Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 57 (2016), No. 3, 397–410

Persistent URL: `http://dml.cz/dmlcz/145843`

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Interaction between cellularity of Alexandroff spaces and entropy of generalized shift maps

Fatemah Ayatollah Zadeh Shirazi, Sahar Karimzadeh Dolatabad, Sara Shamloo

Abstract. In the following text for a discrete finite nonempty set K and a self-map $\varphi : X \to X$ we investigate interaction between different entropies of generalized shift $\sigma_\varphi : K^X \to K^X$, $(x_\alpha)_{\alpha \in X} \mapsto (x_{\varphi(\alpha)})_{\alpha \in X}$ and cellularities of some Alexandroff topologies on X.

Keywords: Alexandroff topology; cellularity; functional Alexandroff topological space; infinite anti-orbit number; infinite orbit number

Classification: 54C70

Introduction

In this paper, we investigate interaction between cellularity of Alexandroff spaces and entropy of generalized shift maps. Briefly, we deal with several concepts including two subclasses of Alexandroff topological spaces, infinite orbit number and infinite anti-orbit number of a self-map, the set-theoretical and the contravariant set-theoretical entropies, the algebraic and the topological entropies, and in particular their interaction regarding generalized shifts' subject.

Binding several concepts, using an idea and investigating new theorems are common and well-known methods in all areas of mathematics. In the following text, infinite orbit number and infinite anti-orbit number of a self-map help us to bind the concepts of "cellularity" and "entropy", e.g. we prove that if K is a finite discrete space with at least two elements, X is an arbitrary space with at least two elements, and $\lambda : X \to X$ does not have any periodic point, then considering K^X under product topology, the topological entropy of generalized shift $\sigma_\lambda : K^X \to K^X$ is equal to $c^*(X, \overline{\tau}_\lambda) \log |K|$, where $c^*(X, \overline{\tau}_\lambda)$ just depends on the cellularity of Alexandroff space $(X, \overline{\tau}_\lambda)$.

Our first two sections are devoted to preliminaries, short historical points in some cases and useful remarks, all of them in a shortened form, i.e., in the first section we have background on Alexandroff topologies, orbit numbers and generalized shifts divided in three subsections, and in the second section we have background on different entropies (set-theoretical, contravariant set-theoretical, topological, algebraic) and the connection between them regarding generalized

DOI 10.14712/1213-7243.2015.174

shifts. Sections 3 and 4 contain the main body of the paper and Section 5 presents main results.

We recall that the *cellularity* of a topological space (X, τ) [11] is defined by

$$c(X, \tau) := \sup\{\mathrm{card}(A)\colon A \text{ is a collection of disjoint nonempty open subsets of } X\}.$$

Moreover, denote

$$c^*(X, \tau) := \begin{cases} c(X, \tau) & \text{if } c(X, \tau) \text{ is finite,} \\ \infty & \text{otherwise.} \end{cases}$$

In the following text $\mathbb{N} = \{1, 2, \ldots\}$ denotes the collection of all natural numbers and $\mathbb{Z} = \{0, \pm1, \pm2, \ldots\}$ denotes the collection of all integers.

1. Background on Alexandroff topologies, orbit numbers and generalized shifts

In this section we have preliminaries on Alexandroff topologies (we deal with two classes of Alexandroff topologies), orbit numbers (infinite orbit number and anti orbit number), and generalized shifts.

1.1 Preliminaries on Alexandroff topologies. We call the topological space (X, τ) *Alexandroff* if for every nonempty family $\{U_\alpha : \alpha \in \Gamma\}$ of open subsets of X, $\bigcap\{U_\alpha : \alpha \in \Gamma\}$ is open (so X is an Alexandroff space and τ is an Alexandroff topology). In particular, in Alexandroff topological space (X, τ) every $x \in X$ has the smallest open neighborhood, we denote this neighborhood by $V(x, \tau)$.

As it has been mentioned in [4], for an arbitrary map $f : X \to \mathcal{P}(X) \setminus \{\varnothing\}$, the family $\{f(x) : x \in X\}$ is a basis for an Alexandroff topology on X such that $f(x)$ is the smallest open neighborhood of x $(x \in X)$ if and only if for all $x, y \in X$ we have:

- $x \in f(x)$;
- if $y \in f(x)$, then $f(y) \subseteq f(x)$.

Now for a self-map $\lambda : X \to X$, consider $f, g : X \to \mathcal{P}(X) \setminus \{\varnothing\}$ with $f(x) = \bigcup\{\lambda^{-n}(x) : n \geq 0\}$ and $g(x) = \{\lambda^n(x) : n \geq 0\}$. Then $\{f(x) : x \in X\}$ and $\{g(x) : x \in X\}$ are the topological basis on X. *Functional Alexandroff topology on X induced by λ* is the topology generated by $\{f(x) : x \in X\}$, we denote this topology on X by τ_λ which has been introduced for the first time in [4] on the basis of a talk in a Conference on 2009 by the first author [2] (for more details on functional Alexandroff spaces see [5]). Also, another Alexandroff topology on X is the topology generated by $\{g(x) : x \in X\}$, we denote this topology on X by $\overline{\tau}_\lambda$ and call it the *Alexandroff topology on X induced by λ*. In [15] a semigroup S of self-maps on X has been considered and the orbit of $x(\in X)$ under S is the smallest open neighborhood of x, we adopt this topology by considering S as a special case $\{\lambda^n : n \geq 0\}$. Let us mention that these two topologies have appeared independently.

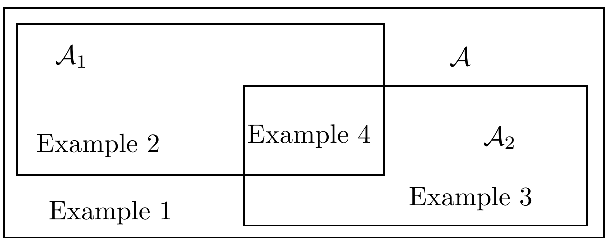
Remark 1.1. Alexandroff topological space (X, τ) is functional Alexandroff if and only if [4, Theorem 3.5]:

- for all $x, y \in X$ we have:

$$V(x, \tau_\lambda) \subseteq V(y, \tau_\lambda) \vee V(y, \tau_\lambda) \subseteq V(x, \tau_\lambda) \vee V(x, \tau_\lambda) \cap V(y, \tau_\lambda) = \varnothing;$$

- for $x \in X$, if there exists $y \in X$ such that $V(x, \tau)$ is a proper subset of $V(y, \tau)$, then for all $z \in X \setminus \{x\}$ we have $V(x, \tau) \neq V(z, \tau)$;
- for all $x, y \in X$, $\{z \in X : V(x, \tau) \subseteq V(z, \tau) \subseteq V(y, \tau)\}$ is finite.

