Mathematica Bohemica

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Mathematica Bohemica, Vol. 141 (2016), No. 3, 327–338

Persistent URL: http://dml.cz/dmlcz/145897

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141 (2016) MATHEMATICA BOHEMICA No. 3, 327–338

APPLICATION OF (L) SETS TO SOME CLASSES OF OPERATORS

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Received December 5, 2014. First published June 16, 2016.
Communicated by Marek Ptak

Abstract. The paper contains some applications of the notion of (L) sets to several classes of operators on Banach lattices. In particular, we introduce and study the class of order (L)-Dunford-Pettis operators, that is, operators from a Banach space into a Banach lattice whose adjoint maps order bounded subsets to an (L) sets. As a sequence characterization of such operators, we see that an operator $T\colon X\to E$ from a Banach space into a Banach lattice is order (L)-Dunford-Pettis, if and only if $|T(x_n)|\to 0$ for $\sigma(E,E')$ for every weakly null sequence $(x_n)\subset X$. We also investigate relationships between order (L)-Dunford-Pettis, AM-compact, weak* Dunford-Pettis, and Dunford-Pettis operators. In particular, it is established that each operator $T\colon E\to F$ between Banach lattices is Dunford-Pettis whenever it is both order (L)-Dunford-Pettis and weak* Dunford-Pettis, if and only if E has the Schur property or the norm of F is order continuous.

Keywords: (L) set; order (L)-Dunford-Pettis operator; weakly sequentially continuous lattice operations; Banach lattice

MSC 2010: 46B42, 46B50, 47B65

1. Introduction and notation

In the sequel, X, Y and Z will denote real Banach spaces, E and F will denote real Banach lattices. B_X denotes the closed unit ball of X, and E^+ denotes the positive cone of E. We will use the term *operator between two Banach spaces* to mean a bounded linear mapping. For an operator $T\colon X\to Y$, the adjoint operator T' is defined from Y' into X' by $T'(f)(x)=f(T(x))$ for each $f\in Y'$ and for each $x\in X$. For terminology concerning Banach lattice theory and positive operators we refer the reader to the excellent book of Aliprantis-Burkinshaw [1].

Recall that a subset A of a Banach space X is called a Dunford-Pettis set whenever every weakly null sequence $(f_n)\subset X'$ converges uniformly to zero on the set A, that

DOI: 10.21136/MB.2016.0076-14

is, $\sup_{x\in A}|f_n(x)|\to 0$. Also, let us recall from [6] that a subset A of the topological dual X', of a Banach space X, is said to be an (L) set whenever every weakly null sequence $(x_n)\subset X$ converges uniformly to zero on the set A, that is, $\sup_{f\in A}|f(x_n)|\to 0$. It is well known that the class of (L) sets contains strictly that of Dunford-Pettis (or relatively compact) sets. But an (L) set is not necessarily Dunford Pettis (and hence not relatively compact). In fact, the closed unit ball B_{ℓ^∞} is an (L) set in ℓ^∞ but fails to be Dunford-Pettis, as ℓ^1 has the Schur property while $(\ell^\infty)'$ does not. An operator $T\colon X\to Y$ is called Dunford-Pettis, if T carries each weakly null sequence $(x_n)\subset X$ to a norm null one in Y, equivalently, T carries relatively weakly compact subsets of X to relatively compact ones in Y. Accordingly, several weak versions of Dunford-Pettis operators are considered in the theory of Banach lattices. Namely, an operator $T\colon E\to F$ is

- ▷ weak Dunford-Pettis, if the sequence $(y'_n(T(x_n)))$ converges to 0 whenever (x_n) converges weakly to 0 in E and (y'_n) converges weakly to 0 in F';
- ▷ weak* Dunford-Pettis, if the sequence $(y'_n(T(x_n)))$ converges to 0 whenever (x_n) converges weakly to 0 in E and (y'_n) converges weak* to 0 in F';
- ▷ almost Dunford-Pettis, if the sequence $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence $(x_n)\subset E$ consisting of pairewise disjoint elements, equivalently, $(\|T(x_n)\|)$ converges to 0 for every weakly null sequence $(x_n)\subset E^+$.

