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Amalendu Ghosh

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COMPLETE RIEMANNIAN MANIFOLDS ADMITTING A PAIR OF EINSTEIN-WEYL STRUCTURES

Amalendu Ghosh, Chandannagar

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Abstract. We prove that a connected Riemannian manifold admitting a pair of non-trivial Einstein-Weyl structures $(g, \pm\omega)$ with constant scalar curvature is either Einstein, or the dual field of ω is Killing. Next, let (M^n, g) be a complete and connected Riemannian manifold of dimension at least 3 admitting a pair of Einstein-Weyl structures $(g, \pm\omega)$. Then the Einstein-Weyl vector field E (dual to the 1-form ω) generates an infinitesimal harmonic transformation if and only if E is Killing.

Keywords: Weyl manifold; Einstein-Weyl structure; infinitesimal harmonic transformation

MSC 2010: 53C25, 53C15, 53C20

1. Introduction

Let M be a smooth manifold with a Riemannian metric g. Then (M, g) is said to be Einstein if there exists a real constant λ such that the Ricci tensor S of g satisfies $S = \lambda g$ (see [1]). In this case, we say that g is an Einstein metric. In recent years, much attention has been given to classification of Riemannian manifolds admitting several generalizations of Einstein metric. An interesting generalization of such metric is the so-called *Einstein-Weyl* metric. This is known as a conformal generalization of Einstein metric, defined in the background of Weyl manifold. A conformal structure on M is a class $[g] = \{e^{2f} g\colon f \in C^\infty(M)\}$ of conformally related Riemannian metrics g of M. By a *Weyl structure* on a manifold M of dimension $\geqslant 3$ we mean a pair $W = ([g], D)$, where D is a torsion free connection which preserves the conformal class $[g]$, i.e. $Dg = -2\omega \otimes g$ for some 1-form ω. A Riemannian manifold together with such a structure is known as a *Weyl manifold*. The Ricci tensor associated with the Weyl connection D is not usually symmetric. Thus, to define

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Einstein-like structures in the framework of conformal geometry one needs to consider the symmetrized Ricci tensor of the Weyl connection D. If this is proportional to a Riemannian metric of the conformal class $[g]$, then we say that the structure is *Einstein-Weyl*. It is evident that every Einstein manifold can be regarded as an Einstein-Weyl manifold. But there exists manifold having no Einstein metric which admits an Einstein-Weyl structure (e.g. see [11], [4] and Example 5.1). As a consequence, the Einstein-Weyl structure can be considered an apt generalization of the Einstein metric (of Riemannian geometry) from the viewpoint of conformal geometry.

Compact Einstein-Weyl structures and their different aspects have been extensively studied by several authors ([4], [5], [6], [9], [11], [14], [15], [20]). In particular, applying a result of Gauduchon [6], Pedersen and Swann [15] proved that on a compact manifold, the Einstein-Weyl equation can be split into a simplified Einstein-Weyl equation and the Killing dual field equation. Such a metric is known as the *Gauduchon metric*. In [15], Pedersen and Swann (see also Higa [9]) first constructed Einstein-Weyl structure on the principal circle bundle over Einstein Kaehler manifolds with positive scalar curvature. Thereafter, the existence and implication of such structures have been confirmed in the framework of Sasakian and K-contact manifolds by Narita [11], Boyer, Galicki and Matzeu [4] and the author [8]. Actually, in the above mentioned papers, several existence results of Einstein-Weyl structures have been constructed. In particular, by deforming the Sasakian Einstein metric (for instance, odd dimensional unit sphere S^{2n+1} with standard metric) one can construct families of Einstein-Weyl structures.

