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GEOMETRY AND INEQUALITIES OF GEOMETRIC MEAN

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Abstract. We study some geometric properties associated with the t-geometric means $A \sharp_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ of two $n \times n$ positive definite matrices A and B. Some geodesical convexity results with respect to the Riemannian structure of the $n \times n$ positive definite matrices are obtained. Several norm inequalities with geometric mean are obtained. In particular, we generalize a recent result of Audenaert (2015). Numerical counterexamples are given for some inequality questions. A conjecture on the geometric mean inequality regarding m pairs of positive definite matrices is posted.

Keywords: geometric mean; positive definite matrix; log majorization; geodesics; geodesically convex; geodesic convex hull; unitarily invariant norm

MSC 2010: 15A45, 15B48

1. Introduction

Let $\mathbb{P}_n$ be the set of $n \times n$ positive definite matrices over $\mathbb{C}$. For $t \in [0,1]$, the *t-geometric mean* of $A, B \in \mathbb{P}_n$ is

$$A \sharp_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}. \tag{1.1}$$

When $t = 1/2$, $A \sharp B := A \sharp_{1/2} B$ is called the geometric mean of A and B; it was first introduced in [17] and is often denoted by $A \sharp B$ in the literature. See [1], [6], [5], and [17]. The t-geometric mean is interesting from the point of view of Riemannian geometry since $\mathbb{P}_n$ is a Riemannian manifold with the Riemannian metric in [6]

$$\delta(A,B) = \left(\sum_{i=1}^{n} \log^2 \lambda_i(A^{-1}B)\right)^{1/2}, \quad A, B \in \mathbb{P}_n,$$

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which is the distance between A and B so that the curve $\beta(t) = A \sharp_t B$, $0 \leqslant t \leqslant 1$, is the unique geodesic joining A and B in $\mathbb{P}_n$, see [6], page 205.

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be in $\mathbb{R}^n$. Let $x^{\downarrow} = (x_{[1]}, x_{[2]}, \ldots, x_{[n]})$ denote a rearrangement of the components of x such that $x_{[1]} \geqslant x_{[2]} \geqslant \ldots \geqslant x_{[n]}$. We say that x is *majorized* by y, denoted by $x \prec y$, if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

We say that x is weakly majorized by y if $\sum\limits_{i=1}^{k} x_{[i]} \leqslant \sum\limits_{i=1}^{k} y_{[i]}$, $k = 1, 2, \ldots, n$, denoted by $x \prec_w y$. If $x > 0$ (i.e., $x_i > 0$ for $i = 1, \ldots, n$) and $y > 0$, we say that x is *log majorized* by y, denoted by $x \prec_{\log} y$, if

$$\prod_{i=1}^{k} x_{[i]} \leqslant \prod_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1, \quad \text{and} \quad \prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]}.$$

In other words, $x \prec_{\log} y$ if and only if $\log x \prec \log y$.

Recall that a norm $|||\cdot|||$ on the algebra $\mathbb{C}_{n\times n}$ of $n \times n$ complex matrices is unitarily invariant if $|||UAV||| = |||A|||$ for any $U, V \in \mathrm{U}(n)$ and any $A \in \mathbb{C}_{n\times n}$, where $\mathrm{U}(n)$ denotes the unitary group; for example, the spectral norm $\|\cdot\|$ is a unitarily invariant norm. Ky Fan dominance theorem, see [7], asserts that given $A, B \in \mathbb{C}_{n\times n}$, $s(A) \prec_w s(B)$ if and only if $|||A||| \leqslant |||B|||$ for all unitarily invariant norms $|||\cdot|||$, where $s(A) = (s_1(A), \ldots, s_n(A))$ denotes the vector of singular values of $A \in \mathbb{C}_{n\times n}$ in descending order.

In this paper we prove several inequalities concerning the geometric mean; some of them are in terms of log majorization. In the next section we study geodesic convexity of the t-geometric means. In Section 3 we consider some specific unitarily invariant norms with t-geometric means and generalize a recent result of Audeaert, see [4]. Some inequalities for the 1-norm are also obtained.

2. Geodesic convexity of t-geometric means

For $H \in \mathbb{C}_{n\times n}$ with positive eigenvalues, let $\lambda(H) = (\lambda_1(H), \ldots, \lambda_n(H))$ denote the vector of eigenvalues of H such that $\lambda_1(H) \geqslant \ldots \geqslant \lambda_n(H)$. We say that $A \prec_{\log} B$, $A, B \in \mathbb{P}_n$ if $\lambda(A) \prec_{\log} \lambda(B)$. We would like to point out that $C \prec_{\log} A$ and $D \prec_{\log} B$ do not imply $C \sharp D \prec_{\log} A \sharp B$, for example, for $C, D = \mathrm{diag}(2, 1/2)$ and $A = \mathrm{diag}(3, 1/3)$ and $B = A^{-1} = \mathrm{diag}(1/3, 3)$ we have $C \sharp D = C$ and $A \sharp B = I$. However, it is true that $X \leqslant U$ and $Y \leqslant V$ imply $X \sharp Y \leqslant U \sharp V$, where $\leqslant$ denotes the Loewner order on $\mathbb{P}_n$, see [11], Theorem 2.2. So, Loewner order is a stronger one, in comparison with $\prec_{\log}$, from the point of view that it respects the geometric mean.

The following interesting results can be found in the paper of Bhatia and Grover [8], page 730.

Theorem 2.1. *Let $A, B \in \mathbb{P}_n$. For any $t \in [0,1]$ and $s > 0$,*

$$\begin{aligned}(2.1)\qquad \lambda(A \sharp_t B) &\prec_{\log} \lambda(\mathrm{e}^{(1-t)\log A + t\log B}) \\ &\prec_{\log} \lambda(B^{ts/2}A^{(1-t)s}B^{ts/2})^{1/s} = \lambda(A^{(1-t)s}B^{ts})^{1/s}.\end{aligned}$$

The first inequality is a result of Ando and Hiai, see [2], Corollary 2.3, as the complementary counterpart of the famous Golden-Thompson inequality $\operatorname{tr}\mathrm{e}^{A+B} \leqslant \operatorname{tr}\mathrm{e}^{A}\mathrm{e}^{B}$ for Hermitian matrices A and B. The second inequality follows from a result of Araki in [3].