Note that for a self-map $\lambda : X \to X$ and $x \in X$ we have $V(x, \tau_\lambda) = \bigcup\{\lambda^{-n}(x) : n \geq 0\}$ and $V(x, \overline{\tau}_\lambda) = \{\lambda^n(x) : n \geq 0\}$. Let us denote the collection of all Alexandroff spaces by $\mathcal{A}$, also consider $\mathcal{A}_1 := \{(X, \tau) \in \mathcal{A} : \exists\lambda\, (\tau = \tau_\lambda)\}$ and $\mathcal{A}_2 := \{(X, \tau) \in \mathcal{A} : \exists\lambda\, (\tau = \overline{\tau}_\lambda)\}$. Then we have the following diagram:



where:

Example 1. Let $X_1 = \{1, 2, 3, 4\}$ with topology $\mathfrak{T}_1$ generated by the basis $\{\{2\}, \{4\}, \{1, 2\}, \{2, 3, 4\}\}$, then $(X_1, \mathfrak{T}_1)$ is obviously an Alexandroff space, also it is neither a functional Alexandroff space (use $V(1, \mathfrak{T}_1) = \{1, 2\}$, $V(3, \mathfrak{T}_1) = \{2, 3, 4\}$ and Remark 1.1) nor a member of $\mathcal{A}_2$ (if $\mathfrak{T}_1 = \overline{\tau}_\lambda$ for some $\lambda : X_1 \to X_1$, then $\{2, 3, 4\} = V(3, \mathfrak{T}_1) = V(3, \overline{\tau}_\lambda) = \{\lambda^n(3) : n \geq 0\}$ and there exist $p \neq q$ with $\lambda^p(3) = 2$, $\lambda^q(3) = 4$, if $p > q$, then $2 = \lambda^p(3) = \lambda^{p-q}(\lambda^q(3)) = \lambda^{p-q}(4) \in V(4, \overline{\tau}_\lambda) = V(4, \mathfrak{T}_1) = \{4\}$ which is a contradiction, the assumption $p < q$ leads to contradiction too).

Example 2. Let $X_2 = \{1, 2, 3\}$ and consider the constant map $c : X_2 \to X_2$ with $c(x) = 1$ (for all $x \in X_2$). Then $V(1, \tau_c) = \{1, 2, 3\}$, $V(2, \tau_c) = \{2\}$, $V(3, \tau_c) = \{3\}$. In addition $(X_2, \tau_c) \notin \mathcal{A}_2$, otherwise there exists $\lambda : X_2 \to X_2$ with $\tau_c = \overline{\tau}_\lambda$, so $2, 3 \in V(1, \tau_c) = V(1, \overline{\tau}_\lambda) = \{\lambda^n(1) : n \geq 0\}$ and there exists $p \neq q$ such that $\lambda^p(1) = 2$, $\lambda^q(1) = 3$ we may suppose $p > q$, thus $2 = \lambda^p(1) = \lambda^{p-q}(\lambda^q(1)) = \lambda^{p-q}(3) \in V(3, \overline{\tau}_\lambda) = V(3, \tau_c) = \{3\}$ which is a contradiction.

Example 3. Let $X_3 = \{1, 2, 3\}$ and consider $\mu : X_3 \to X_3$ with $\mu(1) = 2$, $\mu(2) = 3$, $\mu(3) = 2$, then $V(1, \overline{\tau}_\mu) = \{1, 2, 3\}$, $V(2, \overline{\tau}_\mu) = V(3, \overline{\tau}_\mu) = \{2, 3\}$. Using Remark 1.1, $(X_3, \overline{\tau}_\mu) \notin \mathcal{A}_1$.

Example 4. Consider the nonempty set X_4 and the identity map $id_{X_4} : X_4 \to X_4$ ($id_{X_4}(x) = x$), then both topologies $\tau_{id_{X_4}}$ and $\overline{\tau}_{id_{X_4}}$ are discrete.

1.2 What are $a(\lambda)$ and $o(\lambda)$? For the map $\lambda : X \to X$ if $(x_n : n \geq 0)$ is a one-to-one sequence such that $x_n = \lambda(x_{n+1})$ for all $n \geq 0$, then we call $(x_n : n \geq 0)$ a *strict anti-orbit* with the initial point x_0 (see, e.g. [9, Definition 1.2] and [13, Definition 1.1] too). If $(\lambda^n(x) : n \geq 0)$ is a one-to-one sequence, then we call $(\lambda^n(x) : n \geq 0)$ a *strict orbit* with the initial point x.

For the map $\lambda : X \to X$ define infinite anti-orbit number of λ and infinite orbit number of λ with:

$$a(\lambda) := \sup(\{0\} \cup \{n \in \mathbb{N} : \text{ there exist } n \text{ disjoint strict } \lambda\text{-anti-orbits in } X\})$$

and

$$o(\lambda) = \sup(\{0\} \cup \{n \in \mathbb{N} : \text{ there exist } n \text{ disjoint strict } \lambda\text{-orbits in } X\}).$$

We call $x \in X$ a *periodic point* (of λ) if there exists $n > 0$ with $\lambda^n(x) = x$, also $x \in X$ is a *fixed point* (of λ) if $\lambda(x) = x$. We call $x \in X$ a *quasi-periodic point* (of λ) if there exist $n > m \geq 0$ with $\lambda^n(x) = \lambda^m(x)$. Denote the set of all non-quasi-periodic points of λ with $W(X, \lambda)$ (see [1] and [3]). It is evident that all fixed points of λ are periodic, and all periodic points of λ are quasi-periodic.

1.3 What is a generalized shift? One sided shift $\sigma : \{1, \ldots, k\}^{\mathbb{N}} \to \{1, \ldots, k\}^{\mathbb{N}}$, $(x_n)_{n\geq 1} \mapsto (x_{n+1})_{n\geq 1}$ and two sided shift $\sigma : \{1, \ldots, k\}^{\mathbb{Z}} \to \{1, \ldots, k\}^{\mathbb{Z}}$, $(x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}$ are among the most well-known topics in ergodic theory, dynamical systems, etc. (see e.g., [16] and [14]). For a self-map $\lambda : X \to X$, one may consider a generalized shift map $\sigma_\lambda : K^X \to K^X$ with $\sigma_\lambda((z_t)_{t\in X}) = (z_{\lambda(t)})_{t\in X}$ [6]. If K has topological structure, then $\sigma_\lambda : K^X \to K^X$ is continuous (where K^X is considered with the product topology). If K is a group, then $\sigma_\lambda : K^X \to K^X$ is a group endomorphism. Both of these aspects of generalized shifts have been studied in several texts (e.g. topological (resp. dynamical systems) approach in [3], [7] and algebraic approach in [1], [12]).

