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# NICE CONNECTING PATHS IN CONNECTED COMPONENTS OF SETS OF ALGEBRAIC ELEMENTS IN A BANACH ALGEBRA

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*Dedicated to the memory of Professor Miroslav Fiedler, from two grateful participants in mathematical olympiads*

*Abstract.* Generalizing earlier results about the set of idempotents in a Banach algebra, or of self-adjoint idempotents in a $C^*$-algebra, we announce constructions of nice connecting paths in the connected components of the set of elements in a Banach algebra, or of self-adjoint elements in a $C^*$-algebra, that satisfy a given polynomial equation, without multiple roots. In particular, we prove that in the Banach algebra case every such non-central element lies on a complex line, all of whose points satisfy the given equation. We also formulate open questions.

*Keywords*: Banach algebra; $C^*$-algebra; (self-adjoint) idempotent; connected component of (self-adjoint) algebraic elements; (local) pathwise connectedness; similarity; analytic path; polynomial path; polygonal path; centre of a Banach algebra; distance of connected components

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## 1. Introduction

Let $A$ be a unital complex Banach algebra. Sometimes we will assume that, moreover, $A$ is a $C^*$-algebra.

We let

$$E(A) := \{a \in A\colon\ a^2 = a\}$$

be the set of *idempotents* of $A$, and

$$S(A) := \{a \in A\colon\ a^2 = a = a^*\}$$

the set of *self-adjoint idempotents* for the $C^*$-algebra case.

The *connected components of* $E(A)$ *and of* $S(A)$ have been investigated by many authors. To some of them we will refer later at the respective theorems. An ample literature is given in [1].

*Let*

$$p(\lambda) := \prod_{i=1}^{n} (\lambda - \lambda_i)$$

*be a polynomial over* $\mathbb{C}$*, with all* $\lambda_i$*'s mutually distinct. In the* $C^*$*-algebra case, when considering self-adjoint elements, we will assume that all* $\lambda_i$*'s are real.* (In fact, if $q(\lambda) := \prod\{(\lambda - \lambda_i)\colon\ 1 \leqslant i \leqslant n,\ \lambda_i \in \mathbb{R}\}$, then $p(a) = 0$ and $a = a^*$ imply $q(a) = 0$. Thus below we could use $q(\lambda)$ rather than $p(\lambda)$.) The $\lambda_i$'s are fixed throughout this paper.

We write

$$E_p(A) := \{a \in A\colon\ p(a) = 0\},$$

and

$$S_p(A) := \{a \in A\colon\ p(a) = 0,\ a = a^*\}$$

for the $C^*$-algebra case. Then $E(A)$ and $S(A)$ are special cases of $E_p(A)$ and $S_p(A)$: namely, for $p(\lambda) := \lambda(\lambda - 1)$.

We say that $\{e_1, \ldots, e_n\} \subset A$ is a *partition of unity*, or in the $C^*$-algebra case that $\{e_1, \ldots, e_n\} \subset A$ is a *self-adjoint partition of unity*, if

$$\begin{cases} \{e_1, \ldots, e_n\} \subset E(A),\ \text{or}\ \{e_1, \ldots, e_n\} \subset S(A), \\ \text{and}\ e_i e_j = 0\ \text{for}\ 1 \leqslant i, j \leqslant n\ \text{and}\ i \neq j, \\ \text{and}\ \sum\limits_{i=1}^{n} e_i = 1. \end{cases}$$

The detailed proofs of the statements announced in Section 2 will be published in [7]. The idea of this development originates from personal conversations of the authors at the conference Operator theory and applications: Perspectives and challenges, held in Jurata, Poland, March 18–28, 2010, and from the 2011 lecture by the first named author [6].

## 2. Theorems

The "only if" part of the following Proposition 1 comes from the Riesz decomposition theorem.

