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ON THE STRONG BRILLINGER-MIXING PROPERTY OF α-DETERMINANTAL POINT PROCESSES AND SOME APPLICATIONS

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Abstract. First, we derive a representation formula for all cumulant density functions in terms of the non-negative definite kernel function $C(x, y)$ defining an α-determinantal point process (DPP). Assuming absolute integrability of the function $C_0(x) = C(o, x)$, we show that a stationary α-DPP with kernel function $C_0(x)$ is "strongly" Brillinger-mixing, implying, among others, that its tail-σ-field is trivial. Second, we use this mixing property to prove rates of normal convergence for shot-noise processes and sketch some applications to statistical second-order analysis of α-DPPs.

Keywords: determinantal point process; permanental point process; trivial tail-σ-field; exponential moment; shot-noise process; Berry-Esseen bound; multiparameter K-function; kernel-type product density estimator; goodness-of-fit test

MSC 2010: 60G55, 60F05

1. Introduction

In the last 15 years, determinantal and permanental point processes (DPPs and PPPs, respectively, for short) have attracted increasing attention in pure and applied mathematics as well as of mathematical physics. The first steps to implement this class of point processes were the description of finite random point patterns in bounded subsets of a Euclidean space $\mathbb{R}^d$ of dimension $d \geqslant 1$ in terms of Janossy densities, see Chapters 5.3 and 5.4 of the monograph [3], which also serves us as the main reference on point process theory. A crucial step was done by Macchi [21] by introducing DPPs and PPPs as mathematical models for configurations of fermions and bosons, respectively. Regarding this physical background, DPPs/PPPs provide models that can be used to describe point processes with repulsion/attraction forces between the atoms. From this viewpoint, these PPs are the counterparts to Poisson processes which consist of non-interacting atoms. An embedding of DPPs and PPPs into the theory of Gibbsian PPs can be found in [5].

Next, to avoid ambiguities, we introduce in a rigorous way some definitions and basic notions from PP theory. For the definition of α-DPPs and an overview on the state of art of the theory and the diverse applications of DPPs and PPPs we refer the reader to Camilier & Decreusefond [2] and the list of references therein. The main part of our paper is focused on stationary α-DPPs and is inspired by the recent paper [18] and its extended arXiv versions.

Let $[N, \mathcal{N}]$ denote the measurable space of all locally finite counting measures on $\mathbb{R}^d$ (equipped with its σ-algebra $\mathcal{B}^d$ of Borel sets), where the σ-algebra $\mathcal{N}$ is generated by the family of sets $\{\psi \in N\colon\ \psi(B) = n\}$ for $n \in \mathbb{N} \cup \{0\}$ and $B \in \mathcal{B}_b^d$ (= ring of bounded sets in $\mathcal{B}^d$). A PP on $\mathbb{R}^d$ is defined to be a measurable mapping Ψ from a hypothetical probability space $[\Omega, \mathcal{F}, \mathsf{P}]$ (it always exists!) into $[N, \mathcal{N}]$. Throughout this paper we assume that Ψ is *simple*, i.e. $\mathsf{P}\big(\Psi(\{x\}) \leqslant 1\ \forall\, x \in \mathbb{R}^d\big) = 1$. We briefly write $\Psi = \sum\limits_{i \geqslant 1} \delta_{X_i} \sim P$ with Dirac measure $\delta_x(B) = \mathbf{1}_B(x) = 1$ or 0, if $x \in B$ or $x \notin B$, respectively. Here, the countable set $\{X_i\colon\ i \geqslant 1\}$ of atoms of Ψ (having multiplicity 1) can be considered as locally finite random closed (configuration) in $\mathbb{R}^d$ and the probability measure $P = \mathsf{P} \circ \Psi^{-1}$ induced by Ψ on $[N, \mathcal{N}]$ is called the *distribution of* Ψ. Thanks to the simplicity of Ψ, the distribution of Ψ is uniquely determined by the void probabilities $\mathsf{P}(\Psi(C) = 0)$ for all compact $C \subset \mathbb{R}^d$. Further, let E and Var denote the expectation and variance, respectively, with respect to P.