The present paper is devoted to some applications of the notion of (L) sets to several classes of operators on Banach lattices. More precisely, we establish some characterizations of weak Dunford-Pettis operators (Theorem 2.2) and Banach spaces with the Dunford-Pettis property (see Corollary 2.3). We introduce another weak version of Dunford-Pettis operators, the so-called order (L)-Dunford-Pettis operators (Definition 2.4), and derive some characterizations of this class of operators (Theorem 2.5). Also, we study the relationship between the class of order (L)-Dunford-Pettis operators and that of AM-compact (respectively, order weakly compact, weak Dunford-Pettis) operators. Finally, we characterize Banach lattices E and F on which each operator from E into F which is both order (L)-Dunford-Pettis and weak* Dunford-Pettis, is Dunford-Pettis (Theorem 2.20).

To state our results, we need to fix some notation and recall some definitions. A Banach lattice is a Banach space $(E,\|\cdot\|)$ such that E is a vector lattice and for each $x,y\in E$, $|x|\leqslant|y|$ implies $\|x\|\leqslant\|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each net (x_α) such that $x_\alpha\downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$, where the notation $x_\alpha\downarrow 0$ means that (x_α) is decreasing and $\inf(x_\alpha)=0$. Also, a vector lattice E is Dedekind σ-complete if every majorized countable nonempty subset of E has a supremum. The lattice operations in E are called weakly sequentially continuous if the sequence $(|x_n|)$ converges to 0 in the weak topology, whenever the sequence (x_n) converges weakly to 0 in E. A Banach space X has the Schur property if each weakly null sequence in X converges to zero in the norm. A Banach lattice E has the positive Schur property if weakly null sequences with positive terms are norm null. For example, the Banach lattice $L^1([0,1])$ has the positive Schur property. Note that a Banach space X has the Dunford-Pettis (respectively, Dunford-Pettis*) property if $f_n(x_n) \to 0$ for every weakly null sequence $(x_n) \subset X$ and every weakly (respectively, weak*) null sequence $(f_n) \subset X'$. Finally, we remember that an operator $T\colon E \to X$ is:

- ▷ order weakly compact, if the image by T of each order bounded subset of E is a relatively weakly compact subset of X;
- ▷ order limited, if T carries each order bounded subset in E to a limited one in X, equivalently, $|T'(f_n)| \to 0$ for $\sigma(E', E)$ for each sequence $(f_n) \subset X'$ such that $f_n \to 0$ for $\sigma(X', X)$, [5], Theorem 3.3 (3);
- ▷ AM-compact, if the image by T of each order bounded subset of E is a relatively compact subset of X.

2. Main results

2.1. Weak Dunford-Pettis operators and (L) sets. We will use basically the following operator characterization of (L) sets.

Theorem 2.1 ([7], Theorem 4.4). *Let X be a Banach space. A norm bounded subset A of the dual X' is an (L) set if and only if the adjoint of every weakly compact operator from an arbitrary Banach space Y to X carries A to a relatively compact set.*

In our next result, we give a characterization of the class of weak Dunford-Pettis operators through (L) sets.

Theorem 2.2. *Let $T\colon X \to Y$ be an operator between two Banach spaces. The following statements are equivalent:*

1. *T is a weak Dunfor-Pettis operator;*
2. *T' carries weakly compact sets in Y' to (L) sets in X';*
3. *for an arbitrary Banach space Z and every weakly compact operator $S\colon Z \to X$, the adjoint operator TS' is Dunford-Pettis.*

P r o o f. (1) $\Rightarrow$ (2): Assume that there exist a weakly compact set $A \subset Y'$ such that $T'(A)$ is not an (L) set. Then there exists a weakly null sequence $(x_n) \subset X$, a sequence $(f_n) \subset A$ and some $\varepsilon > 0$ such that $|f_n(T(x_n))| = |T'(f_n)(x_n)| > \varepsilon$ for all n. Since A is weakly compact, we can assume that $f_n \to f$ for $\sigma(Y', Y'')$. As T is a weak Dunford-Pettis operator, hence

$$\varepsilon < |f_n(T(x_n))| \leqslant |(f_n - f)(T(x_n))| + |f(T(x_n))| \to 0,$$

which is impossible.

(2) $\Rightarrow$ (3): Let Z be a Banach space and $S\colon Z \to X$ a weakly compact operator. Let A be a weakly compact set in Y', then by hypothesis $T'(A)$ is an (L) set in X', hence it follows from Theorem 2.1 that $S'(T'(A))$ is a relatively compact set in Z'. This proves that TS' is a Dunford-Pettis operator.