Proposition 2.2. *Let $A, B \in \mathbb{P}_n$ and $t \in [0,1]$ and $s > 0$. For all unitarily invariant norms $|||\cdot|||$ on $\mathbb{C}_{n\times n}$,*

$$(2.2)\qquad |||A \sharp_t B||| \leqslant |||(B^{ts/2}A^{(1-t)s}B^{ts/2})^{1/s}||| \leqslant |||(A^{(1-t)s}B^{ts})^{1/s}|||.$$

In particular, with $s = 1$, $t = 1/2$,

$$|||A^2 \sharp B^2||| \leqslant \min\{|||A^{1/2}BA^{1/2}|||, |||B^{1/2}AB^{1/2}|||\} \leqslant \min\{|||AB|||, |||BA|||\}$$

and

$$|||(A \sharp B)^2||| \leqslant \min\{|||AB|||, |||BA|||\}.$$

P r o o f. Since $A \sharp_t B \in \mathbb{P}_n$ for all $t \in [0,1]$, $s(A \sharp_t B) = \lambda(A \sharp_t B) \prec_{\log} \lambda(B^{ts/2} \times A^{(1-t)s}B^{ts/2})^{1/s}$ for all $t \in [0,1]$ and $s > 0$, by Theorem 2.1. By Weyl's theorem, see [7], on the singular values and the eigenvalues of a matrix, $\lambda(B^{ts/2}A^{(1-t)s} \times B^{ts/2})^{1/s} \prec_{\log} s(B^{ts/2}A^{(1-t)s}B^{ts/2})^{1/s}$. Then applying Ky Fan dominance theorem we have $|||A \sharp_t B||| \leqslant |||(B^{ts/2}A^{(1-t)s}B^{ts/2})^{1/s}|||$. The second inequality follows from

$$\begin{aligned} s(B^{ts/2}A^{(1-t)s}B^{ts/2})^{1/s} &= \lambda(B^{ts/2}A^{(1-t)s}B^{ts/2})^{1/s} \\ &= \lambda(A^{(1-t)s}B^{ts})^{1/s} \\ &\prec_{\log} (A^{(1-t)s}B^{ts})^{1/s}\end{aligned}$$

and the Ky Fan dominance theorem so we just proved (2.2). The last inequality follows from Theorem 2.1. $\square$

For any given $A \in \mathbb{P}_n$, define

$$M(A) := \{B \in \mathbb{P}_n \colon \lambda(B) \prec_{\log} \lambda(A)\} \subset \mathbb{P}_n.$$

The set is convex in the multiplicative sense since the exponential map $\exp\colon \mathbb{H}_n \to \mathbb{P}_n$ is a diffeomorphism, where $\mathbb{H}_n$ is the set of all $n \times n$ Hermitian matrices, so $\log\colon \mathbb{P}_n \to \mathbb{H}_n$ is defined; thus

$$\log(M(A)) = \{H\colon H \in \mathbb{C}_{n\times n} \text{ is Hermitian and } \lambda(H) \prec \lambda(\log A)\}$$

is a convex set. Indeed, it is equal to the convex hull of the set consisting of all Hermitian matrices with spectrum majorized by $\lambda(\log(A))$ according to a result of Thompson in [18], Theorem 12. However, one can easily see that $M(A)$ is not closed under usual matrix addition. So $M(A)$ is not convex in $\mathbb{P}_n$ when $\mathbb{P}_n$ is viewed as a subset of the Euclidean space $\mathbb{C}_{n\times n}$, in which $\mathbb{P}_n$ is a cone. We will see that $M(A)$ is convex, which is associated with the Riemannian structure of $\mathbb{P}_n$. To this end, we say that $C \subset \mathbb{P}_n$ is *geodesically convex* [16], page 67, if all geodesics between any two points lie in $\mathbb{P}_n$, that is, the role of straight lines in the Euclidean space $\mathbb{C}_{n\times n}$ is now replaced by geodesics. Similarly we define the geodesic convex hull, see [16], page 68, of a subset S in $\mathbb{P}_n$ to be the smallest geodesically convex set that contains S. As we have mentioned, $A \sharp_t B$ is the geodesic joining A and B. It turns out that $M(A)$ is also convex with respect to the Riemannian structure of $\mathbb{P}_n$. As a corollary, $M(A)$ is closed under the action of A via the t-geometric means. We would like to point out that the geometry of $\mathbb{P}_n$ (or at least the subset $\mathbb{P}_n^1$ of matrices of determinant 1 in $\mathbb{P}_n$) is hyperbolic since $\mathbb{P}_n^1$ is a symmetric space of the noncompact type.

Theorem 2.3. *Let $A \in \mathbb{P}_n$. The set $M(A) = \{B \in \mathbb{P}_n\colon \lambda(B) \prec_{\log} \lambda(A)\} \subset \mathbb{P}_n$ is geodesically convex with respect to the Riemannian structure of $\mathbb{P}_n$. In other words, if $B, C \in M(A)$, then the geodesic joining B and C lies in $M(A)$. So $M(A) = \{A \sharp_t B\colon t \in [0,1],\ B \in \mathbb{P}_n,\ \lambda(B) \prec_{\log} \lambda(A)\}$.*