If $\mathsf{E}\Psi^k(B) < \infty$ for $B \in \mathcal{B}_b^d$, then there exist the locally finite *k-th factorial moment measure* $\alpha^{(k)}$ and the (signed) *k-th factorial cumulant measure* $\gamma^{(k)}$ on $[\mathbb{R}^{dk}, \mathcal{B}^{dk}]$ defined for any $B_1, \dots, B_k \in \mathcal{B}_b^d$ by

$$\begin{aligned}\alpha^{(k)}\Big(\mathop{\times}_{j=1}^{k} B_j\Big) &:= \int\limits_N \sum_{\substack{x_1,\dots,x_k \\ \in \operatorname{supp}(\psi)}}^{\neq} \prod_{j=1}^{k} \mathbf{1}_{B_j}(x_j) P(\mathrm{d}\psi) \\ &= (-1)^k \lim_{z_1,\dots,z_k \downarrow 0} \frac{\partial^k}{\partial z_1 \dots \partial z_k} \mathbf{G}_P\Big[1 - \sum_{i=1}^{k} z_i \mathbf{1}_{B_i}\Big],\end{aligned}$$

and by

$$\gamma^{(k)}\Big(\mathop{\times}_{j=1}^{k} B_j\Big) := (-1)^k \lim_{z_1,\dots,z_k \downarrow 0} \frac{\partial^k}{\partial z_1 \dots \partial z_k} \log \mathbf{G}_P\Big[1 - \sum_{i=1}^{k} z_i \mathbf{1}_{B_i}\Big], \tag{1.1}$$

respectively, where the probability generating functional $\mathbf{G}_P[f] := \mathsf{E}\Big(\prod\limits_{i \geqslant 1} f(X_i)\Big)$ of $\Psi = \sum\limits_{i \geqslant 1} \delta_{X_i}$ is defined for any measurable $f\colon\ \mathbb{R}^d \to [0,1]$ such that $\operatorname{supp}(1-f) \in \mathcal{B}_b^d$.

For a stationary Poisson process with intensity λ it has the simple exponential shape $\exp\{\lambda \int_{\mathbb{R}^d} (f(x) - 1)\,\mathrm{d}x\}$. Here and below, $\sum^{\neq}$ indicates summation over tuples consisting of pairwise distinct elements. If $\alpha^{(k)}$ is absolutely continuous w.r.t. the Lebesgue measure on $\mathcal{B}^{dk}$, the corresponding Radon-Nikodym density $\varrho^{(k)}\colon \mathbb{R}^{dk} \to [0,\infty]$ is called the *k-th product density*, *k-th correlation function* or *k-point intensity*. These names are motivated by the interpretation that $\mathsf{P}(\Psi(\mathrm{d}x_1) = 1, \ldots, \Psi(\mathrm{d}x_k) = 1) \approx \varrho^{(k)}(x_1,\ldots,x_k)\,\mathrm{d}x_1 \ldots \mathrm{d}x_k$. The existence of the product densities $\varrho^{(1)},\ldots,\varrho^{(k)}$ implies the existence of a Lebesgue density $c^{(k)}\colon \mathbb{R}^{dk} \to [-\infty,\infty]$ of $\gamma^{(k)}$ called the *k-th cumulant density* or the *k-th truncated correlation function*. Since $\gamma^{(k)}$ is a signed measure, the function $c^{(k)}$ may take on positive and negative values. The evaluation of the logarithmic derivatives in (1.1) yields the formula (which goes back to the pioneering paper of Leonov & Shiryaev [20], see also [3], pp. 147–148, or [10])

$$c^{(k)}(x_1,\ldots,x_k) = \sum_{l=1}^{k} (-1)^{l-1}(l-1)! \sum_{\substack{K_1\cup\ldots\cup K_l\\=\{1,\ldots,k\}}} \prod_{j=1}^{l} \varrho^{(\kappa_j)}(x_{k_j}; k_j \in K_j), \tag{1.2}$$

and by inverting the latter relation, we get that

$$\varrho^{(k)}(x_1,\ldots,x_k) = \sum_{l=1}^{k} \sum_{\substack{K_1\cup\ldots\cup K_l\\=\{1,\ldots,k\}}} \prod_{j=1}^{l} c^{(\kappa_j)}(x_{k_j}; k_j \in K_j), \tag{1.3}$$

where the sum $\sum\limits_{K_1\cup\ldots\cup K_l=\{1,\ldots,k\}}$ is taken over all partitions of the set $\{1,\ldots,k\}$ into l disjoint nonempty subsets K_j, and $\kappa_j := \# K_j$ denotes the cardinality of K_j.

