q^{(n)}]$ by

$$\left(\sum_{i,j}a_{ij}(x_i,y_j)\right)^{(A)}=\sum_{i,j}a_{ij}(x_i,y_j)^{(A)},$$

where K is a field of characteristic different from 2 and K contains ε, a primitive 2nd root of unity.

The following propositions and theorems will describe the structure of $O_n(F_q,R)$-invariant ideals of $K[F_q^{(n)}]$ and $\widetilde{G}$-vector invariant ideals of $K[F_q^{(n)}\oplus F_q^{(n)}]$, and also establish the relationship between them. The proofs of these results are similar to those of the corresponding results in Section 2 and Section 3, and thus omitted.

Proposition 5.1. *Let F_q be a finite field of characteristic 2. Suppose that $O_n(F_q,R)$ acts on the F_q-vector space $F_q^{(n)}$. Then there are $(q+1)$ $O_n(F_q,R)$-orbits of maximal ideals of $K[F_q^{(n)}]$, i.e., $\widetilde{\Omega}_{\overline{0}}=\{I_0\}$, $\widetilde{\Omega}_t=\{I_\alpha\colon\ 0\neq\alpha\in F_q^{(n)},\ \alpha R\alpha^{\mathrm{T}}\equiv t\}$, $t\in F_q$.*

Remark 5. We define $I_{\overline{0}}\triangleq I_0$, $I_t\triangleq\bigcap\limits_{I_\alpha\in\widetilde{\Omega}_t}I_\alpha$, $t\in F_q$ and $\widetilde{Q}_1\triangleq\{I_{\overline{0}}\}\cup\{I_t\colon\ t\in F_q\}$.

Theorem 5.2. *Let F_q be a finite field of characteristic 2. Suppose that $O_n(F_q,R)$ acts on the F_q -vector space $F_q^{(n)}$. Then every $O_n(F_q,R)$-invariant ideal I of $K[F_q^{(n)}]$ is an intersection of finitely many ideals in the set $\widetilde{Q}_1$ (the same notation as in Remark 5), i.e., $I=\bigcap\limits_{I'\in\widetilde{Q}_1'}I'$ where $\widetilde{Q}_1'$ is a subset of $\widetilde{Q}_1$.*

Proposition 5.3. *Let F_q be a field with q elements and* $\operatorname{char}F_q=2$. *Suppose that*

$$\widetilde{G}=\left\{(A)=\begin{pmatrix}A&0\\0&A\end{pmatrix}\colon\ A\in O_n(F_q,R)\right\}$$

acts on the vector space $F_q^{(n)}\oplus F_q^{(n)}$. Then there are (q^3+q^2+q+1) $\widetilde{G}$ -orbits of maximal ideals of $K[F_q^{(n)}\oplus F_q^{(n)}]$, i.e.,

$$\begin{aligned}
\widetilde{\Psi}^0&=\{I_{(0,0)}\};\\
\widetilde{\Psi}_t^{0_r}&=\{I_{(\alpha,0)}\colon\ 0\neq\alpha\in F_q^{(n)},\ \alpha R\alpha^{\mathrm{T}}\equiv t\},\quad t\in F_q;\\
\widetilde{\Psi}_t^{0_l}&=\{I_{(0,\beta)}\colon\ 0\neq\beta\in F_q^{(n)},\ \beta R\beta^{\mathrm{T}}\equiv t\},\quad t\in F_q;\\
\widetilde{\Psi}_{t,k}&=\{I_{(\alpha,\beta)}\colon\ 0\neq\alpha\in F_q^{(n)},\ \beta=k\alpha,\ \alpha R\alpha^{\mathrm{T}}\equiv t\},\quad t\in F_q,\ k\in F_q^*;\\
\widetilde{\Psi}_{t,t',t''}&=\{I_{(\alpha,\beta)}\colon\ 0\neq\alpha\in F_q^{(n)},\ \beta\neq k\alpha,\ k\in F_q,\ \alpha(R+R^{\mathrm{T}})\beta^{\mathrm{T}}\equiv t,\ \alpha R\alpha^{\mathrm{T}}\equiv t',\\
&\qquad\beta R\beta^{\mathrm{T}}\equiv t''\},\quad t,t',t''\in F_q.
\end{aligned}$$