Proof. By Proposition 2.2, if $t \in (0,1)$, then

$$\|B \sharp_t C\| \leqslant \|B^{1-t}C^t\| = \|(B^{(1-t)/t})^t C^t\| \leqslant \|B^{(1-t)/t}C\|^t$$

by [7], Theorem IX.2.1. Now

$$\|B \sharp_t C\| \leqslant \|B^{(1-t)/t}C\|^t \leqslant \|B^{(1-t)/t}\|^t\|C\|^t = \|B\|^{1-t}\|C\|^t, \tag{2.3}$$

which is no greater than $\|A\|$ since $\lambda(B), \lambda(C) \prec_{\log} \lambda(A)$. So

$$\lambda_1(B \sharp_t C) = s_1(B \sharp_t C) = \|B \sharp_t C\| \leqslant \|A\| = s_1(A) = \lambda_1(A). \tag{2.4}$$

Denote by $C_k(X)$ the k-th component of $X \in \mathbb{C}_{n\times n}$, $k = 1, \ldots, n$. Note that for any $X, Y \in \mathbb{P}_n$,

$$
\begin{aligned}
C_k(X \sharp_t Y) &= C_k(X^{1/2}(X^{-1/2}YX^{-1/2})^t X^{1/2}) \\
&= C_k(X^{1/2}) C_k((X^{-1/2}YX^{-1/2})^t) C_k(X^{1/2}) \\
&= C_k^{1/2}(X)(C_k^{-1/2}(X)C_k(Y)C_k^{-1/2}(X))^t C_k^{1/2}(X) \\
&= C_k(X) \sharp_t C_k(Y).
\end{aligned} \tag{2.5}
$$

In other words, C_k respects $\sharp_t$ in $\mathbb{P}_n$. Note that the $\binom{n}{k}$ eigenvalues of $C_k(X)$, where $X \in \mathbb{C}_{n\times n}$, are the $\binom{n}{k}$ possible products of any k eigenvalues of X. So

$$
\lambda_1(C_k(B \sharp_t C)) = \prod_{i=1}^{k} \lambda_i(B \sharp_t C), \quad k = 1, \ldots, n-1, \tag{2.6}
$$

and

$$
\det(B \sharp_t C) = (\det B)^{1-t}(\det C)^t = \det A. \tag{2.7}
$$

Applying (2.3) to $C_k(B)$ and $C_k(C)$ that are both positive definite, we have

$$
\begin{aligned}
\prod_{i=1}^{k} \lambda_i(B \sharp_t C) &= \lambda_1(C_k(B) \sharp_t C_k(C)) \quad \text{by (2.6)} \\
&= \|C_k(B) \sharp_t C_k(C)\| \\
&\leqslant \|C_k(B)\|^t \|C_k(C)\|^{1-t} \quad \text{by (2.3)} \\
&= \left(\prod_{i=1}^{k} \lambda_i(B)\right)^t \left(\prod_{i=1}^{k} \lambda_i(C)\right)^{1-t} \\
&\leqslant \prod_{i=1}^{k} \lambda_i(A), \quad i = 1, \ldots, n-1.
\end{aligned}
$$

Together with (2.7), we conclude that $\lambda(B \sharp_t C) \prec_{\log} \lambda(A)$. □

Corollary 2.4. *If $A, B \in \mathbb{P}_n$ are such that $B \prec_{\log} A$, then for all $t \in [0,1]$ we have $A \sharp_t B \prec_{\log} A$, or equivalently, $|||A \sharp_t B||| \leqslant |||A|||$ for all unitarily invariant norms $|||\cdot|||$ on $\mathbb{C}_{n\times n}$.*

We are going to show that $M(A)$ has a nice geometric description and our proof requires a lemma, which is of independent interest. Let $\alpha, \beta \in \mathbb{R}^n$. We say that β is a pinch of α (see [15], page 17), if $\beta = (\lambda I + (1-\lambda)Q)\alpha$, where Q is the permutation matrix that interchanges two coordinates. It is well known that if $\beta \prec \alpha$, then

β can be obtained by applying at most n pinches consecutively, starting from α. The converse is clearly true. Now let $\alpha, \beta \in \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ denotes the set of all positive n-tuples. If $\beta \prec_{\log} \alpha$, then β can be obtained by applying at most n pinches multiplicatively in the following sense. We say that β is a *geometric pinch* of α if

$$\operatorname{diag}(\beta_1, \ldots, \beta_n) = (Q^{\mathrm{T}} \operatorname{diag}(\alpha_1, \ldots, \alpha_n) Q) \sharp_t \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$$

for some $t \in [0,1]$ and some transposition matrix Q.

Lemma 2.5. *Let $\alpha, \beta \in \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ denotes the set of all positive n-tuples. If $\beta \prec_{\log} \alpha$, then β can be obtained by applying at most n geometric pinches consecutively, starting from α.*

Proof. Since $\log \beta \prec \log \alpha$, $\log \beta$ can be obtained by at most n pinches from $\log \alpha$. Let $\log \widehat{\alpha}$ be a pinch of $\log \alpha$. Without loss of generality, we may assume that the pinch occurs on the first two coordinates. So $(\widehat{\alpha}_1, \widehat{\alpha}_2) \prec_{\log} (\alpha_1, \alpha_2)$ and thus $\widehat{\alpha}_1 = \alpha_1^t \alpha_2^{1-t}$ and $\widehat{\alpha}_2 = \alpha_2^t \alpha_1^{1-t}$ for some $t \in [0,1]$. Let P denote the matrix corresponding to the transposition switching the first two coordinates. Then

$$\begin{aligned}
(P^{\mathrm{T}} \operatorname{diag}&(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) P) \sharp_t \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \\
&= \operatorname{diag}(\alpha_2, \alpha_1, \alpha_3, \ldots, \alpha_n) \sharp_t \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \\
&= \operatorname{diag}(\alpha_1^t \alpha_2^{1-t}, \alpha_2^t \alpha_1^{1-t}, \alpha_3, \ldots, \alpha_n) \\
&= \widehat{\alpha}.
\end{aligned}$$

Then repeat the process to conclude that there exist $t_1, \ldots, t_k \in [0,1]$ and transposition matrices $P_1, \ldots, P_k$ such that

$$\operatorname{diag} \alpha^{(i+1)} := (P_i^{\mathrm{T}} (\operatorname{diag} \alpha^{(i)}) P_i) \sharp_{t_i} \operatorname{diag} \alpha^{(i)}, \quad i = 1, \ldots, k,$$

where $\alpha^{(1)} := \alpha$ and $\alpha^{(k+1)} := \beta$. $\square$

Theorem 2.6. *The set $M(A)$ is the geodesic convex hull, denoted by $G(A)$, of the orbit*

$$O(A) := \{UAU^* \colon U \in \mathrm{U}(n)\} = \{B \in \mathbb{P}_n \colon \lambda(B) = \lambda(A)\}$$

consisting of all $B \in \mathbb{P}_n$ whose spectrum coincides with that of A.