If $\Psi \sim P$ is stationary with intensity $\lambda = \mathsf{E}\Psi([0,1]^d) > 0$, both the measures $\alpha^{(k)}$ and $\gamma^{(k)}$ are invariant on the product set $\bigtimes_{j=1}^{k} B_j$ under diagonal shifts, which allows to define implicitly the *k-th reduced factorial moment measure* $\alpha^{(k)}_{\mathrm{red}}$ and the *k-th reduced factorial cumulant measure* $\gamma^{(k)}_{\mathrm{red}}$ on $[\mathbb{R}^{d(k-1)}, \mathcal{B}^{d(k-1)}]$ by disintegration w.r.t. the intensity measure $\lambda|\cdot|$:

$$\alpha^{(k)}\Big(\bigtimes_{j=1}^{k} B_j\Big) = \lambda \int_{B_k} \alpha^{(k)}_{\mathrm{red}}\Big(\bigtimes_{j=1}^{k-1} (B_j - x)\Big)\,\mathrm{d}x$$

and

$$\gamma^{(k)}\Big(\bigtimes_{j=1}^{k} B_j\Big) = \lambda \int_{B_k} \gamma^{(k)}_{\mathrm{red}}\Big(\bigtimes_{j=1}^{k-1} (B_j - x)\Big)\,\mathrm{d}x,$$

see [4], p. 238, for more details. If the k-th product density $\varrho^{(k)}(x_1,\dots,x_k)$ exists then $\alpha^{(k)}_{\mathrm{red}}$ and $\gamma^{(k)}_{\mathrm{red}}$ possesses the Lebesgue densities $\lambda^{-1}\varrho^{(k)}(x_1,\dots,x_{k-1},o)$ and $\lambda^{-1}c^{(k)}(x_1,\dots,x_{k-1},o)$, respectively.

The *total variation measure* $|\gamma^{(k)}_{\mathrm{red}}|$ is defined by $|\gamma^{(k)}_{\mathrm{red}}|(\cdot) = (\gamma^{(k)}_{\mathrm{red}})^+(\cdot) + (\gamma^{(k)}_{\mathrm{red}})^-(\cdot)$, where the measures $(\gamma^{(k)}_{\mathrm{red}})^+$ and $(\gamma^{(k)}_{\mathrm{red}})^-$ are given by the Jordan decomposition $\gamma^{(k)}_{\mathrm{red}}(\cdot) = (\gamma^{(k)}_{\mathrm{red}})^+(\cdot) - (\gamma^{(k)}_{\mathrm{red}})^-(\cdot)$. The *total variation* of $\gamma^{(k)}_{\mathrm{red}}$ is defined by the (finite) number $\|\gamma^{(k)}_{\mathrm{red}}\|_{TV} := |\gamma^{(k)}_{\mathrm{red}}|(\mathbb{R}^{d(k-1)})$.

Definition 1. A stationary PP $\Psi \sim P$ with intensity $\lambda > 0$ such that $\mathsf{E}\Psi^k([0,1]^d) < \infty$ for all $k \geqslant 2$ is said to be *Brillinger-mixing* if the total variation $\|\gamma^{(k)}_{\mathrm{red}}\|_{TV}$ is finite for all $k \geqslant 2$. In the case that the cumulant densities $c^{(k)}$ of any order $k \geqslant 2$ exist; this means that, for all $k \geqslant 2$,

$$\|\gamma^{(k)}_{\mathrm{red}}\|_{TV} = \frac{1}{\lambda}\int_{\mathbb{R}^{d(k-1)}} |c^{(k)}(x_1,\dots,x_{k-1},o)|\,\mathrm{d}(x_1,\dots,x_{k-1}) < \infty, \tag{1.4}$$

see [15], [12], [9], [10]. $\Psi \sim P$ is called *strongly Brillinger-mixing* if $\|\gamma^{(k)}_{\mathrm{red}}\|_{TV} \leqslant a^k k!$ for some $a > 0$ and all $k \geqslant 2$, see [8].

Finally, if the PP $\Psi \sim P$ is stationary and isotropic then its second-order properties are determined by the intensity $\lambda > 0$ and *Ripley's K-function* $K(r) := \lambda^{-1}\alpha^{(2)}_{\mathrm{red}}(B_d(r))$ with $B_d(r) := \{x \in \mathbb{R}^d\colon \|x\| \leqslant r\}$ or, if $\varrho^{(2)}$ exists, by the *pair correlation function* $g\colon (0,\infty) \to [0,\infty]$ defined by $g(r) := \lambda^{-2}\varrho^{(2)}(x,o)$ for $x \in \mathbb{R}^d$ with Euclidean norm $\|x\| = r > 0$. For non-isotropic PPs we suggest to consider the *multiparameter K-function* $K(\mathbf{r}) := \lambda^{-1}\alpha^{(2)}_{\mathrm{red}}(B_d(\mathbf{r}))$ with symmetric rectangles $B_d(\mathbf{r}) = \bigtimes_{i=1}^{d} [-r_i, r_i]$ for $\mathbf{r} = (r_1,\dots,r_d) \in \mathbb{R}^d_+$, see [7].