2. Background on entropies and the connections between them

2.1 Preliminaries on the set-theoretical entropies. For a self-map $\lambda : X \to X$ and the finite subset A of X, the following limit exists:

$$\text{ent}_{set}(\lambda, A) := \lim_{n\to\infty} \frac{|A \cup \lambda(A) \cup \cdots \cup \lambda^{n-1}(A)|}{n}.$$

We call $\text{ent}_{set}(\lambda) := \sup\{\text{ent}_{set}(\lambda, A) : A \text{ is a finite subset of } X\}$ the set-theoretical entropy of λ (for more details on set-theoretical entropy see [3]).

Also if $\lambda : X \to X$ is finite fiber (i.e., for all $x \in X$, $\lambda^{-1}(x)$ is finite) and surjective, then for the finite subset A of X let [8, Proposition 3.2.34]:

$$\text{ent}_{cset}(\lambda, A) := \lim_{n\to\infty} \frac{|A \cup \lambda^{-1}(A) \cup \cdots \cup \lambda^{-n+1}(A)|}{n}.$$

We call $\mathrm{ent}_{cset}(\lambda) := \sup\{\mathrm{ent}_{cset}(\lambda, A) : A$ is a finite subset of $X\}$ the contravariant set-theoretical entropy of λ. Moreover for a finite fiber self-map $\lambda : X \to X$, let $sc(\lambda) := \bigcup\{\lambda^n(X) : n \geq 1\}$ be the *surjective core* of λ, then $\lambda \restriction_{sc(\lambda)}: sc(\lambda) \to sc(\lambda)$ is surjective, and we define the contravariant set-theoretical entropy of λ as $\mathrm{ent}_{cset}(\lambda) := \mathrm{ent}_{cset}(\lambda \restriction_{sc(\lambda)})$ (for more details on contravariant set-theoretical entropy see [8]).

Remark 2.1. For a self-map $\lambda : X \to X$ we have $\mathrm{ent}_{set}(\lambda) = o(\lambda)$ [3, Proposition 2.16], moreover for the finite fiber $\lambda : X \to X$ we have $\mathrm{ent}_{cset}(\lambda) = a(\lambda)$ [8, Theorem 3.2.39].

2.2 Preliminaries on the topological entropy. In the nonvoid compact topological space Z, if $\mathcal{U}$ and $\mathcal{V}$ are two finite open covers of Z, then $\mathcal{U} \vee \mathcal{V} := \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ is a finite open cover of Z too. Moreover suppose $N(\mathcal{U})$ denotes the minimum cardinality of subcovers of $\mathcal{U}$. Then for the continuous map $T : Z \to Z$ the following limit exists:

$$\mathrm{ent}_{top}(T,\mathcal{U}) := \lim_{n\to\infty} \frac{\log(N(\mathcal{U} \vee T^{-1}(\mathcal{U}) \vee \cdots \vee T^{-(n-1)}(\mathcal{U})))}{n}$$

where for $k \geq 1$ let $T^{-k}(\mathcal{U}) := \{T^{-k}(U) : U \in \mathcal{U}\}$. We call $\mathrm{ent}_{top}(T) := \sup\{\mathrm{ent}_{top}(T,\mathcal{U}) : \mathcal{U}$ is a finite open cover of $Z\}$ the topological entropy of T (for more details on topological entropy see [16]).

2.3 Preliminaries on the algebraic entropy. Suppose $(G, +)$ is a group, K is a finite subgroup of G, and $\varphi : G \to G$ is an endomorphism. Then the following limit exists:

$$\mathrm{ent}_{alg}(\varphi, K) := \lim_{n\to\infty} \frac{\log|K + \varphi(K) + \cdots + \varphi^{n-1}(K)|}{n} .$$

We call $\mathrm{ent}_{alg}(\varphi) := \sup\{\mathrm{ent}_{alg}(\varphi, F) : F$ is a finite subgroup of $G\}$ the algebraic entropy of φ (for more details on algebraic entropy see [10], [8]).

2.4 The connection of the entropies of the generalized shifts to the set-theoretical entropies. Now we want to describe the interaction between the above mentioned entropies in generalized shift's approach.

Remark 2.2. Suppose K is a finite discrete space with at least two elements and X is nonempty, consider K^X under the product topology. Then $\sigma_\lambda : K^X \to K^X$ is continuous and $\mathrm{ent}_{top}(\sigma_\lambda) = o(\lambda)\log|K|$ [3, Theorem 4.7], i.e. (by Remark 2.1):

$$\mathrm{ent}_{top}(\sigma_\lambda) = \mathrm{ent}_{set}(\lambda)\log|K|.$$

For a finite abelian nontrivial group K with an identity e and a self-map $\lambda : X \to X$, the algebraic entropy of generalized shift $\sigma_\lambda : K^X \to K^X$, $(z_t)_{t\in X} \mapsto (z_{\lambda(t)})_{t\in X}$, has been computed in [12], also the algebraic entropy of $\sigma_\lambda\restriction_{\bigoplus_X K} : \bigoplus_X K \to \bigoplus_X K$, where $\bigoplus_X K = \{(z_t)_{t\in X}$: there exist $t_1, \ldots, t_n \in X$ such that for all $t \neq t_1, \ldots, t_n$ we have $z_t = e\}$ and $\lambda : X \to X$ is a finite fiber computed in

[1, Theorem 4.14]. The following remark deals with an arbitrary nontrivial finite group K.

Remark 2.3. If K is a nontrivial finite group with at least two elements and X is a nonempty set, $\lambda : X \to X$ is a finite fiber map, then for endomorphism $\sigma_{\lambda\restriction_{\bigoplus_X K}} : \bigoplus_X K \to \bigoplus_X K$ we have $\mathrm{ent}_{alg}(\sigma_{\lambda\restriction_{\bigoplus_X K}}) = a(\lambda)\log|K|$ (see [8, Theorem 7.3.3]), i.e. (by Remark 2.1):

$$\mathrm{ent}_{alg}(\sigma_{\lambda\restriction_{\bigoplus_X K}}) = \mathrm{ent}_{cset}(\lambda)\log|K|.$$

3. Interaction between infinite anti-orbit number of $\lambda : X \to X$ and cellularity of the functional Alexandroff space (X, τ_λ)

In this section we prove $a(\lambda) = c^*(X, \tau_\lambda)$ for a surjective map without any periodic points $\lambda : X \to X$.