Proof. By the spectral theorem, it is easy to see that $O(A) := \{UAU^* \colon U \in \mathrm{U}(n)\}$ is equal to $\{B \in \mathbb{P}_n \colon \lambda(B) = \lambda(A)\} \subset M(A)$. So $G(A) \subset M(A)$ as $M(A)$ is geodesically convex by Theorem 2.3. Thus it suffices to show that $M(A) \subset G(A)$. Let $B \in M(A)$, that is, $\lambda(B) \prec_{\log} \lambda(A)$. Since $M(A)$, $O(A)$ and thus $G(A)$ are invariant under unitary similarity, we may assume that $A = \operatorname{diag}(\alpha_1, \dots, \alpha_n)$. Let $\lambda(B) = (\beta_1, \dots, \beta_n)$ and let $U \in \mathrm{U}(n)$ be such that $B = U^* \operatorname{diag}(\beta_1, \dots, \beta_n) U$. By Lemma 2.5, there exist $t_1, \dots, t_k \in [0,1]$ and transposition matrices $P_1, \dots, P_k$ such that

$$\operatorname{diag} \alpha^{(i+1)} := (P_i^{\mathrm{T}} (\operatorname{diag} \alpha^{(i)}) P_i) \sharp_{t_i} \operatorname{diag} \alpha^{(i)}, \quad i = 1, \dots, k,$$

where $\alpha^{(1)} := \alpha$ and $\alpha^{(k+1)} := \beta$. It is easy to see that $V^*(C \sharp_t D)V = (V^* C V) \sharp_t (V^* D V)$ for all $V \in \mathrm{U}(n)$, $C, D \in \mathbb{P}_n$. So

$$\begin{aligned} B &= U^* \operatorname{diag}(\beta_1, \dots, \beta_n) U \\ &= U^* (P_k^{\mathrm{T}} (\operatorname{diag} \alpha^{(k)}) P_k) \sharp_{t_k} \operatorname{diag} \alpha^{(k)}) U \\ &= (U^* P_k^{\mathrm{T}} (\operatorname{diag} \alpha^{(k)}) P_k U) \sharp_{t_k} (U^* \operatorname{diag} \alpha^{(k)} U). \end{aligned}$$

Then use induction on k to show that $B \in G(A)$ as $U^* P_k^{\mathrm{T}} (\operatorname{diag} \alpha^{(k)}) P_k U$ and $U^* \operatorname{diag} \alpha^{(k)} U \in G(A)$ since $G(A)$ is invariant under unitary similarity. $\square$

3. Norm inequalities for t-geometric means

Bourin and Uchiyama in [10], Theorem 1.1, proved that for any positive semidefinite matrices $A, B \in \mathbb{C}_{n \times n}$ and any nonnegative concave function $f\colon [0, \infty) \to [0, \infty)$,

$$|||f(A+B)||| \leqslant |||f(A) + f(B)|||,$$

where $|||\cdot|||$ denotes any unitarily invariant norm. This is a noncommutative version of the well-known inequality for nonnegative concave functions f on $[0, \infty)$:

$$f(a+b) \leqslant f(a) + f(b), \quad a, b \geqslant 0.$$

Bourin in [9] asked a related question: Given $A, B \geqslant 0$ and $p, q > 0$, is it true that

$$|||A^{p+q} + B^{p+q}||| \leqslant |||(A^p + B^p)(A^q + B^q)|||?$$

Hayajneh and Kittaneh in [12] gave an affirmative answer for the trace norm $\|\cdot\|_1$ and the Hilbert-Schmidt norm $\|\cdot\|_2$. Recently, Audenaert in [4] proved that if $A_i, B_i \in \mathbb{P}_n$, $i = 1, \dots, m$, are such that $A_i B_i = B_i A_i$, then for any unitarily invariant norm $|||\cdot|||$ on $\mathbb{C}_{n \times n}$,

$$\left|\left|\left| \sum_{i=1}^m A_i B_i \right|\right|\right| \leqslant \left|\left|\left| \left(\sum_{i=1}^m A_i^{1/2} B_i^{1/2} \right)^2 \right|\right|\right| \leqslant \left|\left|\left| \left(\sum_{i=1}^m A_i \right) \left(\sum_{i=1}^m B_i \right) \right|\right|\right|. \tag{3.1}$$

In particular, this result confirms a conjecture of Hayajneh and Kittaneh in [12] and answers a question of Bourin. Very recently Lin in [13] gave another proof of inequality (3.1).

Note that the geometric mean is a symmetric operation, that is, $A \sharp B = B \sharp A$, $A, B \in \mathbb{P}_n$ and that $A^2 \sharp B^2 = (A \sharp B)^2 = AB$ for commuting matrices A and B. We also have $|||(A \sharp B)^2||| \leqslant |||AB|||$ and $|||A^2 \sharp B^2||| \leqslant |||AB|||$ by Proposition 2.2. So we ask the following questions.