The remaining part of the paper is organized as follows. Section 2 summarizes the essential properties of α-DPPs. Theorem 1 provides a comparatively simple representation of the cumulant density function in terms of α and kernel C, which forms the basis to prove the strong Brillinger-mixing property of stationary α-DPPs in Section 3. An interesting application of this fact to (integrated) shot-noise processes driven by stationary α-DPPs is formulated in Theorem 3. Section 4 contains further applications of this mixing property to proving asymptotic Gaussianity of empirical (reduced) moment measures (on large sampling windows) of stationary α-DPPs with a special focus on the K-function, the second product density and the pair correlation function. These results enable us, at least in principle, to construct asymptotic goodness-of-fit tests to check hypotheses concerning the kernel function $C_0(x)$.

2. Definition and properties of α-Determinantal Point Processess

The definition of an α-DPP is based on three incredients: (i) an admissible real parameter $\alpha \geqslant -1$, (ii) a Radon measure ν on $\mathcal{B}^d$ which is chosen in this paper to be the Lebesgue measure $\nu(\cdot) = |\cdot|$ and (iii) a (complex-valued) *covariance function* $C\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, that is, C is non-negative definite, i.e.

$$\sum_{i,j=1}^{n} z_i C(x_i, x_j)\overline{z_j} \geqslant 0 \quad \forall\, x_1, \ldots, x_n \in \mathbb{R}^d,\ z_1, \ldots, z_n \in \mathbb{C} \text{ and } n \geqslant 1. \tag{2.1}$$

This implies that $C(x,x) \geqslant 0$, $C(x,y) = \overline{C(y,x)}$ and $|C(x,y)|^2 \leqslant C(x,x)C(y,y)$ for $x, y \in \mathbb{R}^d$.

Definition 2. A simple PP $\Psi_\alpha = \sum_{i \geqslant 1} \delta_{X_i} \sim P_\alpha$ on $\mathbb{R}^d$ is called *α-determinantal PP* for an admissible $\alpha \geqslant -1$ with covariance function C (α-DPP(C) for short) if for all $k \geqslant 1$, the k-th product density $\varrho_\alpha^{(k)}$ is given by

$$\varrho_\alpha^{(k)}(x_1, \ldots, x_k) = \sum_{\pi \in \mathcal{P}(1,\ldots,k)} \alpha^{k-n(\pi)} \prod_{i=1}^{k} C(x_i, x_{\pi(i)}) \tag{2.2}$$

for all $x_1, \ldots, x_k \in \mathbb{R}^d$, where the sum is taken over the set $\mathcal{P}(1, \ldots, k)$ of all permutations of $\{1, \ldots, k\}$, and $n(\pi)$ is the number of cycles in the permutation π. Here, a *cycle* in the permutation $\pi \in \mathcal{P}(1, \ldots, k)$ is an ordered subset $\{i_1, \ldots, i_l\}$ of $\{1, \ldots, k\}$ that is cyclically permuted, i.e., $\pi(i_1) = i_2, \ldots, \pi(i_{l-1}) = i_l$, $\pi(i_l) = i_1$ if the cycle length is $l \geqslant 2$, and $\pi(i_1) = i_1$ for $l = 1$. Note that any $\pi \in \mathcal{P}(1, \ldots, k)$ has a unique decomposition in $n(\pi)$ pairwise disjoint cycles covering $\{1, \ldots, k\}$, see [23], Chapt. 6. The number $\operatorname{sgn}(\pi) := (-1)^{k-n(\pi)}$ is called the *sign* of π.