Lemma 3.1. *Consider a self-map $\lambda : X \to X$. For a strict λ-anti-orbit sequence $(x_n : n \geq 0)$, the following statements are valid.*

1. *For all $m \geq 0$ we have $\lambda^m(x_m) = x_0$ and $\{x_n : n \geq m\} \subseteq V(x_m, \tau_\lambda)$, thus $V(x_n, \tau_\lambda) \subseteq V(x_m, \tau_\lambda)$ for all $n \geq m$.*
2. *For $p \geq 0$, if x_p is periodic, then x_0 is periodic and $x_p \in \{\lambda^n(x_0) : n \geq 0\}$.*
3. *There exists $q \geq 0$ such that for all $n \geq q$, x_n is not periodic.*

Proof: (1) By definition of λ-anti-orbit it is clear that for all $k \geq m$ we have $x_m = \lambda^{m-k}(x_k)$, thus $x_k \in \lambda^{-(m-k)}(x_m) \subseteq \bigcup\{\lambda^{-n}(x_m) : n \geq 0\} = V(x_m, \tau_\lambda)$, so

$$(*) \qquad \{x_n : n \geq m\} \subseteq V(x_m, \tau_\lambda).$$

On the other hand for all $n \geq m$, by (*) we have $x_n \in V(x_m, \tau_\lambda)$, which leads to $V(x_n, \tau_\lambda) \subseteq V(x_m, \tau_\lambda)$.

(2) Consider $p \geq 0$ and $m \geq 1$ with $\lambda^m(x_p) = x_p$. If $m < p$, then $x_p = \lambda^m(x_p) = x_{p-m}$ which is a contradiction since $(x_n : n \geq 0)$ is one-to-one and $p - m < p$. Thus $m \geq p$ and

$$x_p = \lambda^m(x_p) = \lambda^{m-p}(\lambda^p(x_p)) = \lambda^{m-p}(x_0) \in \{\lambda^n(x_0) : n \geq 0\},$$

moreover $\lambda^m(x_0) = \lambda^m(\lambda^p(x_p)) = \lambda^p(\lambda^m(x_p)) = \lambda^p(x_p) = x_0$ and x_0 is periodic.

(3) If x_0 is not periodic, then by item (2) for all $n \geq 0$, x_n is not periodic and we are done. Otherwise x_0 is periodic and there exists $m \geq 1$ such that $\lambda^m(x_0) = x_0$. Let $K := \{n \geq 0 : x_n \in \{x_0, \lambda(x_0), \ldots, \lambda^m(x_0)\}$, then K is finite since $(x_n : n \geq 0)$ is one-to-one. The set K is a nonempty finite subset of $\mathbb{N} \cup \{0\}$ (note: $0 \in K$). Suppose $r = \max K$, then for all $n \geq r + 1$ we have $n \notin K$ and (by $\lambda^m(x_0) = x_0$ we have $\{x_0, \lambda(x_0), \ldots, \lambda^m(x_0)\} = \{\lambda^s(x_0) : s \geq 0\}$):

$$x_n \notin \{x_0, \lambda(x_0), \ldots, \lambda^m(x_0)\} = \{\lambda^s(x_0) : s \geq 0\},$$

by (2), x_n is not periodic. □

Lemma 3.2. *Consider a self-map $\lambda : X \to X$ and two disjoint strict λ-anti-orbits $(x_n : n \geq 0)$, $(y_n : n \geq 0)$. Then there exist $m, k \geq 0$ such that $V(x_m, \tau_\lambda)$ and $V(y_k, \tau_\lambda)$ are disjoint.*

Proof: By Lemma 3.1(3) we may choose $q \geq 0$ such that for all $n \geq q$, x_n and y_n are not periodic. If $V(x_q, \tau_\lambda)$ and $V(y_q, \tau_\lambda)$ are disjoint, we are done, otherwise by Remark 1.1 $V(x_q, \tau_\lambda) \subseteq V(y_q, \tau_\lambda)$ or $V(y_q, \tau_\lambda) \subseteq V(x_q, \tau_\lambda)$. Suppose $V(x_q, \tau_\lambda) \subseteq V(y_q, \tau_\lambda)$. Thus

$$x_q \in V(x_q, \tau_\lambda) \subseteq V(y_q, \tau_\lambda) = \bigcup \{\lambda^{-n}(y_q) : n \geq 0\},$$

and there exists $r \geq 0$ with $x_q \in \lambda^{-r}(y_q)$. We claim that $V(x_q, \tau_\lambda) \cap V(y_{q+r}, \tau_\lambda) = \varnothing$, otherwise by Remark 1.1 we have the following cases:

• Case I: $V(y_{q+r}, \tau_\lambda) \subseteq V(x_q, \tau_\lambda)$. In this case $y_{q+r} \in V(x_q, \tau_\lambda)$, thus there exists $s \geq 0$ with $\lambda^s(y_{q+r}) = x_q$. Now we have:

$$\lambda^s(y_q) = \lambda^s(\lambda^r(y_{q+r})) = \lambda^r(\lambda^s(y_{q+r})) = \lambda^r(x_q) = y_q\,.$$

Since y_q is not periodic, we have $s = 0$. Therefore $x_q = \lambda^0(y_{q+r}) = y_{q+r}$, which is a contradiction by disjointness of two sequences $(x_n : n \geq 0)$ and $(y_n : n \geq 0)$.

• Case II: $V(x_q, \tau_\lambda) \subseteq V(y_{q+r}, \tau_\lambda)$. In this case $x_q \in V(y_{q+r}, \tau_\lambda)$, thus there exists $s \geq 0$ with $\lambda^s(x_q) = y_{q+r}$. We have the following two subcases:

Subcase II-a. Let $s \leq r$. Using $\lambda^s(x_q) = y_{q+r}$, for all $t \in \{0, \dots, r\}$ we have $y_{q+t} = \lambda^{r-t}(y_{q+r}) = \lambda^{r-t+s}(x_q)$. So

$$\begin{aligned} \{y_q, y_{q+1}, \dots, y_{q+r}\} &= \{y_{q+t} : 0 \leq t \leq r\} \\ &= \{\lambda^{r-t+s}(x_q) : 0 \leq t \leq r\} \\ &= \{\lambda^{s+t}(x_q) : 0 \leq t \leq r\}. \end{aligned}$$

The set $\{y_q, y_{q+1}, \dots, y_{q+r}\}$ has exactly $r+1$ elements since $(y_n : n \geq 0)$ is a one-to-one sequence. Therefore $\{\lambda^{s+t}(x_q) : 0 \leq t \leq r\}$ has exactly $r+1$ elements too. Moreover we have:

$$\lambda^{s+r}(x_q) = \lambda^r(\lambda^s(x_q)) = \lambda^r(y_{q+r}) = y_q = \lambda^r(x_q) = \lambda^{s+(r-s)}(x_q)$$

since $r, r-s \in \{0, \dots, r\}$, $\lambda^{s+r}(x_q) = \lambda^{s+(r-s)}(x_q) \in \{\lambda^{s+t}(x_q) : 0 \leq t \leq r\}$ which has exactly $r+1$ elements. Thus $r = r-s$ and $s = 0$, therefore $x_q = y_{q+r}$, which is a contradiction by disjointness of two sequences $(x_n : n \geq 0)$ and $(y_n : n \geq 0)$.