(3) $\Rightarrow$ (1): Let $(x_n) \subset X$ and $(f_n) \subset Y'$ be weakly null sequences. Consider the operator

$$(*) \qquad S\colon \ell^1 \to X, \quad (\lambda_n)_n \mapsto \sum_{n=1}^{\infty} \lambda_n x_n.$$

S is weakly compact ([1], Theorem 5.26). On the other hand, the adjoint operator S' is defined by

$$S'\colon X' \to \ell^\infty, \quad f \mapsto \{f(x_n)\}_{n\geqslant 1},$$

and note that $S'(X') \subset c_0$.

Thus, by our hypothesis TS' is a Dunford-Pettis operator and then,

$$|f_n(T(x_n))| = |T'(f_n)(x_n)| \leqslant \|S'T'(f_n)\|_\infty \to 0.$$

This shows that T is a weak Dunfor-Pettis operator. $\square$

As a consequence of Theorem 2.2, we have the following characterizations of Banach spaces with the Dunford-Pettis property.

Corollary 2.3. *Let X be a Banach space. The following statements are equivalent:*

(1) *X has the Dunford-Pettis property;*
(2) *weakly compact subsets of X' are* (L) *sets;*
(3) *weakly compact operators from an arbitrary Banach space Z into X have a Dunford-Pettis adjoint.*

2.2. The class of order (L)-Dunford-Pettis operators. It can be shown easily that an operator $T\colon X \to Y$ is Dunford-Pettis if and only if T' maps each norm bounded set in Y' to an (L) set in X'. Therefore, operators defined from a Banach space into a Banach lattice whose adjoint carries order bounded subsets to an (L) sets arise naturally.

Definition 2.4. An operator $T\colon X \to E$, from a Banach space into a Banach lattice, is said to be order (L)-Dunford-Pettis if its adjoint T' carries each order bounded subset of E' into an (L) set in X'.

The order (L)-Dunford-Pettis operators present the dual counterpart of order limited operators, and are at the same time a weak version of Dunford-Pettis operators defined from a Banach space into a Banach lattice, as each Dunford-Pettis operator $T\colon X \to E$ is order (L)-Dunford-Pettis. Note that the converse is false in general. In fact, since the lattice operations in ℓ^∞ are weakly sequentially continuous, the identity operator of the Banach lattice ℓ^∞ is order (L)-Dunford-Pettis (see Corollary 2.6), but is not Dunford-Pettis. On the other hand, there are operators which are not order (L)-Dunford-Pettis. The identity operator of the Banach lattice $L^2([0,1])$ is an example (the lattice operations in $L^2([0,1])$ are not weakly sequentially continuous). Clearly, $T\colon X \to E$ is order (L)-Dunford-Pettis if and only if $T'([-f,f])$ is an (L) set in X' for each $f \in (E')^+$. The next result characterizes the class of order (L)-Dunford-Pettis operators.

Theorem 2.5. *Let $T\colon X \to E$ be an operator from a Banach space into a Banach lattice. The following statements are equivalent:*
(1) *T is an order* (L)*-Dunford-Pettis operator;*
(2) *for every weakly compact operator S from an arbitrary Banach space Y into X, $(T \circ S)'$ is an* AM*-compact operator;*
(3) *for every weakly null sequence $(x_n) \subset X$, $|T(x_n)| \to 0$ for $\sigma(E, E')$.*

Proof. (1) $\Rightarrow$ (2): Let S be a weakly compact operator from an arbitrary Banach space Y into X. It follows from (1) that for each $f \in (E')^+$, $T'[-f,f]$ is an (L) set in X' and then, by Theorem 2.1, $S'T'[-f,f]$ is a relatively compact set in Y'. Hence $(T \circ S)'$ is an AM-compact operator.