Using the sign $\operatorname{sgn}(\pi)$ of a permutation π and the usual (Leibniz) definition of a determinant, see e.g. [23], p. 221, we get from (2.2) the representations

$$\varrho_{-1}^{(k)}(x_1, \ldots, x_k) = \det\bigl(C(x_i, x_j)\bigr)_{i,j=1}^{k} = \begin{vmatrix} C(x_1,x_1) & \cdots & C(x_1,x_k) \\ \vdots & \ddots & \vdots \\ C(x_k,x_1) & \cdots & C(x_k,x_k) \end{vmatrix} \tag{2.3}$$

of any order $k \geqslant 1$ defining the *determinantal* PP with kernel function C (DPP(C) for short). For $\alpha = 1$, the corresponding PP with k-th product density

$$\varrho_1^{(k)}(x_1, \ldots, x_k) = \operatorname{per}\bigl(C(x_i, x_j)\bigr)_{i,j=1}^{k} := \sum_{\pi \in \mathcal{P}(1,\ldots,k)} \prod_{i=1}^{k} C(x_i, x_{\pi(i)}) \tag{2.4}$$

for $k \geqslant 1$ is called the *permanental* PP with kernel function C (PPP(C) for short).

Notice that a real parameter α is admissible if there exists a covariance function C such that the functions $\varrho_\alpha^{(k)}$ defined by (2.2) turn out to be product densities of some PP $\Psi_\alpha \sim P_\alpha$. For example, $\alpha < -1$ is not admissible since $\varrho_\alpha^{(2)}(x,y) = \alpha|C(x,y)|^2 + C(x,x)C(y,y)$ takes on negative values on the diagonal. Notice that it is still a nontrivial problem to determine the set of admissible parameters, see e.g. [19], [24], [2]. To avoid technical problems, we assume throughout that the covariance function C is continuous on $(\mathbb{R}^d)^2$. Below we will show that $\Psi \sim P$ (if it exists) is uniquely determined by its factorial moments $\alpha^{(k)}(B^k) = \int_{B^k} \varrho_\alpha^{(k)}(x)\,\mathrm{d}x$ for all $B \in \mathcal{B}_b^d$ and $k \geqslant 1$. In [2], the set of admissible parameters $\alpha \in \{2/m\colon m \in \mathbb{N}\} \cup \{-1/m\colon m \in \mathbb{N}\}$ is discussed in connection with fractional powers $-1/\alpha$ of certain Fredholm determinants expressing the Laplace functional $\mathbf{G}_P[\exp\{-h\}]$ of α-DPPs for positive, compactly supported h on $\mathbb{R}^d$. The best understood cases are DPPs and PPPs, see e.g. [2] for details. The crucial role is played by the selfadjoint integral operator $(\mathcal{C}_D f)(\cdot) = \int_D C(x,\cdot) f(x)\,\mathrm{d}x$ mapping from $L^2(D)$ into $L^2(D)$, which turns out to be a Hilbert-Schmidt operator for any compact $D \subset \mathbb{R}^d$ (locally trace-class operator) with norm $\|\mathcal{C}_D\|_2 := \bigl(\int_{D\times D} |C(x,y)|^2\,\mathrm{d}(x,y)\bigr)^{1/2}$. In particular, a DPP(C), i.e. $\alpha = -1$, with continuous C exists if and only if for each compact D, all real, non-negative eigenvalues of $\mathcal{C}_D$ belong to $[0,1]$, see [18]. For a stationary DPP(C), this spectral condition can be expressed more explicitly, see Section 3.

Lemma 1. *For an admissible $\alpha \geqslant -1$ let $\Psi_\alpha \sim P_\alpha$ be an α-DPP(C) on $\mathbb{R}^d$ with covariance function C such that $\|C_B\| := \int_B C(x,x)\mathrm{d}x < \infty$ for $B \in \mathcal{B}_b^d$. Then for any $B \in \mathcal{B}_b^d$ there exists a number $s = s(\alpha, B, C) > 0$ such that $\mathsf{E}e^{s\Psi_\alpha(B)} < \infty$. In other words, $\Psi_\alpha(B)$ possesses an exponential moment, which implies the uniqueness of the distribution P_α.*

P r o o f. We may assume the existence of at least one PP $\Psi_\alpha \sim P_\alpha$ with product densities defined by (2.2) for a certain covariance function C. Let B^k denote the k-fold cartesian product of B. Then $\alpha^{(k)}(B^k)$ coincides with the k-th factorial moment $\mathsf{E}\bigl[\Psi_\alpha(B)(\Psi_\alpha(B)-1)\ldots(\Psi_\alpha(B)-k+1)\bigr]$ of $\Psi_\alpha(B)$. On the other hand, (2.2) yields that

$$\begin{aligned}\alpha^{(k)}(B^k) &= \int_{B^k} \varrho_\alpha^{(k)}(x)\,\mathrm{d}x\\ &\leqslant k!(\alpha\vee 1)^{k-1} \max_{\pi\in\mathcal{P}(1,\ldots,k)} \int_B \ldots \int_B \prod_{i=1}^k |C(x_i, x_{\pi(i)})|\,\mathrm{d}x_1\ldots\mathrm{d}x_k.\end{aligned}$$