**Proposition 1.** *Let $A$ be a unital complex Banach algebra ($C^*$-algebra). Let $a \in A$. Then $a \in E_p(A)$ ($a \in S_p(A)$) if and only if there exists a (self-adjoint) partition of unity $\{e_1, \dots, e_n\}$ such that*

$$a = \sum_{i=1}^{n} \lambda_i e_i.$$

*In the "only if" part, for $a \in E_p(A)$ (for $a \in S_p(A)$) one can choose the $e_i$'s as polynomials of $a$, with complex (real) coefficients, which depend only on the $\lambda_i$'s.*

This representation provides the tool for reducing questions about $E_p(A)$ (about $S_p(A)$) to those about $E(A)$ (about $S(A)$). Of course, for the respective proofs for $E_p(A)$ (for $S_p(A)$) one still has substantial work to do. As an illustration, we include a sketch of proof of Theorem 7 in Section 3.

The distinctness of the $\lambda_i$'s is essential in order that $a$ should have such a simple form. For $T \in A := B(l^2 \oplus l^2)$, having a block matrix form $(T_{ij})_{i,j=1}^2$, which is subdiagonal (i.e., strictly lower triangular), we have $T^2 = 0$, but $T_{21} \in B(l^2)$ can be as complicated as an element of $B(l^2)$ can be.

A *path in a topological space* $X$ is a continuous map $f\colon [0,1] \to X$. We will say that $f(0), f(1) \in X$ *are connected by this path* $f$. By a small abuse of language we will also say that $f([0,1]) \subset X$ *is a path in* $X$ (e.g., for polygonal paths). A topological space $X$ is *pathwise connected* if any two of its points are connected by a path in $X$. A topological space $X$ is *locally pathwise connected* if each point $x \in X$ has a base of (not necessarily open) neighbourhoods consisting of pathwise connected sets.

**Theorem 2.** *Let $A$ be a unital complex Banach algebra and $C$ a connected component of $E_p(A)$. Then $C$ is a relatively open subset of $E_p(A)$. Further, $C$ is locally pathwise connected via each of the following types of paths:*

1) *similarity via an exponential function, i.e., $t \mapsto \mathrm{e}^{-ct} a \mathrm{e}^{ct}$;*
2) *a polynomial path of degree at most three;*
3) *a polygonal path of $n$ segments.*

For $E(A)$, relative openness of $C$ was proved by Zemánek [11], 1) was proved by Zemánek [11], 2) was proved by Esterle [3] and Tremon [10], 3) was proved by Kovarik [4] (cf. also [11]).

**Theorem 3.** *Under the hypotheses of Theorem* 2, *$C$ is pathwise connected via each of the following types of paths:*

1) *similarity via a finite product of exponential functions, i.e., $t \mapsto \mathrm{e}^{-c_m t} \ldots \mathrm{e}^{-c_1 t} \times a\mathrm{e}^{c_1 t} \ldots \mathrm{e}^{c_m t}$;*
2) *a polynomial path;*
3) *a polygonal path.*

*In fact, there is a path satisfying* 1) *and* 2) *simultaneously.*

For $E(A)$, 1) was proved by Zemánek [11], 2) was proved by Esterle [3] and Tremon [10], 3) was proved by Kovarik [4] (cf. also [11]), and the last sentence was proved by [3] and [10].

**Problem.** Does there exist a uniform bound on the "minimum degree" of these polynomial connections, possibly depending on $n$, valid for all Banach algebras? Does such a bound exist, depending on $n$ and on $A$ (or even on $C$)? Even the case of a uniform bound for polynomial connections of idempotents is open, even if we allow dependence of the bound on $A$ (or even on $C$). For some particular cases, see [10] and [8]. ([9] announced a further partial result, but his proof seems to be incorrect.)

Even the "simplest" case $A := B(l^2)$ is open. (The case $A =: B(\mathbb{C}^n)$ is solved affirmatively by [10], the uniform bound being 3, which is sharp. Here the connected components of $E(A)$ consist of the projections of the same rank.) For $A = B(l^2)$, the connected components of $E(A)$ are $\{e \in A\colon \dim N(e) = \alpha,\ \dim R(e) = \beta\}$, where $0 \leqslant \alpha,\ \beta \leqslant \aleph_0$ are cardinalities with $\alpha + \beta = \aleph_0$, cf. [1] ($N(\cdot)$ is the null-space and $R(\cdot)$ is the range). By [8], for $\min\{\alpha, \beta\} < \aleph_0$, in the respective connected component there exists an at most third degree polynomial path between any two elements of that component. But even the case $\alpha = \beta = \aleph_0$ here is open.