(2) $\Rightarrow$ (3): Let (x_n) be a weakly null sequence in X and let S be defined as in $(*)$. According to our hypothesis, $S'T'[-f,f]$ is a relatively compact set in c_0 for each $f \in (E')^+$. Hence, it follows from [1], Exercise 14 in Section 3.2 that for each $f \in (E')^+$,

$$\begin{aligned} f|T(x_n)| &= \sup\{|g(T(x_n))|,\ |g| \leqslant f\} = \sup\{|T'(g)(x_n)|,\ |g| \leqslant f\} \\ &= \sup\{|h(x_n)|,\ h \in T'[-f,f]\} \to 0. \end{aligned}$$

(3) $\Rightarrow$ (1): follows immediately from the equality

$$f|T(x_n)| = \sup\{|h(x_n)|,\ h \in T'[-f,f]\}.$$

$\square$

Notice that there is a weakly compact operator from a Banach space into a Banach lattice whose adjoint is not AM-compact. In fact, the identity operator of the Banach lattice $L^2([0,1])$ is weakly compact but its adjoint, which is again the identity operator of the Banach lattice $L^2([0,1])$ is not AM-compact (because $L^2([0,1])$ is not discrete). However, the following result is a consequence of Theorem 2.5.

Corollary 2.6. *Let E be a Banach lattice. The following statements are equivalent:*

(1) *the identity operator of E is order* (L)*-Dunford-Pettis;*
(2) *all order intervals of E' are* (L) *sets.*
(3) *weakly compact operators from an arbitrary Banach space X into E have an* AM*-compact adjoint;*
(4) *the lattice operations in E are weakly sequentially continuous.*

We need the following lemma for the next result.

Lemma 2.7. *Let A be a norm bounded subset of the dual of a Banach space X. If for each $\varepsilon > 0$ there exists an* (L) *set A_ε in X' such that $A \subset A_\varepsilon + \varepsilon B_{X'}$ then A is an* (L) *set in X'.*

Proof. Let Y be a Banach space and let $S\colon Y \to X$ be a weakly compact operator. We shall prove that $S'(A)$ is relatively compact in Y'. Let $\varepsilon > 0$, then by our hypothesis there exists an (L) set A_ε in X' such that $A \subset A_\varepsilon + \varepsilon B_{X'}$, hence $S'(A) \subset S'(A_\varepsilon) + \varepsilon\|S'\|B_{Y'}$. As A_ε is an (L) set, $S'(A_\varepsilon)$ is relatively compact in Y' and hence by Theorem 3.1 of [1], $S'(A)$ is relatively compact in Y'. This shows by Theorem 2.1 that A is an (L) set in X'. □

Other properties of order (L)-Dunford-Pettis operators are given in the following propositions.

Proposition 2.8. *The class of order* (L)*-Dunford-Pettis operators from a Banach space X into a Banach lattice E is a norm closed vector subspace of the space $L(X,E)$ of all operators from X into E.*

Proof. Clearly, the class of order (L)-Dunford-Pettis operators from X into E is a vector subspace of $L(X,E)$. Now, let S be in the norm closure of the class of order (L)-Dunford-Pettis operators from X into E. Let f be nonzero in $(E')^+$ and let $\varepsilon > 0$. Choose an order (L)-Dunford-Pettis operator T from X into E such that $\|S' - T'\| = \|S - T\| \leqslant \varepsilon/\|f\|$, and observe that $S'([-f,f]) \subset T'([-f,f]) + \varepsilon B_{X'}$ holds. Since T is order (L)-Dunford-Pettis, $T'([-f,f])$ is an (L) set in X' and hence by Lemma 2.7, $S'([-f,f])$ is an (L) set in X'. This shows that S is order (L)-Dunford-Pettis. And we are done. □

Proposition 2.9. *Consider the scheme of operators* $X \xrightarrow{S} E \xrightarrow{T} F$, *where* X *is a Banach space and* E, F *are two Banach lattices. Then the composed operator* $T \circ S$ *is order* (L)*-Dunford-Pettis whenever one of the following holds:*

(a) T *is order* (L)*-Dunford-Pettis.*

(b) T *is order bounded and* S *is order* (L)*-Dunford-Pettis.*

Proof. (a) Follows immediately from Theorem 2.5 (3).