Question 1. Let $A_i, B_i \in \mathbb{P}_n$. Is it true that

$$\left|\left|\left| \sum_{i=1}^{m} A_i^2 \sharp B_i^2 \right|\right|\right| \leqslant \left|\left|\left| \sum_{i=1}^{m} A_i B_i \right|\right|\right| \tag{3.2}$$

and

$$\left|\left|\left| \sum_{i=1}^{m} A_i^2 \sharp B_i^2 \right|\right|\right| \leqslant \left|\left|\left| \left(\sum_{i=1}^{m} A_i\right)\left(\sum_{i=1}^{m} B_i\right) \right|\right|\right|? \tag{3.3}$$

Question 2. For $A_i, B_i \in \mathbb{P}_n$, is it true that

$$\left|\left|\left| \sum_{i=1}^{m} (A_i \sharp B_i)^2 \right|\right|\right| \leqslant \left|\left|\left| \sum_{i=1}^{m} A_i B_i \right|\right|\right| \tag{3.4}$$

and

$$\left|\left|\left| \sum_{i=1}^{m} (A_i \sharp B_i)^2 \right|\right|\right| \leqslant \left|\left|\left| \left(\sum_{i=1}^{m} A_i\right)\left(\sum_{i=1}^{m} B_i\right) \right|\right|\right|? \tag{3.5}$$

Inequalities (3.2) and (3.4) are false in general (we thank the referee for pointing this out). See the Appendix for their counterexamples. Moreover, the left hand sides in (3.2) and (3.4) are not comparable. However, in the next theorem we prove inequality (3.5) which generalizes inequality (3.1) as the matrices in $\mathbb{P}_n$ in inequality (3.5) are not necessarily commuting pairwisely.

Theorem 3.1. *Let $A_i, B_i \in \mathbb{P}_n$, $i = 1, \ldots, m$. Then for any unitarily invariant norm $|||\cdot|||$ on $\mathbb{C}_{n\times n}$, we have*

$$\begin{aligned} \left|\left|\left| \sum_{i=1}^{m} (A_i \sharp B_i)^2 \right|\right|\right| &\leqslant \left|\left|\left| \left(\sum_{i=1}^{m} A_i\right)^{1/2} \left(\sum_{i=1}^{m} B_i\right)\left(\sum_{i=1}^{m} A_i\right)^{1/2} \right|\right|\right| \\ &\leqslant \left|\left|\left| \left(\sum_{i=1}^{m} A_i\right)\left(\sum_{i=1}^{m} B_i\right) \right|\right|\right|. \end{aligned} \tag{3.6}$$

P r o o f. Applying [10], Theorem 1.2, to the convex function t^2, we have

$$\Big|\Big|\Big|\sum_{i=1}^{m}(A_i \sharp B_i)^2\Big|\Big|\Big| \leqslant \Big|\Big|\Big|\Big(\sum_{i=1}^{m} A_i \sharp B_i\Big)^2\Big|\Big|\Big|. \tag{3.7}$$

On the other hand, by the concavity of the geometric mean, we have

$$\Big|\Big|\Big|\sum_{i=1}^{m} A_i \sharp B_i\Big|\Big|\Big| \leqslant \Big|\Big|\Big|\Big(\sum_{i=1}^{m} A_i\Big) \sharp \Big(\sum_{i=1}^{m} B_i\Big)\Big|\Big|\Big|.$$

Consequently,

$$\Big|\Big|\Big|\Big(\sum_{i=1}^{m} A_i \sharp B_i\Big)^2\Big|\Big|\Big| \leqslant \Big|\Big|\Big|\Big(\Big(\sum_{i=1}^{m} A_i\Big) \sharp \Big(\sum_{i=1}^{m} B_i\Big)\Big)^2\Big|\Big|\Big|. \tag{3.8}$$

Combining (3.7) and (3.8) and Proposition 2.2, we get (3.6). □

Remark 3.2. When $A_iB_i = B_iA_i$ for all $i = 1, \ldots, m$, inequalities in (3.1) follow from the proof of Theorem 3.1.

For the trace norm $\|\cdot\|_1$ we have the following corollary.

Corollary 3.3. *Let $A_i, B_i \in \mathbb{P}_n$, $i = 1, \ldots, m$. We have*

$$\begin{aligned} \Big\|\sum_{i=1}^{m} A_i^2 \sharp B_i^2\Big\|_1 &\leqslant \Big\|\Big(\sum_{i=1}^{m} A_i\Big)^{1/2}\Big(\sum_{i=1}^{m} B_i\Big)\Big(\sum_{i=1}^{m} A_i\Big)^{1/2}\Big\|_1 \\ &\leqslant \Big\|\Big(\sum_{i=1}^{m} A_i\Big)\Big(\sum_{i=1}^{m} B_i\Big)\Big\|_1. \end{aligned} \tag{3.9}$$

P r o o f. We have a special case of the Ando-Hiai log majorization

$$\lambda(A^2 \sharp B^2) \prec_{\log} \lambda(A \sharp B)^2.$$

Consequently,

$$\Big\|\sum_{i=1}^{m} A_i^2 \sharp B_i^2\Big\|_1 \leqslant \Big\|\sum_{i=1}^{m}(A_i \sharp B_i)^2\Big\|_1.$$

By Theorem 3.1, we get the conclusion. □

Proposition 3.4. *Let $A_i, B_i \in \mathbb{P}_n$, $i = 1, \ldots, m$. We have*

$$\left\| \sum_{i=1}^{m} A_i^2 \sharp B_i^2 \right\|_1 \leqslant \left\| \sum_{i=1}^{m} A_i B_i \right\|_1. \tag{3.10}$$

The inequality in (3.10) *becomes equality if and only if*

(1) $(A_1^{1/2}B_1^{1/2}, \ldots, A_m^{1/2}B_m^{1/2})$ *and* $(A_1^{1/2}U_1B_1^{1/2}, \ldots, A_m^{1/2}U_mB_m^{1/2})$ *are linearly dependent, where* $U_i := (A_i^{-1}B_i^2A_i^{-1})^{1/2}A_iB_i^{-1}$, $i = 1, \ldots, m$, *and*

(2) $\operatorname{tr}(U_i^*A_iU_iB_i) = \operatorname{tr}(A_iU_iB_i)$ *for all* $i = 1, \ldots, m$, *and*