**Theorem 4.** *Let $A$ be a unital complex $C^*$-algebra, and $C$ a connected component of $S_p(A)$. Then $C$ is a relatively open subset of $S_p(A)$. Further, $C$ is locally pathwise connected by similarities via exponential functions, i.e., $t \mapsto \mathrm{e}^{-\mathrm{i}ct} a \mathrm{e}^{\mathrm{i}ct}$, where additionally $c = c^*$.*

For $S(A)$, Theorem 4 was proved by Maeda [5] (cf. also [11]).

**Theorem 5.** *Under the hypotheses of Theorem* 4, *$C$ is pathwise connected by similarities via finite products of exponential functions, i.e., $t \mapsto \mathrm{e}^{-\mathrm{i}c_m t} \ldots \mathrm{e}^{-\mathrm{i}c_1 t} a \times \mathrm{e}^{\mathrm{i}c_1 t} \ldots \mathrm{e}^{\mathrm{i}c_m t}$, where additionally $c_1 = c_1^*, \ldots, c_m = c_m^*$.*

For $S(A)$, Theorem 5 was proved by Maeda [5] (cf. also [11]).

For the $C^*$-algebra case, the analogues of 2) and 3) from Theorems 2 and 3 are false for $S_p(A)$. In fact, already the connected component of $S(B(\mathbb{C}^2))$ consisting of all rank-one orthogonal projections does not contain any non-constant polynomial path. (The connected components of $S(B(\mathbb{C}^n))$ consist of orthogonal projections of the same rank.)

**Theorem 6.** *Let $A$ be a unital complex Banach algebra ($C^*$-algebra). Let $a \in E_p(A)$ (let $a \in S_p(A)$). Then $a$ belongs to the centre of $A$ if and only if its connected component in $E_p(A)$ (in $S_p(A)$) is $\{a\}$.*

Theorem 6 for $E(A)$ was proved by Zemánek [11], for $S(A)$ by Maeda [5]. In Theorem 6, of course, the "only if" part for $S_p(A)$ follows from the "only if" part for $E_p(A)$.

**Theorem 7.** *Let $A$ be a unital complex Banach algebra, and $C$ a connected component of $E_p(A)$. If $C$ is disjoint from the centre of $A$, then any element of $C$ belongs to a complex line entirely contained in $C$. In particular, $C$ is unbounded.*

For $E(A)$, Theorem 7 was proved by Zemánek [11].

In the $C^*$-algebra case even the entire $S_p(A)$ has a distance $\max\{|\lambda_i|\colon\ 1 \leqslant i \leqslant n\}$ from 0, so the analogue of Theorem 7 for $S_p(A)$ is false for *each* $C^*$-algebra $A$.

Theorem 6 and Theorem 7 yield the next Corollary 8.

**Corollary 8.** *Let $A$ be a unital complex Banach algebra. Then $E_p(A)$ is a union of its isolated points and of complex lines.*

**Theorem 9.** *There exists an explicit constant $c(\lambda_1, \ldots, \lambda_n) > 0$ (depending on $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, and invariant under any map $(\lambda_1, \ldots, \lambda_n) \mapsto (a + b\lambda_1, \ldots, a + b\lambda_n)$ with $a, b \in \mathbb{R}$ and $b \neq 0$) such that the following holds. If $A$ is a unital complex $C^*$-algebra, and $C_1$, $C_2$ are distinct connected components of $S_p(A)$, then the distance of $C_1$ and $C_2$ is at least $c(\lambda_1, \ldots, \lambda_n) \min\{|\lambda_i - \lambda_j|\colon\ 1 \leqslant i, j \leqslant n,\ i \neq j\}$.*

**Conjecture.** Let $A$ be a unital complex Banach algebra ($C^*$-algebra) and $C_1, C_2$ distinct connected components of $E_p(A)$ (of $S_p(A)$). Then the distance of $C_1$ and $C_2$ is at least $\min\{|\lambda_i - \lambda_j|\colon\ 1 \leqslant i, j \leqslant n,\ i \neq j\}$.