(b) As $T\colon E \to F$ is an order bounded operator, its adjoint $T'\colon F' \to E'$ is likewise order bounded ([1], Theorem 1.73). Hence, for each $f \in (F')^+$, $T'([-f, f])$ is an order bounded subset of E' and since S is order (L)-Dunford-Pettis, $S'(T'([-f, f]))$ is an (L) set in X'. This proves that $T \circ S$ is order (L)-Dunford-Pettis. $\square$

Corollary 2.10. *Let* E *and* F *be two Banach lattices. Then the following holds:*

(1) *If the lattice operations in* F *are weakly sequentially continuous, then each operator from* E *into* F *is order* (L)*-Dunford-Pettis.*

(2) *If the lattice operations in* E *are weakly sequentially continuous, then each order bounded operator from* E *into* F *is order* (L)*-Dunford-Pettis.*

Clearly, if T is an operator from a Banach space X into Banach lattice E such that T' is AM-compact then T is an order (L)-Dunford-Pettis operator. But the converse is false in general. In fact, the identity operator of the Banach lattice ℓ^∞ is order (L)-Dunford-Pettis, but its adjoint operator, which is the identity operator of $(\ell^\infty)'$, is not AM-compact (because $(\ell^\infty)'$ is not discrete).

Corollary 2.11. *Let* T *be an operator from a Banach space* X *into a Banach lattice* E *such that* X *is reflexive. If* T *is order* (L)*-Dunford-Pettis then* T' *is* AM-*compact.*

Proof. Let $T\colon X \to E$ be an order (L)-Dunford-Pettis operator. Since X is reflexive then the identity operator $I\colon X \to X$ is weakly compact. Hence, by Theorem 2.5, $T' = (T \circ I)'$ is AM-compact. $\square$

It should be noted that the adjoint of an order (L)-Dunford-Pettis operator is not necessary order weakly compact. In fact, the identity operator of the Banach lattice ℓ^1 is Dunford-Pettis (order (L)-Dunford-Pettis) but its adjoint, which is the identity operator of the Banach lattice ℓ^∞ is not order weakly compact (the norm of ℓ^∞ is not order continuous).

Theorem 2.12. *Let $T\colon E \to F$ be an operator between Banach lattices such that the norm of E' is order continuous. If T is order* (L)*-Dunford-Pettis then the adjoint T' is order weakly compact.*

P r o o f. Consider an order (L)-Dunford-Pettis operator $T\colon E \to F$. Then $T'[-f,f]$ is an (L) set in E' for each $f \in (F')^+$. Hence, [3], Theorem 3.1 implies that $T'[-f,f]$ is a relatively weakly compact set in E' for each $f \in (F')^+$, that is, T' is order weakly compact. $\square$

Note that, if T is an operator from a Banach space into a Banach lattice such that T is weak Dunford-Pettis (or T' is order weakly compact) then T is not necessarily order (L)-Dunford-Pettis. In fact, the identity operator of the Banach lattice $L^1([0,1])$ is weak Dunford-Pettis (because $L^1([0,1])$ has the Dunford-Pettis property), but is not order (L)-Dunford-Pettis (because the lattice operations in $L^1([0,1])$ are not weakly sequentially continuous). Also, the identity operator of the Banach lattice $L^2([0,1])$ is not order (L)-Dunford-Pettis (as the lattice operations in $L^2([0,1])$ are not weakly sequentially continuous), however, its adjoint is order weakly compact.

Theorem 2.13. *Consider the scheme of operators $Y \xrightarrow{S} X \xrightarrow{T} E$, where X, Y are two Banach spaces and E is a Banach lattice. Then, if S is weak Dunford-Pettis and the adjoint T' is order weakly compact, then $T \circ S$ is order* (L)*-Dunford-Pettis.*

P r o o f. Let $f \in (E')^+$, then $T'[-f,f]$ is a weakly relatively compact set in X'. Since S is weak Dunford-Pettis, Theorem 2.2 implies that $S'(T'[-f,f])$ is an (L) set in Y' and then $T \circ S$ is order (L)-Dunford-Pettis. $\square$

The following consequence of Theorem 2.13 gives a sufficient condition under which the class of order (L)-Dunford-Pettis operators contains strictly that of weak Dunford-Pettis operators.

Corollary 2.14. *Let X be a Banach space and let E be a Banach lattice such that the norm of E' is order continuous. Then each weak Dunford-Pettis operator $T\colon X \to E$ is order* (L)*-Dunford-Pettis.*

Another consequence of Theorem 2.13 gives a sufficient condition under which each operator from a Banach space into a Banach lattice is order (L)-Dunford-Pettis, whenever its adjoint is order weakly compact.