Recall that a Riemannian manifold admits both Einstein-Weyl structures $(g, \pm\omega)$ if and only if it satisfies some additional conditions on the Ricci tensor and the dual field of ω is conformal Killing. In this direction, Boyer, Galicki and Matzeu [4] proved that any η-Einstein (i.e. the Ricci tensor S satisfies $S = \alpha g + \beta\eta\otimes\eta$ for some constants α, β and η is a contact form) K-contact manifold admits an Einstein-Weyl structure $(g, f\eta)$ with $\beta < 0$ and some constant f. In this case, it also admits an Einstein-Weyl structure $(g, -f\eta)$. More generally, they proved that "*A Sasakian manifold* $M^{2n+1}(\varphi, \xi, \eta, g)$ *admits a pair of Einstein Weyl structures* $(g, \pm\omega)$ (*where* ω *is the* 1*-form associated with the metric* g *of the conformal class* $[g]$) *if and only if it is* η*-Einstein with Einstein constants* (α, β)*, where* $\beta < 0$ *and* $\omega = \pm\mu\eta$ *such that* $\mu^2 = -\beta/(2n-1)$". Later on, the author in [8] extended this result by proving that "*A* K*-contact manifold admits a pair of Einstein-Weyl structure* $(g, \pm\omega)$ *if and only if it is* η*-Einstein with Einstein constants* (α, β)*, where* $\beta < 0$". These results imply that there is a strong connection between the pair of Einstein-Weyl structures $(g, \pm\omega)$ and the η-Einstein K-contact (Sasakian) geometry.

In the literature, various aspects of compact Einstein-Weyl structure with Gauduchon metric have been studied in depth by several authors. On the other hand, Riemannian manifolds admitting a pair of Einstein-Weyl structures have been studied mostly on K-contact and Sasakian manifolds. Thus, we are motivated to consider complete Riemannian manifolds admitting a pair of the Einstein-Weyl structures $(g, \pm\omega)$. The outline of the paper is as follows. In Section 2, we present a brief review of Einstein-Weyl structures, infinitesimal harmonic transformations and contact geometry. Section 3 will be devoted to proving that a complete and connected Riemannian manifold admitting a pair of Einstein-Weyl structures $(g, \pm\omega)$ with constant scalar curvature is either Einstein or the dual field of ω is Killing. Finally, we show that the vector field E of a Riemannian manifold (of dimension at least 3) admitting a pair of Einstein-Weyl structures $(g, \pm\omega)$ generates an infinitesimal harmonic transformation if and only if it is Killing.

2. Einstein-Weyl structures, infinitesimal harmonic transformations and contact geometry

Throughout this paper, we denote by R the Riemann curvature tensor, by S the $(0,2)$ Ricci tensor and by Q the $(1,1)$ Ricci operator associated with S. We have already mentioned that a Weyl structure on a Riemannian manifold M of dimension $\geqslant 3$ is defined by the pair $W = ([g], D)$ satisfying

$$Dg = -2\omega \otimes g. \tag{2.1}$$

Equation (2.1) implies

$$D_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X - g(X,Y)E,$$

where ∇ denotes the Levi-Civita connection of g and E the dual vector field of ω with respect to g. We refer to the vector field E as Einstein-Weyl vector field. The pair (g, ω) induces the same Weyl connection as the pair $(\mathrm{e}^{2f} g, \omega + df)$, $f \in C^\infty(M)$. This implies that if the 1-form ω is closed, then the Weyl connection becomes locally Riemannian connection. A Weyl manifold $(M, [g], D)$ is said to be Einstein-Weyl if there exists a smooth function Λ on M such that:

$$S^D(X,Y) + S^D(Y,X) = \Lambda g(X,Y). \tag{2.2}$$

Let M^n admit the Weyl structure (g, ω). Then the Ricci tensor S^D of D and S of ∇ are connected by the following equation (see Higa [9]):

$$\begin{aligned} S^D(X,Y) = S(X,Y) - (n-1)(\nabla_X \omega)Y + (\nabla_Y \omega)X \\ + (n-2)\omega(X)\omega(Y) + (\delta\omega - (n-2)|E|^2)g(X,Y), \end{aligned} \tag{2.3}$$

where $\delta\omega$ is the co-differential of ω and $|E|$ is the length of the vector field with respect to g. Thus, if M admits an Einstein-Weyl structure, then by virtue of (2.2) and (2.3) it follows that (see Higa [9])

$$S(X,Y) - \frac{n-2}{2}((\nabla_X\omega)Y + (\nabla_Y\omega)X) + (n-2)\omega(X)\omega(Y) = \sigma g(X,Y) \tag{2.4}$$

for every part of vector fields X,Y on M, where σ is a smooth function on M. Moreover, if E vanishes, then M becomes Einstein, and the Einstein-Weyl structure is said to be trivial (Einstein).