Subcase II-b. Let $s > r$. We have:

$$y_{q+r} = \lambda^s(x_q) = \lambda^{s-r}(\lambda^r(x_q)) = \lambda^{s-r}(y_q) = \lambda^{s-r}(\lambda^r(y_{q+r})) = \lambda^s(y-q+r)$$

and y_{q+r} is a periodic point of λ, which is a contradiction by the way we choose q.

Using the above two subcases we have $V(x_q, \tau_\lambda) \not\subseteq V(y_{q+r}, \tau_\lambda)$.

Considering cases I and II (and Remark 1.1) we have $V(x_q, \tau_\lambda) \cap V(y_{q+r}, \tau_\lambda) = \varnothing$ which completes the proof. $\square$

Lemma 3.3. *Consider a self-map $\lambda : X \to X$ and $k \geq 2$ pairwise disjoint strict λ-anti-orbits $(x_n^1 : n \geq 0), \dots, (x_n^k : n \geq 0)$. Then there exist $m_1, \dots, m_k \geq 0$ such that $V(x_{m_i}^i, \tau_\lambda) \cap V(x_{m_j}^j, \tau_\lambda) = \varnothing$ for distinct $i, j \in \{1, \dots, k\}$.*

Proof: If $k = 2$, then the lemma is valid by Lemma 3.2. Suppose $p \geq 2$ and the lemma is valid whenever $k \in \{2, \dots, p\}$. If $(x_n^1 : n \geq 0), \dots, (x_n^{p+1} : n \geq 0)$ are $p+1$ disjoint strict λ-anti-orbits, then using our hypothesis there exist $r_1, \dots, r_p \geq 0$ such that $V(x_{r_1}^1, \tau_\lambda), \dots, V(x_{r_p}^p, \tau_\lambda)$ are pairwise disjoint. For $i \in \{3, \dots, p+1\}$ two strict λ-anti-orbits $(x_n^i : n \geq r_i)$ and $(x_n^{p+1} : n \geq 0)$ are disjoint, thus by Lemma 3.2 there exist $m_i \geq r_i$ and $t_i \geq 0$ such that $V(x_{m_i}^i, \tau_\lambda) \cap V(x_{t_i}^{p+1}, \tau_\lambda) = \varnothing$. Let $m_{p+1} = \max(t_1, \dots, t_p)$. For all $i \in \{1, \dots, p\}$ since $t_i \leq m_{p+1}$ by Lemma 3.1(1) we have $x_{m_{p+1}} \in V(x_{t_i}^{p+1}, \tau_\lambda)$ hence $V(x_{m_{p+1}}^{p+1}, \tau_\lambda) \subseteq V(x_{t_i}^{p+1}, \tau_\lambda)$ and:

$$V(x_{m_i}^i, \tau_\lambda) \cap V(x_{m_{p+1}}^{p+1}, \tau_\lambda) \subseteq V(x_{m_i}^i, \tau_\lambda) \cap V(x_{t_i}^{p+1}, \tau_\lambda) = \varnothing.$$

On the other hand for distinct $i, j \in \{1, \dots, p\}$ we have:

$$V(x_{m_i}^i, \tau_\lambda) \cap V(x_{m_j}^j, \tau_\lambda) \subseteq V(x_{r_i}^i, \tau_\lambda) \cap V(x_{r_j}^j, \tau_\lambda) = \varnothing\,,$$

since $V(x_{m_i}^i, \tau_\lambda) \subseteq V(x_{r_i}^i, \tau_\lambda)$ and $V(x_{m_j}^j, \tau_\lambda) \subseteq V(x_{r_j}^j, \tau_\lambda)$ by Lemma 3.1(1). Thus $V(x_{m_1}^1, \tau_\lambda), \dots, V(x_{m_p}^p, \tau_\lambda), V(x_{m_{p+1}}^{p+1}, \tau_\lambda)$ are pairwise disjoint and we are done. □

For $\lambda : X \to X$, we call a subset A of X *λ-invariant* if $\lambda(A) \subseteq A$.

Lemma 3.4. *For $\lambda : X \to X$ and*

$$Y := \{x \in X : \text{ there exists a strict } \lambda\text{-anti-orbit with the initial point } x\},$$

we have:

1. *Y is λ-invariant.*
2. *If K is a λ-invariant subset of X, then for all $x \in K$ we have $V(x, \tau_{\lambda\restriction_K}) = V(x, \tau_\lambda) \cap K$ (in the corresponding functional Alexandroff spaces $(K, \tau_{\lambda\restriction_K})$ and (X, τ_λ)). In other words the functional Alexandroff topology on K induced by $\lambda\restriction_K : K \to K$ coincides with subspace topology on K inherited from the functional Alexandroff topology on X induced by $\lambda : X \to X$, i.e. $\tau_{\lambda\restriction_K} = \tau_\lambda \restriction_K$.*

Proof: (1) Suppose $x \in Y$. There exists a strict anti-orbit $(x_n : n \geq 0)$ in X with the initial point $x_0 = x$. We have the following cases:

• First case: for all $n \geq 0$, $\lambda(x) \neq x_n$. Therefore for $x_{-1} = \lambda(x)$, $(x_n : n \geq -1)$ is a one-to-one sequence, therefore it is a strict anti-orbit with the initial point x_{-1}, hence $\lambda(x) = x_{-1} \in Y$.

• Second case: There exists $m \geq 0$ such that $\lambda(x) = x_m$. Therefore $(x_n : n \geq m)$ is a one-to-one sequence so it is a strict anti-orbit with the initial point x_m which shows $\lambda(x) = x_m \in Y$.

By the above two cases, Y is λ-invariant.