Corollary 2.15. *Let X be a Banach space with the Dunford-Pettis property and let E be a Banach lattice. Then each operator $T\colon X \to E$ with order weakly compact adjoint is order* (L)*-Dunford-Pettis.*

As a consequence, we obtain the following result, which gives a sufficient condition guaranteeing weakly sequentially continuous lattice operations in a Banach lattice.

Corollary 2.16. *Let E be a Banach lattice. If E has the Dunford-Pettis property and the norm of E' is order continuous, then the lattice operations in E are weakly sequentially continuous.*

Note that there exists an almost Dunford-Pettis operator which is not Dunford-Pettis. Indeed, the identity operator of the Banach lattice $L^1([0,1])$ is almost Dunford-Pettis, but is not Dunford-Pettis.

Theorem 2.17. *Let X and Y be two Banach spaces and let E be a Banach lattice. For the scheme of operators $X \xrightarrow{T} E \xrightarrow{S} Y$, if T is order* (L)*-Dunford-Pettis and S is almost Dunford-Pettis, then $S \circ T$ is a Dunford-Pettis operator.*

Proof. Let (x_n) be a weakly null sequence in X. Since T is order (L)-Dunford-Pettis, we have $|T(x_n)| \to 0$ for $\sigma(E, E')$. From the inequalities $0 \leqslant (T(x_n))^+ \leqslant |T(x_n)|$ and $0 \leqslant (T(x_n))^- \leqslant |T(x_n)|$, we get $(T(x_n))^+ \to 0$ and $(T(x_n))^- \to 0$ for $\sigma(E, E')$. As $S\colon E \to Y$ is almost Dunford-Pettis, we have

$$\|S(T(x_n))\| \leqslant \|S((T(x_n))^+)\| + \|S((T(x_n))^-)\| \to 0.$$

This shows that $S \circ T$ is Dunford-Pettis as desired. □

As a consequence of Theorem 2.17, we obtain the following result [2], Theorem 2.2 (1).

Corollary 2.18. *Let E and Y be a Banach lattice and a Banach space, respectively. If the lattice operations in E are weakly sequentially continuous, then each almost Dunford-Pettis operator from E into Y is Dunford-Pettis.*

The following consequence of Theorem 2.17 gives a sufficient condition under which the class of order (L)-Dunford-Pettis operators coincides with that of Dunford-Pettis operators.

Corollary 2.19. *Let X and E be a Banach space and a Banach lattice, respectively. If E has the positive Schur property, then each order* (L)*-Dunford-Pettis operator from X into E is Dunford-Pettis.*

Clearly, every Dunford-Pettis operator $T\colon X \to E$ is simultaneously order (L)-Dunford-Pettis and weak* Dunford-Pettis. However, there exists an operator $T\colon X \to E$ which is both order (L)-Dunford-Pettis and weak* Dunford-Pettis, but fails to be Dunford-Pettis (e.g. the identity operator $I\colon \ell^\infty \to \ell^\infty$). In our last major result, we characterize Banach lattices E and F on which each operator $T\colon E \to F$ is Dunford-Pettis whenever it is both order (L)- and weak* Dunford-Pettis.

Theorem 2.20. *Let E and F be two Banach lattices such that F is Dedekind σ-complete. Then the following assertions are equivalent:*

(1) *each operator $T\colon E \to F$ is Dunford-Pettis whenever it is both order* (L)*-Dunford-Pettis and weak* Dunford-Pettis;*

(2) *one of the following is valid:*
   (a) *E has the Schur property;*
   (b) *the norm of F is order continuous.*

Proof. (1) $\Rightarrow$ (2): Assume that (2) is false, i.e., the norm of F is not order continuous and E does not have the Schur property. We will construct an operator $T\colon E \to F$ which is weak* Dunford-Pettis and order (L)-Dunford-Pettis but fails to be Dunford-Pettis. To this end, as E does not have the Schur property, there are a weakly null sequence $(x_n) \subset E$, some $\varepsilon > 0$, and a sequence $(f_n) \subset (B_{E'})$ such that $|f_n(x_n)| > \varepsilon$ for all n. Now, consider the operator $P\colon E \to \ell^\infty$ defined by