By the inequality $|C(x,y)|^2 \leqslant C(x,x)C(y,y)$ with $C(x,x) \geqslant 0$ for all $x \in \mathbb{R}^d$, it follows that

$$\prod_{i=1}^k |C(x_i, x_{\pi(i)})| \leqslant \prod_{i=1}^k C(x_i,x_i) \quad \text{for each } \pi \in \mathcal{P}(1,\ldots,k).$$

Hence, applying Fubini's theorem leads to $\alpha^{(k)}(B^k) \leqslant k!(\alpha \vee 1)^{k-1}\|C_B\|^k$. Using the factorial moment generating function of $\Psi_\alpha(B)$, we arrive at

$$\mathsf{E}(1+s)^{\Psi_\alpha(B)} = 1+\sum_{k=1}^{\infty} \alpha^{(k)}(B^k)\frac{s^k}{k!} \leqslant \frac{1}{1-s\|C_B\|(\alpha \vee 1)} \quad \text{for } 0 \leqslant s < \frac{1}{\|C_B\|(\alpha \vee 1)}.$$

Finally, the elementary estimate $\mathrm{e}^s \leqslant 1+2s$ for $0 \leqslant s \leqslant 2/3$ yields

$$\mathsf{E}\mathrm{e}^{s\Psi_\alpha(B)} \leqslant \frac{1}{1-\|C_B\|(\alpha \vee 1)} \quad \text{for } 0 \leqslant s < \frac{2}{3} \wedge \frac{1}{2\|C_B\|(\alpha \vee 1)},$$

which completes the proof of Lemma 1. □

Remark 1. It should be mentioned that in the case of DPPs, uniqueness has been stated in [24], Theorem 3, and the existence of exponential moments is shown in [13], Lemma 4.2.6. Further, note that for a DPP(C) Ψ_{-1} the above bounds of the factorial moments $\alpha^{(k)}(B^k)$ can be improved. The positive definiteness of the Hermitian matrix $(C(x_i,x_j))_{i,j=1}^k$ defining $\varrho_{-1}^{(k)}$ in (2.3) allows to show, see [23], Theorem 9.4.10, that $\varrho_{-1}^{(k)}(x_1,\ldots,x_k) \leqslant \varrho_{-1}^{(k-1)}(x_1,\ldots,x_{k-1})C(x_k,x_k) \leqslant \prod_{i=1}^k C(x_i,x_i)$, which implies that $\alpha^{(k)}\Bigl(\mathop{\times}\limits_{i=1}^k B_i\Bigr) \leqslant \prod_{i=1}^k \|C_{B_i}\|^k$ for all $k \geqslant 1$ and $B_1,\ldots,B_k \in \mathcal{B}_b^d$. Thus, Ψ_{-1} turns out to be a so-called *sub-Poisson process* with an entire factorial moment generating function

$$M_B(z) := \mathsf{E}(1+z)^{\Psi_{-1}(B)} = 1+\sum_{k=1}^{\infty} \alpha^{(k)}(B^k)\frac{z^k}{k!} \quad \text{for } z \in \mathbb{C},\ B \in \mathcal{B}_b^d$$

providing the *void probabilities* $\mathsf{P}(\Psi_{-1}(B)=0) = M_B(-1)$ and the bound $|M_B(z)| \leqslant \exp\{|z|\|C_B\|\}$.

The following result is crucial for proving the strong Brillinger-mixing property of stationary α-DPPs in the next section.