Recall that M admits a pair of Einstein-Weyl structures $(g, \pm\omega)$ if and only if the following two equations hold for every part of vector fields X,Y in M (Higa [9]):

$$(\nabla_X\omega)Y + (\nabla_Y\omega)X + \frac{2}{n}\delta\omega g(X,Y) = 0, \tag{2.5}$$

$$S(X,Y) - \frac{r}{n}g(X,Y) = \frac{n-2}{n}|E|^2 g(X,Y) - (n-2)\omega(X)\omega(Y), \tag{2.6}$$

where r is the scalar curvature of g. Equation (2.5) shows that ω is conformal Killing form. This structure enjoys a nice property when M is compact. In fact, Gauduchon [6] proved that *on a compact Weyl manifold, up to homothety, there is a unique metric g_0 in the conformal class $[g]$ such that the corresponding 1-form ω_0 is co-closed* (*i.e.* $\delta\omega_0 = 0$). We shall refer to this metric as the *Gauduchon metric.* Moreover, Pedersen and Swann [15] (see also Tod [20]) proved that on a compact Einstein-Weyl manifold this co-closed 1-form becomes the dual of a Killing field. It turns out that on every compact manifold one can split the Einstein-Weyl equation (2.4) into the simplified Einstein-Weyl equation and the Killing dual field equation.

We now present the notion of an *infinitesimal harmonic transformation* on a Riemannian manifold. By an infinitesimal harmonic transformation on a Riemannian manifold (M^n, g) we mean a vector field V such that the local 1-parameter group of infinitesimal point transformations generated by V forms a group of harmonic transformations (see [17], [12], [18]). An interesting characterization of such vector field was given by Stepanov-Shandra in [17]. They proved the following:

Theorem 2.1 (Stepanov-Shandra). *A necessary and sufficient condition for a vector field V to be an infinitesimal harmonic transformation on a Riemannian manifold (M^n, g) is that it satisfies $\Delta V = 2QV$.*

The operator Δ is known as the Laplacian and is determined by the Weitzenbock formula

$$\Delta V = \nabla^*\nabla V + QV,$$

where ∇^* is the formal adjoint of ∇. The rough Laplacian $\overline{\Delta}$ of a vector field V is defined by $\overline{\Delta}V = -\operatorname{tr}\nabla^2 V$, where

$$(\nabla^2 V)(X,Y) = \nabla_X\nabla_Y V - \nabla_{\nabla_X Y}V.$$

In terms of any local orthonormal frame field $\{e_i\}$, we have

$$\overline{\Delta}V = \sum_i \{\nabla_{\nabla_{e_i}e_i} - \nabla_{e_i}\nabla_{e_i}\}V. \tag{2.7}$$

It is well-known that $\overline{\Delta}V = \nabla^*\nabla V$, and therefore $\Delta V = \overline{\Delta}V + QV$. Some well-known examples of infinitesimal harmonic transformations are as follows:

▷ Any Killing vector field on a Riemannian manifold generates an infinitesimal harmonic transformation (see [17]).

▷ The potential vector field V of the Ricci soliton is necessarily an infinitesimal harmonic transformation (see [18]).

▷ The Reeb vector field ξ of a K-contact manifold is Killing, and hence it generates an infinitesimal harmonic transformation (see [16]).

▷ Any vector field V on a contact metric manifold $M^{2n+1}(\varphi,\xi,\eta,g)$ that leaves the tensor φ invariant (i.e., $\pounds_V\varphi = 0$) is necessarily an infinitesimal harmonic transformation (see [7]).

Next, we recall the notion of K-contact and Sasakian manifold. Let M be an odd dimensional Riemannian manifold with (φ,ξ,η,g) as an almost contact metric structure, where

$$\varphi^2 X = -X + \eta(X)\xi, \quad g(X,\xi) = \eta(X), \quad g(\xi,\xi) = 1$$

for any vector field X. This structure is said to be contact metric if $d\eta(X,Y) = g(X,\varphi Y)$. If the vector field ξ (called the Reeb vector field) of the almost contact metric structure (φ,ξ,η,g) is Killing, then M is said to be K-contact. Moreover, this almost contact metric structure will be Sasakian if it satisfies