(2) Suppose K is a λ-invariant subset of X and $x \in K$, then we have:

$$\begin{aligned} V(x, \tau_{\lambda\restriction_K}) &= \bigcup\{\lambda \restriction_K{}^{-n}(x) : n \geq 0\} \\ &= \{y \in K : \exists n \geq 0 \ \ \lambda \restriction_K{}^{n}(y) = x\} \\ &= \{y \in K : \exists n \geq 0 \ \ \lambda^{n}(y) = x\} \\ &= \{y \in X : \exists n \geq 0 \ \ \lambda^{n}(y) = x\} \cap K \\ &= \bigcup\{\lambda^{-n}(x) : n \geq 0\} \cap K = V(x, \tau_\lambda) \cap K, \end{aligned}$$

which leads to the desired result. □

Now we are ready to obtain the main result of this section in the following theorem.

Theorem 3.5. *For $\lambda : X \to X$ and*

$$Y := \{x \in X : \text{ there exists a strict } \lambda\text{-anti-orbit with the initial point } x\},$$

in the functional Alexandroff topological spaces (X, τ_λ) and $(Y, \tau_{\lambda\restriction_Y})$ we have:

(1) $a(\lambda) = c^*(Y, \tau_{\lambda\restriction_Y})$.

(2) *Consider the following statements:*

  (a) $a(\lambda) = c^*(X, \tau_\lambda)$;

  (b) $X = Y$;

  (c) $\lambda : X \to X$ *is surjective and Y contains all periodic points of λ.*

Then we have (c)⇔(b)⇒(a). Moreover if $c^*(X, \tau_\lambda)$ is finite, then (a), (b), and (c) are equivalent.

Proof: (1) For $k \in \{0, 1, 2, \ldots\}$ we show:

$$a(\lambda) \geq k \Rightarrow c^*(Y, \tau_{\lambda\restriction_Y}) \geq k.$$

It is clear that $c^*(Y, \tau_{\lambda\restriction_Y}) \geq 0$. If $a(\lambda) \geq 1$, then there exists a strict λ-anti-orbit $(x_n : n \geq 0)$ in X, thus $x_0 \in Y$ and $Y \neq \varnothing$, therefore $c^*(Y, \tau_{\lambda\restriction_Y}) \geq 1$. If $a(\lambda) \geq k \geq 2$, then there exist $k \geq 2$ disjoint strict λ-anti-orbits $(x^1_n : n \geq 0), \ldots, (x^k_n : n \geq 0)$. It is clear that for all $i \in \{1, \ldots, k\}$ we have $\{x^i_n : n \geq 0\} \subseteq Y$, therefore $(x^1_n : n \geq 0), \ldots, (x^k_n : n \geq 0)$ are k disjoint strict $\lambda\restriction_Y$-anti-orbits (by Lemma 3.4, Y is λ-invariant). By Lemma 3.3 there exist $m_1, \ldots, m_k \geq 0$ such that $V(x^1_{m_1}, \tau_{\lambda\restriction_Y}), \ldots, V(x^k_{m_k}, \tau_{\lambda\restriction_Y})$ are pairwise nonempty open subsets of $(Y, \tau_{\lambda\restriction_Y})$. Hence $c^*(Y, \tau_{\lambda\restriction_Y}) \geq k$ which leads to $c^*(Y, \tau_{\lambda\restriction_Y}) \geq a(\lambda)$.

Conversely, for $k \in \{0, 1, 2, \ldots\}$ we show:

$$c^*(Y, \tau_{\lambda\restriction_Y}) \geq k \Rightarrow a(\lambda) \geq k.$$

It is clear that $a(\lambda) \geq 0$. If $c^*(Y, \tau_{\lambda\restriction_Y}) \geq k \geq 1$, then there exist k disjoint nonempty open subsets $U_1, \ldots, U_k$ of $(Y, \tau_{\lambda\restriction_Y})$. For each $i \in \{1, \ldots, k\}$ choose $y_i \in U_i$. Since $y_i \in Y$, there is a strict λ-anti-orbit $(y^i_n : n \geq 0)$ in X with initial point $y_0^i = y_i$. It is clear that $\{y_n^i : n \geq 0\} \subseteq Y$ and $(y_n^i : n \geq 0)$ is a strict $\lambda\restriction_Y$-anti-orbit in Y, so by Lemma 3.1(1) we have

$$\{y_n^i : n \geq 0\} \subseteq V(y_i, \tau_{\lambda\restriction_Y}) \subseteq U_i$$

for all $i \in \{1, \ldots, k\}$. Since $U_1, \ldots, U_k$ are pairwise disjoint, $(y_n^1 : n \geq 0), \ldots, (y_n^k : n \geq 0)$ are k pairwise disjoint strict λ-anti-orbits and $a(\lambda) \geq k$. Thus $a(\lambda) \geq c^*(Y, \tau_{\lambda\restriction_Y})$ which completes the proof.

(2) For "(b)⇒(a)" use (1), also the implication "(b)⇒(c)" is obvious. In order to show "(c)⇒(b)", suppose $x \in X$, using the subjectivity of $\lambda : X \to X$ there exists $(x_n : n \geq 0)$ with $x_0 = x$ and $\lambda(x_{n+1}) = x_n$ for all $n \geq 0$. If $(x_n : n \geq 0)$ is one-to-one, then using the definition of Y we have $x \in Y$, otherwise there exists $n > m \geq 0$ with $x_n = x_m$, hence $x = \lambda^n(x_n) = \lambda^n(x_m) = \lambda^{n-m}(\lambda^m(x_m)) = \lambda^{n-m}(x)$ and x is a periodic point of λ. In order to complete the proof of (2), we prove the following two claims.

Claim 1. If $c^*(X, \tau_\lambda) = a(\lambda)$ is finite, then $\lambda : X \to X$ is surjective.

Proof of Claim 1: Suppose $c^*(X, \tau_\lambda) = a(\lambda) = k > 0$ is finite, then there exist disjoint strict λ-anti-orbits $(x_n^1 : n \geq 0), \ldots, (x_n^k : n \geq 0)$, and by Lemma 3.3 there exist $m_1, \ldots, m_k \geq 0$ such that $V(x_{m_i}^i, \tau_\lambda) \cap V(x_{m_j}^j, \tau_\lambda) = \varnothing$ for distinct $i, j \in \{1, \ldots, k\}$. Without loss of generality we may suppose $m_1 = m_2 = \cdots = m_k = 0$, so

$$\forall i \neq j \ (V(x_0^i, \tau_\lambda) \cap V(x_0^j, \tau_\lambda) = \varnothing).$$

If $\lambda : X \to X$ is not surjective, then there exists $x \in X \setminus \lambda(X)$, so $\{x\}$ is an isolated point of (X, τ_λ). We have the following cases.