Theorem 1. *For any admissible $\alpha \geqslant -1$ and $k \geqslant 2$, the k-th cumulant density $c_\alpha^{(k)}(x_1,\ldots,x_k)$ defined by (1.2) with product densities $\varrho_\alpha^{(1)},\ldots,\varrho_\alpha^{(k)}$ given in (2.2) can be expressed as*

$$\begin{aligned} (2.5)\quad & c_\alpha^{(k)}(x_1,\ldots,x_k) \\ &= \alpha^{k-1} \sum_{\pi \in \mathcal{P}(\{2,\ldots,k\})} C(x_1,x_{\pi(2)})C(x_{\pi(2)},x_{\pi(3)})\ldots C(x_{\pi(k)},x_1) \\ (2.6)\quad &= \alpha^{k-1} \sum_{\pi \in \mathcal{P}(\{1,\ldots,k\}\setminus\{j\})} C(x_j,x_{\pi(1)})\ldots C(x_{\pi(j-1)},x_{\pi(j+1)})\ldots C(x_{\pi(k)},x_j) \end{aligned}$$

for any $x_1,\dots,x_k \in \mathbb{R}^d$ and $j = 2,\dots,k$, where the set $\mathcal{P}(k_1,\dots,k_l)$ contains all $l!$ permutations of the distinct numbers $k_1,\dots,k_l$. (In (2.6) set $\pi(k+1) := k$ for $j = k$.)

P r o o f. For notational ease we put $C_{ij} = C(x_i,x_j)$ for $x_1,\dots,x_k \in \mathbb{R}^d$. To begin with we show that each summand on the r.h.s. of (2.5) occurs in the sum (2.6) and vice versa. For fixed $\pi \in \mathcal{P}(\{2,\dots,k\})$ and $j \in \{2,\dots,k\}$ there exists an $i \in \{2,\dots,k\}$ such that $\pi(i) = j$. This gives

$$\begin{aligned}&C_{1\pi(2)}C_{\pi(2)\pi(3)}\dots C_{\pi(i-1)j}C_{j\pi(i+1)}\dots C_{\pi(k)1}\\&\qquad = C_{j\pi(i+1)}C_{\pi(i+1)\pi(i+2)}\dots C_{\pi(k)1}C_{1\pi(2)}\dots C_{\pi(i-2)\pi(i-1)}C_{\pi(i-1)j}.\end{aligned}$$

Obviously, the numbers $1,\pi(2),\dots,\pi(i-1),\pi(i+1),\dots,\pi(k)$ form a permutation of the set $\{1,\dots,k\}\setminus\{j\}$, confirming the first part. With the same argument it follows that every summand in the sum (2.6) shows up on the r.h.s. of (2.5). The representation (2.5) can be verified by induction on $k \geqslant 2$.

For $k = 2$ the formulae (1.2) and (2.2) combined with $\varrho_\alpha^{(1)}(x_i) = C(x_i,x_i)$ yield

$$c_\alpha^{(2)}(x_1,x_2) = \varrho_\alpha^{(2)}(x_1,x_2) - \varrho_\alpha^{(1)}(x_1)\varrho_\alpha^{(1)}(x_2) = \alpha C(x_1,x_2)C(x_2,x_1). \tag{2.7}$$

This coincides with (2.5) for $k = 2$. Together with the trivial relation $c_\alpha^{(1)}(x) = C(x,x)$ we now assume the validity of (2.5) for cumulant densities $c_\alpha^{(\kappa)}$ of order $\kappa = 2,\dots,k-1$. For $k \geqslant 3$, an obvious rearrangement of (1.3) gives the relation

$$c_\alpha^{(k)}(x_1,\dots,x_k) = \varrho_\alpha^{(k)}(x_1,\dots,x_k) - \sum_{l=2}^{k}\sum_{\substack{K_1\cup\ldots\cup K_l\\=\{1,\dots,k\}}}\prod_{j=1}^{l} c_\alpha^{(\kappa_j)}(x_{k_j};k_j\in K_j) \tag{2.8}$$

for any $x_1,\dots,x_k \in \mathbb{R}^d$, where the sum on the r.h.s. and the κ_j's are the same as in (1.3).

For $j = 1,\dots,l$, let $K_j = \{k_{j1},k_{j2},\dots,k_{j\kappa_j}\}$ with $k_{j1} < k_{j2} < \ldots < k_{j\kappa_j}$, and $\pi_j \in \mathcal{P}(K_j\setminus\{k_{j1}\})$ runs over all $(\kappa_j-1)!$ permutations of the set $K_j\setminus\{k_{j1}\}$ (which is empty if $\kappa_j = 1$). If $l \geqslant 2$, the above assumption enables us to write

$$\begin{aligned}&c_\alpha^{(\kappa_j)}(x_q;q\in K_j)\\&\qquad = \alpha^{\kappa_j-1}\sum_{\pi_j\in\mathcal{P}(K_j\setminus\{k_{j1}\})} C_{k_{j1}\pi_j(k_{j2})}C_{\pi_j(k_{j2})\pi_j(k_{j3})}\dots C_{\pi_j(k_{j\kappa_j})k_{j1}},\end{aligned} \tag{2.9}$$