Remark 6. For convenience, we define

$$\begin{aligned}
&I^0\triangleq I_{(0,0)};\\
&I_t^{0_r}\triangleq\bigcap_{I_{(\alpha,\beta)}\in\widetilde{\Psi}_t^{0_r}}I_{(\alpha,\beta)},\ t\in F_q;\quad I_{t,k}\triangleq\bigcap_{I_{(\alpha,\beta)}\in\widetilde{\Psi}_{t,k}}I_{(\alpha,\beta)},\ t\in F_q,\ k\in F_q^*;\\
&I_t^{0_l}\triangleq\bigcap_{I_{(\alpha,\beta)}\in\widetilde{\Psi}_t^{0_l}}I_{(\alpha,\beta)},\ t\in F_q;\quad I_{t,t',t''}\triangleq\bigcap_{I_{(\alpha,\beta)}\in\widetilde{\Psi}_{t,t',t''}}I_{(\alpha,\beta)},\ t,t',t''\in F_q;\\
&\widetilde{Q}_2\triangleq\{I^0,\ I_t^{0_r},\ t\in F_q,\ I_t^{0_l},\ t\in F_q,\ I_{t,k},\ t\in F_q,\ k\in F_q^*,\ I_{t,t',t''},\ t,t',t''\in F_q\}.
\end{aligned}$$

Theorem 5.4. *Let F_q be a field with q elements and* $\operatorname{char} F_q = 2$. *Suppose that*

$$\widetilde{G} = \left\{ (A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : \ A \in O_n(F_q, R) \right\}$$

acts on the vector space $F_q^{(n)} \oplus F_q^{(n)}$. Then every $\widetilde{G}$-invariant ideal I of $K[F_q^{(n)} \oplus F_q^{(n)}]$ is an intersection of finitely many ideals in the set $\widetilde{Q}_2$ (the same notation as in Remark 6*), i.e., $I = \bigcap_{I' \in \widetilde{Q}'_2} I'$ where $\widetilde{Q}'_2$ is a subset of $\widetilde{Q}_2$.*

Theorem 5.5. *Let F_q be a finite field of characteristic* 2. *Suppose that $O_n(F_q, R)$ acts on the F_q-vector space $F_q^{(n)}$ and*

$$\widetilde{G} = \left\{ (A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : \ A \in O_n(F_q, R) \right\}$$

acts on the vector space $F_q^{(n)} \oplus F_q^{(n)}$. Then the invariant ideals of $K[F_q^{(n)}]$ are precisely the intersections of the natural projections of some vector invariant ideals of $K[F_q^{(n)} \oplus F_q^{(n)}]$.

Acknowledgement. The authors are grateful to the anonymous referees for their valuable comments and suggestions. The authors also thank Dr. Miaomiao Ren for helping with the preparation of the final version of this article.

References

[1] *C. J. B. Brookes, D. M. Evans*: Augmentation modules for affine groups. Math. Proc. Camb. Philos. Soc. *130* (2001), 287–294.
[2] *J. Nan, L. Zeng*: Vector invariant ideals of abelian group algebras under the action of the symplectic groups. J. Algebra Appl. *12* (2013), Article ID 1350046, 12 pages.
[3] *J. M. Osterburg, D. S. Passman, A. E. Zalesskiĭ*: Invariant ideals of abelian group algebras under the multiplicative action of a field II. Proc. Am. Math. Soc. *130* (2002), 951–957.
[4] *D. S. Passman*: Invariant ideals of abelian group algebras under the action of simple linear groups. Resen. Inst. Mat. Estat. Univ. São Paulo *5* (2002), 377–390.
[5] *D. S. Passman*: Finitary actions and invariant ideals. Turk. J. Math. *31,* Suppl.*,* (2007), 113–130.
[6] *D. S. Passman*: Invariant ideals of abelian group algebras under the torus action of a field I. J. Algebra *324* (2010), 3035–3043.
[7] *D. S. Passman*: Invariant ideals of abelian group algebras under the torus action of a field II. J. Algebra *331* (2011), 362–377.
[8] *D. S. Passman, A. E. Zalesskiĭ*: Invariant ideals of abelian group algebras under the multiplicative action of a field I. Proc. Am. Math. Soc. *130* (2002), 939–949.
[9] *D. R. Richman*: On vector invariants over finite fields. Adv. Math. *81* (1990), 30–65.
[10] *D. R. Richman*: Explicit generators of the invariants of finite groups. Adv. Math. *124* (1996), 49–76.

[11] *D. R. Richman*: Invariants of finite groups over fields of characteristic p. Adv. Math. *124* (1996), 25–48.

[12] *Z. Wan*: Geometry of classical groups over finite fields and its applications. Discrete Math. *174* (1997), 365–381.

Authors' addresses: Lingli Zeng (corresponding author), Department of Mathematics, Northwest University, 229 North Taibai Road, Xi'an 710127, Shaanxi, P. R. China (current address), and School of Mathematical Sciences, Dalian University of Technology, No. 2 Linggong Road, Dalian 116024, Ganjingzi, Liaoning P. R. China (primary address), e-mail: zengll929@163.com; Jizhu Nan, School of Mathematical Sciences, Dalian University of Technology, No. 2 Linggong Road, Dalian 116024, Ganjingzi, Liaoning, P. R. China, e-mail: jznan@163.com.