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(X)Y$$

for any vector field X, Y on M. From the definition it follows that $\eta\wedge(d\eta)^n$ is nonvanishing everywhere on M, which is also the volume form of M. In other words, a contact metric manifold $M(\varphi,\xi,\eta,g)$ is Sasakian if and only if the metric cone $C(M)(dr^2 + r^2 g, d(r^2\eta))$ is Kaehler. A Sasakian manifold is K-contact, but the converse is not true, except in dimension 3. A contact metric manifold M of dimension $(2n+1)$ is said to be η-Einstein in the wider sense, if there exist two functions α, β such that the Ricci tensor can be expressed as

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y).$$

It is well-known (Okumura [13]) that α and β are constant if M is K-contact manifold of dimension at least 5. For details about contact manifolds and Sasakian geometry we refer to Blair [2] and Boyer-Galicki [3].

3. Einstein-Weyl structures with constant scalar curvature

Theorem 3.1. *Let M be a connected Riemannian manifold admitting a pair of nontrivial Einstein-Weyl structures $(g, \pm\omega)$. If the scalar curvature of M is constant, then the vector field E is Killing.*

Proof. Let us suppose that M admits a pair of Einstein-Weyl structures $(g, \pm\omega)$. Then from (2.6) we have

$$S(X,Y) = fg(X,Y) - (n-2)\omega(X)\omega(Y), \tag{3.1}$$

where

$$f = \frac{r + (n-2)|E|^2}{n}. \tag{3.2}$$

Covariant differentiation of (3.1) along Z gives

$$(\nabla_Z S)(X,Y) = (Zf)g(X,Y) - (n-2)\{\omega(X)(\nabla_Z\omega)Y + \omega(Y)(\nabla_Z\omega)X\}.$$

Taking cyclic sum over $\{X, Y, Z\}$ and recalling (2.5) we get

$$\bigoplus_{X,Y,Z} \Big\{(\nabla_Z S)(X,Y) - (Zf)g(X,Y) - \frac{2(n-2)\delta\omega}{n}\omega(Z)g(X,Y)\Big\} = 0, \tag{3.3}$$

where $\bigoplus_{X,Y,Z}$ denotes the cyclic sum over X, Y, Z. Contracting (3.3) over Y and Z provides

$$\frac{2}{n+2}(Xr) = (Xf) + \frac{2(n-2)\delta\omega}{n}\omega(X). \tag{3.4}$$

Using (3.4) in (3.3) and recalling the hypothesis that the scalar curvature r is constant, we get

$$\bigoplus_{X,Y,Z} (\nabla_Z S)(X,Y) = 0.$$

We now rewrite the equation (3.1) as

$$S(X,Y) = fg(X,Y) + b\nu(X)\nu(Y), \tag{3.5}$$

where $b = -(n-2)|E|^2$, $\nu(X) = g(X, V)$ and $V(= E/|E|)$ is a unit vector field. Next, we differentiate (3.5) along an arbitrary vector field Z to obtain

$$(\nabla_Z S)(X,Y) = (Zf)g(X,Y) + (Zb)\nu(X)\nu(Y) + b\{\nu(Y)g(\nabla_Z\nu)X + \nu(X)g(\nabla_Z\nu)Y\}. \tag{3.6}$$

Taking cyclic sum over $\{X, Y, Z\}$ and then recalling that the Ricci tensor is cyclic parallel, the foregoing equation leads to

$$\bigoplus_{X,Y,Z}[(Zf)g(X,Y)+(Zb)\nu(X)\nu(Y)+b\{\nu(Y)g(\nabla_Z\nu)X+\nu(X)g(\nabla_Z\nu)Y\}] = 0. \tag{3.7}$$