(3) $(\operatorname{tr}(A_1B_1), \ldots, \operatorname{tr}(A_mB_m))$ *and* $(\operatorname{tr}(A_1U_1B_1), \ldots, \operatorname{tr}(A_mU_mB_m))$ *in* $\mathbb{R}^m$ *are linearly dependent.*

Proof. By [6], Proposition 4.1.8, there exists a unique $U_i \in \mathrm{U}(n)$ such that $A_i^2 \sharp B_i^2 = A_iU_iB_i$ for each $i = 1, \ldots, m$. Indeed, $U_i = (A_i^{-1}B_i^2A_i^{-1})^{1/2}A_iB_i^{-1}$ as $(A_i^{-1}B_i^2A_i^{-1})^{1/2} = A_i^{1/2}(A_i^{-1}B^2)^{1/2}A^{-1/2}$ and $A_i^2 \sharp B_i^2 = A_i^2(A_i^{-2}B_i^2)^{1/2}$, cf. [6], page 109. Since each $A_i^2 \sharp B_i^2$ is positive definite, so is their sum. Thus

$$\begin{aligned} \left\| \sum_{i=1}^{m} A_i^2 \sharp B_i^2 \right\|_1 &= \operatorname{tr}\left(\sum_{i=1}^{m} A_i^2 \sharp B_i^2 \right) \\ &= \operatorname{tr}\left(\sum_{i=1}^{m} A_iU_iB_i \right) = \sum_{i=1}^{m} \operatorname{tr}(B_i^{1/2}A_i^{1/2}A_i^{1/2}U_iB_i^{1/2}). \end{aligned} \tag{3.11}$$

Applying Cauchy-Schwarz's inequality, we have

$$\begin{aligned} \sum_{i=1}^{m} \operatorname{tr}(B_i^{1/2}A_i^{1/2}A_i^{1/2}U_iB_i^{1/2}) &\leqslant \sum_{i=1}^{m} (\operatorname{tr}(A_iB_i))^{1/2}(\operatorname{tr}(B_i^{1/2}U_i^*A_i^{1/2}A_i^{1/2}U_iB_i^{1/2}))^{1/2} \\ &= \sum_{i=1}^{m} (\operatorname{tr}(A_iB_i))^{1/2}(\operatorname{tr}(U_i^*A_iU_iB_i))^{1/2} \\ &\leqslant \sum_{i=1}^{m} (\operatorname{tr}(A_iB_i))^{1/2}(\operatorname{tr}(A_iU_iB_i))^{1/2} \end{aligned}$$

since each $A_iU_iB_i$ is positive definite. Applying Cauchy-Schwarz's inequality again, we obtain

$$\sum_{i=1}^{m} (\operatorname{tr}(A_iB_i))^{1/2}(\operatorname{tr}(A_iU_iB_i))^{1/2} \leqslant \left(\sum_{i=1}^{m} (\operatorname{tr}(A_iB_i)) \right)^{1/2} \left(\sum_{i=1}^{m} \operatorname{tr}(A_iU_iB_i) \right)^{1/2}.$$

Hence we have

$$\left\| \sum_{i=1}^{m} A_i^2 \sharp B_i^2 \right\|_1 = \sum_{i=1}^{m} \|A_i^2 \sharp B_i^2\|_1 \leqslant \left(\sum_{i=1}^{m} \|A_iB_i\|_1 \right)^{1/2} \left(\sum_{i=1}^{m} \|A_i^2 \sharp B_i^2\|_1 \right)^{1/2}. \tag{3.12}$$

Note that (3.11) implies that $\left\|\sum\limits_{i=1}^{m} A_i^2 \sharp B_i^2\right\|_1 = \sum\limits_{i=1}^{m} \|A_i^2 \sharp B_i^2\|_1$. So the inequality (3.10) follows from (3.12). The conditions for equality to occur are readily seen. $\square$

Because of the symmetric property of the geometric mean, we have the following corollary. The results are generalizations of some results in [12], Theorem 3.1.

Corollary 3.5. *Let $A_i, B_i \in \mathbb{P}_n$, $i = 1, \ldots, m$. We have*

$$\begin{aligned}
\left\|\sum_{i=1}^{m}(A_i^2 \sharp B_i^2)\right\|_1 &\leqslant \left\|\sum_{i=1}^{m}(A_i \sharp B_i)^2\right\|_1 \\
&\leqslant \min\left\{\left\|\left(\sum_{i=1}^{m} A_i\right)^{1/2}\left(\sum_{i=1}^{m} B_i\right)\left(\sum_{i=1}^{m} A_i\right)^{1/2}\right\|_1,\right. \\
&\qquad\left.\left\|\left(\sum_{i=1}^{m} B_i\right)^{1/2}\left(\sum_{i=1}^{m} A_i\right)\left(\sum_{i=1}^{m} B_i\right)^{1/2}\right\|_1\right\} \\
&\leqslant \min\left\{\left\|\left(\sum_{i=1}^{m} A_i\right)\left(\sum_{i=1}^{m} B_i\right)\right\|_1, \left\|\left(\sum_{i=1}^{m} B_i\right)\left(\sum_{i=1}^{m} A_i\right)\right\|_1\right\}.
\end{aligned}$$

Remark 3.6. Let us consider again the matrices $C, D = \operatorname{diag}(2, 1/2)$ and $A = \operatorname{diag}(3, 1/3)$ and $B = \operatorname{diag}(1/3, 3)$. So, $C \prec_{\log} A$ and $D \prec_{\log} B$, but the inequality $C + D \prec_{\log} A + B$ is not true. In other words,

$$\lambda(A_i^2 \sharp B_i^2) \prec_{\log} \lambda(A_i \sharp B_i)^2$$

does not imply

$$\left\|\sum_{i=1}^{m} A_i^2 \sharp B_i^2\right\|_2 \leqslant \left\|\sum_{i=1}^{m}(A_i \sharp B_i)^2\right\|_2. \tag{3.13}$$

But we still believe that inequality (3.13) is true. As a consequence, we hope that the following conjecture is true.