(i) For all $i \in \{1, \ldots, k\}$, $x \notin V(x_0^i, \tau_\lambda)$. In this case $\{V(x_0^i, \tau_\lambda) : i \in \{1, \ldots, k\}\} \cup \{\{x\}\}$ is a collection of $k+1$ disjoint open subsets of (X, τ_λ), thus $c^*(X, \tau_\lambda) \geq k+1 > a(\lambda)$, which is a contradiction.

(ii) There exists $i \in \{1, \ldots, k\}$, with $x \in V(x_0^i, \tau_\lambda)$. First of all note that in this case there exists a unique $i \in \{1, \ldots, k\}$ with $x \in V(x_0^i, \tau_\lambda)$, so we may suppose $x \in V(x_0^1, \tau_\lambda)$ and $x \notin V(x_0^i, \tau_\lambda)$ for all $i > 1$. Since $x \in V(x_0^1, \tau_\lambda)$, thus there exists $p \geq 0$ with $\lambda^p(x) = x_0^1$ and $\lambda^s(x) \neq x_0^1$ for all $s < p$ (using $x \notin \lambda(X)$, we have $x \neq \lambda(x_1^1) = x_0^1 = \lambda^p(x)$, thus $p \geq 1$). We have the following cases.

- $x \notin V(x_p^1, \tau_\lambda)$. In this case $\{V(x_0^i, \tau_\lambda) : i \in \{2, \ldots, k\}\} \cup \{V(x_p^1, \tau_\lambda)\} \cup \{\{x\}\}$ is a collection of $k+1$ disjoint open subsets of (X, τ_λ), thus $c^*(X, \tau_\lambda) \geq k+1 > a(\lambda)$, which is a contradiction.
- $x \in V(x_p^1, \tau_\lambda)$. In this case there exists $q \geq 0$ with $\lambda^q(x) = x_p^1$, thus $x_0^1 = \lambda^p(x_p^0) = \lambda^p(\lambda^q(x)) = \lambda^q(\lambda^p(x)) = \lambda^q(x_0^1)$ (note that using $x \notin \lambda(X)$, we have $x \neq \lambda(x_{p+1}^1) = x_p^1 = \lambda^q(x)$, thus $q \geq 1$) so x_0^1 is a periodic point of λ. So $\{\lambda^n(x) : n \geq 0\} = \{\lambda^n(x) : 0 \leq n \leq p\} \cup \{\lambda^n(x_0^1) : n \geq 0\} = \{\lambda^n(x) : 0 \leq n \leq p\} \cup \{\lambda^n(x_0^1) : 0 \leq n \leq q\}$ is finite, but $(x_n^1 : n \geq 0)$ is a one-to-one sequence, thus there exists

$t \geq 0$ with $x_t^1 \notin \{\lambda^n(x) : n \geq 0\}$, so $x \notin V(x_t^1, \tau_\lambda)$. Similarly to the previous item $\{V(x_0^i, \tau_\lambda) : i \in \{2, \dots, k\}\} \cup \{V(x_t^1, \tau_\lambda)\} \cup \{\{x\}\}$ is a collection of $k+1$ disjoint open subsets of (X, τ_λ), thus $c^*(X, \tau_\lambda) \geq k+1 > a(\lambda)$, which is a contradiction.

Cases (i) and (ii) complete the proof of Claim 1.

Claim 2. If $c^*(X, \tau_\lambda) = a(\lambda)$ is finite, then Y contains all periodic points of λ.

Proof of Claim 2. Suppose $c^*(X, \tau_\lambda) = a(\lambda) = k > 0$ is finite, then similarly to the proof of Claim 1, there exist disjoint strict λ-anti-orbits $(x_n^1 : n \geq 0), \dots, (x_n^k : n \geq 0)$ with

$$\forall i \neq j \; (V(x_0^i, \tau_\lambda) \cap V(x_0^j, \tau_\lambda) = \varnothing).$$

If $x \in X \setminus Y$ is a periodic point of λ, then $\{\lambda^n(x) : n \geq 0\} \subseteq V(x, \tau_\lambda)$. Since Y is λ-invariant (use Lemma 3.4) and $x \notin Y$, hence $V(x, \tau_\lambda) \cap Y = \varnothing$, and for $i \in \{1, \dots, k\}$ we have $x_0^i \in Y \setminus V(x, \tau_\lambda)$, thus $V(x_0^i, \tau_\lambda) \not\subseteq V(x, \tau_\lambda)$. Moreover $x_0^i \in Y \setminus V(x, \tau_\lambda) \subseteq Y \setminus \{\lambda^n(x) : n \geq 0\}$, so $x \notin V(x_0^i, \tau_\lambda)$ which leads to $V(x, \tau_\lambda) \not\subseteq V(x_0^i, \tau_\lambda)$. Using Remark 1.1 we have $V(x, \tau_\lambda) \cap V(x_0^i, \tau_\lambda) = \varnothing$ for all $i \in \{1, \dots, k\}$, thus $\{V(x_0^i, \tau_\lambda) : i \in \{1, \dots, k\}\} \cup \{V(x, \tau_\lambda)\}$ is a collection of $k+1$ disjoint open subsets of (X, τ_λ), which is a contradiction and leads to the desired result. $\square$

4. Interaction between infinite orbit number of $\lambda : X \to X$ and cellularity of $(X, \overline{\tau}_\lambda)$

In this section we prove $o(\lambda) = c^*(X, \overline{\tau}_\lambda)$ for a self-map $\lambda : X \to X$ without any periodic points.

Theorem 4.1. *For $\lambda : X \to X$, $W(X, \lambda)$ is λ-invariant and*

(1) $o(\lambda) = c^*(W(X, \lambda), \overline{\tau}_{\lambda \restriction_{W(X,\lambda)}})$.

(2) *Consider the following statements:*

  (a) $o(\lambda) = c^*(X, \overline{\tau}_\lambda)$;

  (b) $X = W(X, \lambda)$;

  (c) *the map $\lambda : X \to X$ does not have any periodic points.*

Then we have (c)$\Leftrightarrow$(b)$\Rightarrow$(a). *Moreover if $c^*(X, \overline{\tau}_\lambda)$ is finite, then* (a), (b), *and* (c) *are equivalent.*

Proof: It is clear that $W(X, \lambda)$ is λ-invariant.

(1) Let $W := (W(X, \lambda), \overline{\tau}_{\lambda \restriction_{W(X,\lambda)}})$. For $k \in \{0, 1, 2, \dots\}$ we prove:

$$o(\lambda) \geq k \Rightarrow c^*(W) \geq k.$$

It is clear that $c^*(W) \geq 0$. If $o(\lambda) \geq k \geq 1$, then there exist $x_1, \dots, x_k \in X$ such that $(\lambda^n(x_1) : n \geq 0), \dots, (\lambda^n(x_k) : n \geq 0)$ are pairwise disjoint one-to-one sequences in X. Thus $x_1, \dots, x_k \in W(X, \lambda)$ and $\{\lambda^n(x_1) : n \geq 0\}, \dots, \{\lambda^n(x_k) : n \geq 0\} \in \overline{\tau}_{\lambda \restriction_{W(X,\lambda)}}$ are pairwise disjoint which leads to $c^*(W) \geq k$. Hence $c^*(W) \geq o(\lambda)$.