For $n = 2$ this conjecture is equivalent to the statement that this distance for $E_p(A) := E(A)$ (for $S_p(A) := S(A)$) is at least 1, which is due to Zemánek [11] (due to Maeda [5]). For $n \geqslant 3$ we do not even know whether this distance for the Banach algebra case is positive.

If true, this conjecture would be sharp, for any Banach algebra: consider $\lambda_i \cdot 1$ and $\lambda_j \cdot 1$.

The Conjecture for the case of $S_p(A)$ would follow from the Conjecture in the case of $E_p(A)$. In fact, different connected components of $S_p(A)$ are subsets of different connected components of $E_p(A)$, by [2], Section 1, Applications, 2), also taking into consideration our Proposition 1 and Theorem 3.

## 3. A proof

P r o o f of Theorem 7. The proof of Theorem 7 follows from Theorem 3 and Theorem 6. If $C$ is disjoint from the centre, then by Theorem 6 it has more than one elements. Let $a_0 \in C$ be an arbitrary element of $C$, and let $a_1 \in C$, with $a_1 \neq a_0$. Then, by Theorem 3, 3), there exists a non-constant polygonal path connecting $a_0$ to $a_1$ in $C$. Its first non-constant segment (counted from $a_0$) is the graph of a non-constant polynomial of degree 1, say of

$$\lambda \mapsto a_0 + b\lambda, \quad \text{from} \quad [0,1] \quad \text{to} \quad C \ (\subset E_p(A) \subset A).$$

Hence

$$b \neq 0 \quad \text{and we have for all } \lambda \in [0,1] \text{ identically } p(a_0 + b\lambda) = 0. \tag{1}$$

Then the equation in (1) is a polynomial equation, with coefficients from $A$ and of degree at most $n$, for $\lambda \in \mathbb{C}$. (Attention: here the coefficient of $\lambda^n$ is $b^n$, which may be 0 even for $b \neq 0$.)

*We make an indirect assumption. If the polynomial*

$$\mathbb{C} \ni \lambda \mapsto p(a_0 + b\lambda) \in A \tag{2}$$

*were not identically* 0 *for all* $\lambda \in \mathbb{C}$, then for some $\lambda_0 \in \mathbb{C}$ we would have

$$p(a_0 + b\lambda_0) \neq 0.$$

Then for some continuous linear functional $a'$ on $A$ we would have

$$\langle p(a_0 + b\lambda_0), a' \rangle \neq 0.$$

The polynomial

$$\mathbb{C} \ni \lambda \mapsto \langle p(a_0 + b\lambda), a' \rangle \in \mathbb{C} \tag{3}$$

is a $\mathbb{C}$-valued polynomial on $\mathbb{C}$ of degree at most $n$, which would not vanish at $\lambda_0 \in \mathbb{C}$. Hence *the polynomial in* (3) *would have at most* $n$ *distinct roots.*

However, by (1) we have that *the polynomial in* (3) *vanishes for all* $\lambda \in [0,1]$ *identically.* This is a contradiction, showing that our indirect assumption is false.

That is, the polynomial in (2) is identically 0 for all $\lambda \in \mathbb{C}$. In other words, for all $\lambda \in \mathbb{C}$ we have

$$p(a_0 + b\lambda) = 0, \quad \text{i.e.,} \quad a_0 + b\lambda \in E_p(A),$$

which implies by connectedness of $\mathbb{C}$ that for all $\lambda \in \mathbb{C}$ we have even

$$a_0 + b\lambda \in C.$$

Since by (1) $b \neq 0$, we see that $C$ *contains a complex line passing through its arbitrary point* $a_0$. □

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