Conjecture. Let $A_i, B_i \in \mathbb{P}_n$, $i = 1, \ldots, m$. For any unitarily invariant norm $|||\cdot|||$ on $\mathbb{C}_{n\times n}$, the following inequalities are true:

$$\left|\left|\left|\sum_{i=1}^{m} A_i^2 \sharp B_i^2\right|\right|\right| \leqslant \left|\left|\left|\left(\sum_{i=1}^{m} A_i\right)^{1/2}\left(\sum_{i=1}^{m} B_i\right)\left(\sum_{i=1}^{m} A_i\right)^{1/2}\right|\right|\right|.$$

4. Appendix

In this section we generate counterexamples for inequalities (3.2) and (3.4) by MATLAB. Consider the real positive definite matrices

$$A = \begin{pmatrix} 7.0306 & -1.7934 & -1.6084 \\ -1.7934 & 1.3863 & -1.5663 \\ -1.6084 & -1.5663 & 4.5745 \end{pmatrix}, \quad B = \begin{pmatrix} 0.3082 & 0.6024 & 0.7812 \\ 0.6024 & 2.1386 & 2.1567 \\ 0.7812 & 2.1567 & 3.6532 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0.8033 & -0.6712 & -0.1708 \\ -0.6712 & 2.6880 & -3.8800 \\ -0.1708 & -3.8800 & 7.6512 \end{pmatrix}, \quad D = \begin{pmatrix} 3.7785 & -1.5586 & 0.8780 \\ -1.5586 & 4.1198 & -0.2474 \\ 0.8780 & -0.2474 & 1.0951 \end{pmatrix}.$$

By MATLAB,

$$s((A \sharp B)^2 + (C \sharp D)^2) = (33.7025, 9.5523, 0.2768)$$

and

$$s(AB + CD) = (20.6082, 5.2004, 0.1149).$$

Therefore

$$s_j((A \sharp B)^2 + (C \sharp D)^2) \not\prec_w s_j(AB + CD).$$

In other words, inequality (3.4) is false.

Consider the following real positive definite matrices which yield a counterexample for inequality (3.2):

$$A = \begin{pmatrix} 1.8074 & -0.5670 & -0.9254 \\ -0.5670 & 2.5836 & -0.1312 \\ -0.9254 & -0.1312 & 0.5493 \end{pmatrix}, \quad B = \begin{pmatrix} 1.7959 & -0.5089 & -0.3335 \\ -0.5089 & 1.5947 & -0.4223 \\ -0.3335 & -0.4223 & 1.8003 \end{pmatrix},$$

$$C = \begin{pmatrix} 2.0032 & -0.3516 & -0.4131 \\ -0.3516 & 1.9916 & 0.9086 \\ -0.4131 & 0.9086 & 2.7557 \end{pmatrix}, \quad D = \begin{pmatrix} 8.6985 & 3.9885 & -3.1856 \\ 3.9885 & 3.8386 & 0.8276 \\ -3.1856 & 0.8276 & 4.0948 \end{pmatrix},$$

and

$$E = \begin{pmatrix} 6.8082 & -4.5263 & 4.6906 \\ -4.5263 & 4.3174 & -2.8839 \\ 4.6906 & -2.8839 & 3.3008 \end{pmatrix}, \quad F = \begin{pmatrix} 3.3452 & 2.6243 & -1.2761 \\ 2.6243 & 2.5186 & -0.1382 \\ -1.2761 & -0.1382 & 2.5263 \end{pmatrix}.$$

MATLAB gives

$$s((A^2 \sharp B^2) + (C^2 \sharp D^2) + (E^2 \sharp F^2)) = (54.8001, 25.0031, 3.5943)$$

and

$$s(AB + CD + EF) = (30.2699, 21.0650, 5.6523).$$

Therefore,

$$s((A \sharp B)^2 + (C \sharp D)^2 + (E \sharp F)^2) \not\prec_w s(AB + CD + EF).$$

In other words, inequality (3.2) is false.

Finally, we would like to point out that the left hand sides in inequality (3.2) and inequality (3.4) are not comparable. Consider the real positive definite matrices

$$A = \begin{pmatrix} 2.9414 & -1.2083 & -0.3679 \\ -1.2083 & 1.4373 & 0.2056 \\ -0.3679 & 0.2056 & 1.4924 \end{pmatrix}, \quad B = \begin{pmatrix} 3.6033 & -2.8792 & 1.7006 \\ -2.8792 & 3.9028 & -3.8431 \\ 1.7006 & -3.8431 & 6.4632 \end{pmatrix},$$
$$C = \begin{pmatrix} 1.4116 & -1.3675 & -1.2256 \\ -1.3675 & 4.2278 & 1.2205 \\ -1.2256 & 1.2205 & 2.0843 \end{pmatrix}, \quad D = \begin{pmatrix} 2.6449 & 1.2291 & -1.8794 \\ 1.2291 & 1.8298 & -0.6273 \\ -1.8794 & -0.6273 & 1.4651 \end{pmatrix},$$
$$E = \begin{pmatrix} 1.8504 & -2.9027 & -0.4251 \\ -2.9027 & 7.1646 & 0.9936 \\ -0.4251 & 0.9936 & 3.9361 \end{pmatrix}, \quad F = \begin{pmatrix} 4.7157 & -0.9068 & 0.8326 \\ -0.9068 & 0.3987 & -0.1025 \\ 0.8326 & -0.1025 & 1.1471 \end{pmatrix}.$$

By MATLAB

$$s(A^2 \sharp B^2 + C^2 \sharp D^2 + E^2) = (41.2706, 13.4935, 6.4385)$$

and

$$s((A \sharp B)^2 + (C \sharp D)^2 + (E \sharp F)^2) = (31.3871, 16.7946, 5.3813).$$