Conversely for $k \in \{0,1,2,\ldots\}$ we prove:

$$c^*(W) \geq k \Rightarrow o(\lambda) \geq k.$$

It is clear that $o(\lambda) \geq 0$. If $c^*(W) \geq k \geq 1$, then there exist k disjoint nonempty open subsets of W we denote by $U_1,\ldots,U_k$. For each $i \in \{1,\ldots,k\}$ choose $y_i \in U_i$. Since $y_i \in W(X,\lambda)$, $(\lambda^n(y_i) : n \geq 0)$ is a one-to-one sequence (and a subset of $W(X,\lambda)$). It is clear that $\{\lambda^n(y_i) : n \geq 0\} = V(y_i, \overline{\tau}_{\lambda \restriction_{W(X,\lambda)}}) \subseteq U_i$, thus $(\lambda^n(y_1) : n \geq 0),\ldots,(\lambda^n(y_k) : n \geq 0)$ are k disjoint strict λ-orbits and $o(\lambda) \geq k$. Hence $o(\lambda) \geq c^*(W)$, which leads to the desired result.

(2) Considering (1), (c)$\Leftrightarrow$(b)$\Rightarrow$(a) is obvious. In order to complete the proof suppose $c^*(X,\overline{\tau}_\lambda) = o(\lambda) = k \geq 1$ is finite, we prove λ has no periodic points. There exist $x_1,\ldots,x_k \in X$ such that $(\lambda^n(x_i) : n \geq 0)$ are k disjoint one-to-one sequences, thus $V(x_i,\overline{\tau}_\lambda) = \{\lambda^n(x_i) : n \geq 0\}$ are disjoint open subsets of $(X,\overline{\tau}_\lambda)$. If $x \in X$ is a periodic point of λ, there exists $p \geq 1$ with $\lambda^p(x) = x$, hence $V(x,\overline{\tau}_\lambda) = \{\lambda^i(x) : 0 \leq i \leq p\}$. Moreover for all $i \in \{1,\ldots,k\}$ and $n \geq 0$, $\lambda^n(x_i)$ are not periodic, thus $\{\lambda^n(x_i) : n \geq 0, 1 \leq i \leq k\} \cap \{\lambda^i(x) : 0 \leq i \leq p\} = \varnothing$. Therefore $\{V(x_i,\overline{\tau}_\lambda) : 1 \leq i \leq k\} \cup \{V(x,\overline{\tau}_\lambda)\}$ is a collection of $k+1$ disjoint nonempty open subsets of $(X,\overline{\tau}_\lambda)$, which is a contradiction and leads to the desired result. □

Here we mention that in Theorem 4.1(2) (resp. Theorem 3.5(2)) one can easily find examples with $o(\lambda) = c^*(X,\overline{\tau}_\lambda) = \infty$ (resp. $a(\lambda) = c^*(X,\tau_\lambda) = \infty$) where both (c) and (b) fail.

5. Main results

Now we are ready to establish our main results.

Theorem 5.1. *Consider* $\lambda : X \to X$.
1. Interaction of $c^*(X,\overline{\tau}_\lambda)$ and the set-theoretical entropy of λ. *We have* $\mathrm{ent}_{set}(\lambda) = c^*(W(X,\lambda), \overline{\tau}_{\lambda \restriction_{W(X,\lambda)}})$. *In particular if λ does not have any periodic points, then* $\mathrm{ent}_{set}(\lambda) = c^*(X,\overline{\tau}_\lambda)$.
2 Interaction of $c^*(X,\tau_\lambda)$ and the contravariant set-theoretical entropy of λ. *For $Y = \{x \in X :$ there exists a strict λ-anti-orbit with the initial point $x\}$, $\mathrm{ent}_{cset}(\lambda) = c^*(Y, \tau_{\lambda \restriction_Y})$. In particular if $\lambda : X \to X$ is surjective and Y contains all periodic points of λ, then $Y = X$ and* $\mathrm{ent}_{cset}(\lambda) = c^*(X,\tau_\lambda)$.

PROOF: Use Remark 2.1, Theorem 3.5 and Theorem 4.1. □

The following corollary completes our investigations.

Corollary 5.2. *Consider* $\lambda : X \to X$.
1. Interaction of $c^*(X,\overline{\tau}_\lambda)$ and the topological entropy of σ_λ. *If K is finite and discrete with at least two elements and X is arbitrary with at least two elements, and K^X is endowed with the product topology, then for $\sigma_\lambda : K^X \to K^X$ we have:*

$$\mathrm{ent}_{top}(\sigma_\lambda) = c^*(W(X,\lambda), \overline{\tau}_{\lambda \restriction_{W(X,\lambda)}}) \log |K|.$$

In particular if λ does not have any periodic points, then $\mathrm{ent}_{top}(\sigma_\lambda) = c^*(X,\overline{\tau}_\lambda)\log|K|$.

2. Interaction of $c^*(X,\tau_\lambda)$ and the algebraic entropy of σ_λ. *If K is a nontrivial finite group with at least two elements, X is nonempty and $\lambda : X \to X$ is a finite fiber map, then for $\sigma_\lambda \restriction_{\bigoplus_X K} : \bigoplus_X K \to \bigoplus_X K$ we have*

$$\mathrm{ent}_{alg}(\sigma_\lambda \restriction_{\bigoplus_X K}) = c^*(Y,\tau_{\lambda\restriction_Y})\log|K|,$$

where $Y = \{x \in X :$ there exists a strict λ-anti-orbit with the initial point $x\}$. In particular if $\lambda : X \to X$ is surjective and Y contains all periodic points of λ, then $Y = X$ and

$$\mathrm{ent}_{alg}(\sigma_\lambda \restriction_{\bigoplus_X K}) = c^*(X,\tau_\lambda)\log|K|.$$

Proof: Use Remark 2.3, Remark 2.2, and Theorem 5.1. □

Acknowledgment. The authors wish to express their deep thanks to the referee who helped us to improve the text and the theorems with her/his useful suggestions.

The authors are grateful to the research division of the University of Tehran, for the grant which supported this research under the ref. no. 6103027/1/08.

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(*Received* November 9, 2013, *revised* March 9, 2016)