$$P(x) = \{f_n(x)\}_n.$$

On the other hand, since the norm of F is not order continuous, it follows from Theorem 4.51, [1] that ℓ^∞ is lattice embeddable in F, i.e., there exists a lattice homomorphism $S\colon \ell^\infty \to F$ satisfying

$$m\|(\lambda_k)_k\|_\infty \leqslant \|S((\lambda_k)_k)\| \leqslant M\|(\lambda_k)_k\|_\infty$$

for some positive constants M and m, and for all $(\lambda_k)_k \in \ell^\infty$. Put $T = S \circ P\colon E \to \ell^\infty \to F$, and note that T is weak* Dunford-Pettis (ℓ^∞ has the DP* property), and also, T is order (L)-Dunford-Pettis (as the lattice operations in ℓ^∞ are weakly sequentially continuous). However, for the weakly null sequence $(x_n) \subset E$, we have

$$\|T(x_n)\| = \|S \circ P(x_n)\| = \|S((f_k(x_n))_k)\| \geqslant m\|(f_k(x_n))\|_\infty \geqslant m|f_n(x_n)| \geqslant m\varepsilon$$

for every n. This shows that T is not Dunford-Pettis.

(2.a) $\Rightarrow$ (1): Obvious.

(2.b) $\Rightarrow$ (1): Consider an operator $T\colon E \to F$ which is weak* Dunford-Pettis and order (L)-Dunford-Pettis. Let $(x_n) \subset E$ be a weakly null sequence. We shall show that $\|T(x_n)\| \to 0$. By Corollary 2.6 in [4], it suffices to prove that $|T(x_n)| \to 0$ for $\sigma(F, F')$ and $f_n(T(x_n)) \to 0$ for every disjoint and norm bounded sequence $(f_n) \subset (F')^+$. Indeed:

— as T is order (L)-Dunford-Pettis, hence $|T(x_n)| \to 0$ for $\sigma(F, F')$;

— as the norm of F is order continuous, it follows from Corollary 2.4.3 in [8] that for each disjoint and norm bounded sequence $(f_n) \subset (F')^+$ we have $f_n \to 0$ for $\sigma(F', F)$ and thus $f_n(T(x_n)) \to 0$ since T is weak* Dunford-Pettis operator. This completes the proof. □

Remark. The assumption "F is Dedekind σ-complete" is essential in Theorem 2.20. In fact, if we take $E = \ell^\infty$ and $F = c$ (which is not Dedekind σ-complete), it is known that each operator from ℓ^∞ into c is Dunford-Pettis. But neither ℓ^∞ has the Schur property nor the norm of c is order continuous.

As consequences of Theorem 2.20, we obtain the following results.

Corollary 2.21. *Let E be a Dedekind σ-complete Banach lattice. Then the following assertions are equivalent:*

(1) *each operator $T\colon E \to E$ is Dunford-Pettis whenever it is both order* (L)- *and weak* Dunford-Pettis;*

(2) *the norm of E is order continuous.*

Corollary 2.22. *Let E and F be two Banach lattices such that F is Dedekind σ-complete and discrete. Then the following assertions are equivalent:*

(1) *each weak* Dunford-Pettis operator $T\colon E \to F$ is Dunford-Pettis;*

(2) *one of the following is valid:*

  (a) *E has the Schur property;*

  (b) *the norm of F is order continuous.*

Proof. Only (2.b) $\to$ 1 needs the proof. Indeed, it follows from [8], Proposition 2.5.23, that lattice operations in F are weakly sequentially continuous, i.e., each operator $T\colon E \to F$ is order (L)-Dunford-Pettis. Therefore, the desired conclusion follows from Theorem 2.20. □

Corollary 2.23. *Let E and F be two Banach lattices such that F is Dedekind σ-complete and E or F has the Dunford-Pettis* property. Then the following assertions are equivalent:*

(1) *each order* (L)*-Dunford-Pettis operator $T\colon E \to F$ is Dunford-Pettis;*

(2) *one of the following is valid:*

   (a) *E has the Schur property;*

   (b) *the norm of F is order continuous.*

Corollary 2.24. *Let E be a Dedekind σ-complete Banach lattice. Then the following assertions are equivalent:*

(1) *each order* (L)*-Dunford-Pettis operator $T\colon l^\infty \to E$ is Dunford-Pettis;*

(2) *the norm of E is order continuous.*

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