So

$$s((A \sharp B)^2 + (C \sharp D)^2 + (E \sharp F)^2) \prec_w s(A^2 \sharp B^2 + C^2 \sharp D^2 + E^2).$$

Thus for all unitarily invariant norms $|||\cdot|||$,

$$|||(A \sharp B)^2 + (C \sharp D)^2 + (E \sharp F)^2||| \leqslant |||A^2 \sharp B^2 + C^2 \sharp D^2 + E^2 \sharp F^2|||.$$

On the other hand, consider the real positive definite matrices

$$A = \begin{pmatrix} 7.5408 & 0.5924 & -1.4512 \\ 0.5924 & 3.6894 & -2.1791 \\ -1.4512 & -2.1791 & 3.0029 \end{pmatrix}, \quad B = \begin{pmatrix} 5.3431 & -7.5310 & 2.4967 \\ -7.5310 & 11.0391 & -3.8396 \\ 2.4967 & -3.8396 & 1.4136 \end{pmatrix},$$
$$C = \begin{pmatrix} 1.7759 & -2.7332 & -1.6756 \\ -2.7332 & 5.0135 & 2.8238 \\ -1.6756 & 2.8238 & 2.8247 \end{pmatrix}, \quad D = \begin{pmatrix} 3.0511 & 0.4364 & 2.0034 \\ 0.4364 & 0.2126 & 0.4251 \\ 2.0034 & 0.4251 & 3.2135 \end{pmatrix},$$
$$E = \begin{pmatrix} 2.9873 & 1.2895 & -0.3725 \\ 1.2895 & 5.2250 & -0.2709 \\ -0.3725 & -0.2709 & 0.2721 \end{pmatrix}, \quad F = \begin{pmatrix} 1.6925 & 1.9769 & 0.2839 \\ 1.9769 & 2.3873 & 0.2145 \\ 0.2839 & 0.2145 & 1.1522 \end{pmatrix}.$$

By MATLAB

$$s(A^2 \sharp B^2 + C^2 \sharp D^2 + E^2) = (64.4274, 15.1265, 4.4290)$$

and

$$s((A \sharp B)^2 + (C \sharp D)^2 + (E \sharp F)^2) = (78.2216, 19.8741, 5.2668).$$

So

$$s(A^2 \sharp B^2 + C^2 \sharp D^2 + E^2) \prec_w s((A \sharp B)^2 + (C \sharp D)^2 + (E \sharp F)^2).$$

Thus for all unitarily invariant norms $|||\cdot|||$,

$$|||A^2 \sharp B^2 + C^2 \sharp D^2 + E^2 \sharp F^2||| \leqslant |||(A \sharp B)^2 + (C \sharp D)^2 + (E \sharp F)^2|||.$$

Remark 4.1. The following is another proof of inequality (3.5) given by the referee. A 2×2 block matrix $M = z \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is called *positive partial transpose* (PPT) if M and $\begin{pmatrix} A & X^* \\ X & B \end{pmatrix}$ are positive semidefinite. Since

$$\begin{pmatrix} A_i & A_i \sharp B_i \\ A_i \sharp B_i & B_i \end{pmatrix}, \quad i = 1, \ldots, m$$

are PPT (see for example [19], Lemma 1.21, and [6], Proposition 4.1.8), so is the matrix

$$\sum_{i=1}^m \begin{pmatrix} A_i & A_i \sharp B_i \\ A_i \sharp B_i & B_i \end{pmatrix} = \begin{pmatrix} \sum\limits_{i=1}^m A_i & \sum\limits_{i=1}^m A_i \sharp B_i \\ \sum\limits_{i=1}^m A_i \sharp B_i & \sum\limits_{i=1}^m B_i \end{pmatrix}.$$

By a result of Bourin in [14], Theorem 4.1, one gets the first inequality of the relation

$$\lambda\Big(\Big(\sum_{i=1}^m A_i \sharp B_i\Big)^2\Big) \prec_{\log} \lambda\Big(\Big(\sum_{i=1}^m A_i\Big)\Big(\sum_{i=1}^m B_i\Big)\Big) \prec_{\log} s\Big(\Big(\sum_{i=1}^m A_i\Big)\Big(\sum_{i=1}^m B_i\Big)\Big).$$

The second inequality follows from the well known fact that $|\lambda(X)| \prec_{\log} s(X)$ for all $X \in \mathbb{C}_{n\times n}$ and the fact that the eigenvalues of XY are positive for $X, Y \in \mathbb{P}_n$. Note that $\sum\limits_{i=1}^m (A_i \sharp B_i)^2 \in \mathbb{P}_n$ so that $\lambda\Big(\Big(\sum\limits_{i=1}^m A_i \sharp B_i\Big)^2\Big) = s\Big(\Big(\sum\limits_{i=1}^m A_i \sharp B_i\Big)^2\Big)$. Now we have

$$s\Big(\Big(\sum_{i=1}^m A_i \sharp B_i\Big)^2\Big) \prec_{\log} s\Big(\Big(\sum_{i=1}^m A_i\Big)\Big(\sum_{i=1}^m B_i\Big)\Big).$$

It is well-known that $a \prec_{\log} b$ implies $a \prec_w b$ for all nonnegative vectors $a, b \in \mathbb{R}^n$, see [19], Theorem 2.7. So

$$s\Big(\Big(\sum_{i=1}^m A_i \sharp B_i\Big)^2\Big) \prec_w \lambda\Big(\Big(\sum_{i=1}^m A_i\Big)\Big(\sum_{i=1}^m B_i\Big)\Big).$$

Thus

$$\Big|\Big|\Big|\Big(\sum_{i=1}^m A_i \sharp B_i\Big)^2\Big|\Big|\Big| \leqslant \Big|\Big|\Big|\Big(\sum_{i=1}^m A_i\Big)\Big(\sum_{i=1}^m B_i\Big)\Big|\Big|\Big|.$$

Then by (3.7) we get the desired result.

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