Setting $Y = Z = V$ in (3.7) we deduce that

$$2b(\nabla_V\nu)X + Xf + Xb + 2(Vf + Vb)\nu(X) = 0. \tag{3.8}$$

From (3.2) and since the scalar curvature is constant, we have

$$n(Xf) + Xb = 0. \tag{3.9}$$

Further, contracting (3.7) over Y, Z and remembering the fact that r is constant, we find

$$Xf + (Vb)\nu(X) + b(\nabla_V\nu)X - b\nu(X)\delta\nu = 0. \tag{3.10}$$

On the other hand, putting $X = V$ in (3.8) gives $Vf + Vb = 0$. Combining this with (3.9) shows that $Vf = Vb = 0$. Moreover, putting $X = V$ in (3.10) and using the last equation we find that $b\delta\nu = 0$. By virtue of this and $Vb = 0$, equation (3.10) provides $Xf + b(\nabla_V\nu)X = 0$. Using this and $Vf = Vb = 0$, it follows from (3.8) that $Xb - Xf = 0$ and hence, by virtue of (3.9), we ultimately conclude that $Xf = 0 = Xb$. Consequently, the functions f and b are constant on M. This implies that $|E|^2$ is constant on M. Therefore from (2.5) we deduce

$$\frac{1}{n}\delta\omega|E|^2 = -g(\nabla_E E, E) = 0.$$

Since $|E|^2$ is constant, we have either $|E|^2 = 0$, or $|E|^2 \neq 0$. By hypothesis, the former is not possible while the latter implies that E is Killing. This completes the proof. □

Example 3.1. We shall now exhibit an example which satisfies the hypothesis and conclusion of Theorem 5.1. It is known that any unit sphere S^{2n+1} with standard contact metric admits K-contact (Sasakian) structure (e.g. see [2]). Since the unit sphere S^{2n+1} is of constant sectional curvature 1, it is also Einstein. Following Tanno [19], one can deform the metric of the unit sphere so that the resulting metric will be η-Einstein. For this we set

$$\overline{g} = ag + a(a-1)\eta \otimes \eta, \quad \overline{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \overline{\varphi} = \varphi$$

for a positive constant a. Here, we choose $0 < a < 1$. Then

$$g = \frac{1}{a}\overline{g} + \frac{1}{a}\Big(\frac{1}{a} - 1\Big)\overline{\eta} \otimes \overline{\eta}.$$

Applying this D-homothetic deformation, one can prove that the Ricci tensor $\overline{S}$ satisfies (see [19])

$$\overline{S}(X,Y) = \frac{1}{a}[2(n-a+1)\overline{g} + 2(a-1)(n+1)]\overline{\eta}(X)\overline{\eta}(Y),$$

which shows that the Sasakian structure $(\overline{\varphi}, \bar{\xi}, \overline{\eta}, \overline{g})$ is η-Einstein (but not of constant curvature) with constant scalar curvature and $\alpha = 2(n-a+1)/a$, $\beta = 2(a-1)(n+1)/a$. By a suitable choice of the D-homothetic constant a (e.g., $a = 4/5$), it can be shown that the deformed sphere metric admits a pair of Einstein-Weyl structures with $E = 2\bar{\xi}(a-1)(n+1)/a$ (as $\beta < 0$, for details see [4]). In this case, E is nonzero and the deformed sphere metric has constant scalar curvature. Moreover, the Einstein-Weyl vector field E is also Killing.

4. Infinitesimal harmonic transformations and Einstein-Weyl structures

In this section, we study a pair of Einstein-Weyl structures when its Einstein-Weyl vector field E generates an infinitesimal harmonic transformation. It is well known (see [15]) that for a compact Einstein-Weyl manifold with Gauduchon metric, the Einstein-Weyl vector field E is Killing. Therefore, using the formula (which holds for every Killing vector field) (see page 24 of [21])

$$\nabla_X \nabla_Y E - \nabla_{\nabla_X Y} E = R(X,E)Y,$$

and recalling the definition of rough Laplacian, we deduce $\overline{\Delta} E = QE$. Making use of this in the Weitzenbock formula

$$\Delta E = \overline{\Delta} E + QE$$

provides $\Delta E = 2QE$. This shows that E is an infinitesimal harmonic transformation. Thus, we have the following:

Proposition 4.1. *Let (M, g) be a compact Riemannian manifold. If g represents a Gauduchon metric, then the Einstein-Weyl vector field E generates an infinitesimal harmonic transformation.*

In fact, any Killing vector field on a Riemannian manifold generates an infinitesimal harmonic transformation (e.g. see [18]). So one may ask whether the last result is valid for a Riemannian manifold admitting a pair of Einstein-Weyl structures. Precisely we have:

Theorem 4.1. *Let (M^n, g), $n \geqslant 3$ be a complete and connected Riemannian manifold admitting a pair of Einstein-Weyl structure $(g, \pm\omega)$. Then the Einstein-Weyl vector field E generates an infinitesimal harmonic transformation if and only if E is Killing.*

Proof. Since M admits a pair of Einstein-Weyl structures, it follows from (2.5) that the vector field E is conformal Killing. That is, $\pounds_E g = 2\varrho g$, where $\varrho = -\delta\omega/n$. First, we prove that a conformal Killing field gives rise to an infinitesimal harmonic transformation if and only if ϱ $(= -\delta\omega/n)$ is constant. Now, for a conformal vector field we know that (see Yano [21])

$$(\pounds_E \nabla)(X, Y) = (X\varrho)Y + (Y\varrho)X - g(X, Y)D\varrho, \tag{4.1}$$

where D is the gradient operator. Thus, using the identity (see [21])

$$(\pounds_E \nabla)(X, Y) = \nabla_X \nabla_Y E - \nabla_{\nabla_X Y} E - R(X, E)Y$$

in (4.1), we receive

$$\nabla_X \nabla_Y E - \nabla_{\nabla_X Y} E - R(X, E)Y = (X\varrho)Y + (Y\varrho)X - g(X, Y)D\varrho. \tag{4.2}$$

At this point, we take an orthonormal frame $\{e_i \colon i = 1, 2, \ldots, n\}$ and then replace X and Y by e_i in (4.2) to achieve

$$\overline{\Delta} E - QE = (n-2)D\varrho. \tag{4.3}$$

Now suppose that E generates an infinitesimal harmonic transformation. Then $\Delta E = 2QE$. Using this in the Weitzenbock formula gives $\overline{\Delta} E = QE$. Therefore (4.3) shows that $D\varrho = 0$, i.e. ϱ is constant on M. Conversely, if ϱ is constant, then from (4.3) and Weitzenbock formula it follows that E generates an infinitesimal harmonic transformation. To complete the proof we must show that ϱ is actually zero. Now, (3.4) can be written as

$$d\Big(r - \frac{(n+2)}{2} f\Big) = (n+2)\varrho\omega.$$

Applying d to this equation and using $d^2 = 0$, we find $\varrho d\omega = 0$. Since ϱ is constant, we have either $\varrho = 0$, or $\varrho \neq 0$. First, suppose that $\varrho \neq 0$, then ω is closed, i.e. $g(\nabla_X E, Y) - g(\nabla_Y E, X) = 0$. In view of this, (2.5) provides

$$\nabla_X E = \varrho X. \tag{4.4}$$

Making use of this in (4.2) and noting that ϱ is constant, we have $R(X,Y)E = 0$. This gives $S(Y,E) = 0$. By virtue of this, (3.1) implies $(f - (n-2)|E|^2)|E|^2 = 0$. If $E = 0$, then as $\pounds_E g = 2\varrho g$, we see that $\varrho = 0$. This leads to a contradiction. Hence $f = (n-2)|E|^2$. Differentiating this along an arbitrary vector field X and recalling (4.4) yields

$$Xf = 2(n-2)g(\nabla_X E, E) = 2(n-2)\varrho g(X,E),$$

from which it follows that $Df = 2(n-2)\varrho E$. Again, differentiating this and using (4.4) shows that

$$\nabla_X Df = 2(n-2)\varrho^2 X. \tag{4.5}$$

This shows that M admits a special concircular field $f = (n-2)|E|^2$. Since M is complete and $\varrho \neq 0$, applying a result of Ishihara [10], we can conclude that M is locally isometric to the Euclidean space. Hence $R = 0$ and this implies $S = 0$. As a consequence, (3.1) entails that $fg(X,Y) - (n-2)\omega(X)\omega(Y) = 0$. Tracing this yields $nf - (n-2)|E|^2 = 0$. Combining this with $f = (n-2)|E|^2$, we observe that $f = (n-2)|E|^2 = 0$. Again we arrive at a contradiction. Thus the only possibility is that $\varrho = 0$, and hence E is Killing. This completes the proof. $\square$

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Author's address: *Amalendu Ghosh*, Department of Mathematics, Chandernagore College, Strand Road, Chandannagar, District Hooghly, 712 136, West Bengal, India, e-mail: aghosh_70@yahoo.com.