for $j = 1,\dots,l$. Here, the sum stretches over all cyclic permutations of the cycle $k_{j1},k_{j2},\dots,k_{j\kappa_j}$ of the length $\kappa_j \geqslant 2$ which begins and ends with the smallest element k_{j1} and the remaining elements are permuted. Let us denote such cycle by $\sigma(k_{j1},k_{j2},\dots,k_{j\kappa_j})$. We put $c_\alpha^{(1)}(x_{k_{j1}}) = C_{k_{j1}k_{j1}}$ if $\kappa_j = 1$, in accordance with a cycle of length 1.

Next we make use of the fact that each permutation $\pi \in \mathcal{P}(1,\dots,k)$ can be written in a unique way as a union of cyclic permutations of pairwise disjoint cycles (ordered according to ascending smallest cycle elements). To obtain all $\pi \in \mathcal{P}(1,\dots,k)$ we decompose $\{1,\dots,k\}$ into nonempty, pairwise disjoint sets $K_1,\dots,K_l$ for $l=1,\dots,k$, and take the union of the k-tuples $\bigcup\limits_{j=1}^{l} \sigma(k_{j1},\pi_j(k_{j2}),\dots,\pi_j(k_{j\kappa_j}))$ over all cyclic permutations π_j of the cycles $K_j=\{k_{j1},\dots,k_{j\kappa_j}\}$ for $j=1,\dots,l$. With the above notation this can be formally expressed by

$$\mathcal{P}(1,\dots,k) = \bigcup_{l=1}^{k} \bigcup_{\substack{K_1\cup\ldots\cup K_l\\ =\{1,\dots,k\}}} \bigcup_{\substack{\pi_j\in\mathcal{P}(K_j\setminus\{k_{j1}\})\\ j=1,\dots,l}} \bigcup_{j=1}^{l} \sigma(k_{j1},\pi_j(k_{j2}),\dots,\pi_j(k_{j\kappa_j})),$$

implying the following alternative representation of (1.3):
(2.10)

$$\begin{aligned}\varrho_\alpha^{(k)}(x_1,\dots,x_k) &= \sum_{\pi\in\mathcal{P}(1,\dots,k)} \alpha^{k-n(\pi)} \prod_{i=1}^{k} C_{i\pi(i)}\\ &= \sum_{l=1}^{k} \sum_{\substack{K_1\cup\ldots\cup K_l\\ =\{1,\dots,k\}}} \alpha^{k-l} \prod_{j=1}^{l} \sum_{\pi_j\in\mathcal{P}(K_j\setminus\{k_{j1}\})} C_{k_{j1}\pi_j(k_{j2})} C_{\pi_j(k_{j2})\pi_j(k_{j3})} \ldots C_{\pi_j(k_{j\kappa_j})k_{j1}}.\end{aligned}$$

Finally, inserting the latter expression of $\varrho_\alpha^{(k)}(x_1,\dots,x_k)$ (with $k-l=\sum\limits_{j=1}^{l}(\kappa_j-1)$) and the formula (2.9) to the r.h.s. of (2.8), we recognize that $c_\alpha^{(k)}(x_1,\dots,x_k)$ is just equal to the first summand (for $l=1$) of (2.10), which coincides with (2.5). This completes the proof of Theorem 1. □

Remark 2. Let C be a covariance function and $m \geqslant 2$ an integer such that the DPP $(\Psi_m)_{-1}$ with covariance function C/m exists. According to (2.5), its k-th cumulant density $(c_m)_{-1}^{(k)}(x_1,\dots,x_k)$ satisfies the relation

$$\begin{aligned}&m(c_m)_{-1}^{(k)}(x_1,\dots,x_k)\\ &\qquad= \Bigl(-\frac{1}{m}\Bigr)^{k-1} \sum_{\pi\in\mathcal{P}(\{2,\dots,k\})} C(x_1,x_{\pi(2)})C(x_{\pi(2)},x_{\pi(3)})\ldots C(x_{\pi(k)},x_1)\end{aligned}$$

for every $k\in\mathbb{N}$. Obviously, the expression on the r.h.s. is exactly the k-th cumulant density of an α-DPP(C) with $\alpha=-1/m$. This PP can be obtained by superposition of m independent copies of the DPP $(\Psi_m)_{-1}$. An analogous construction holds for $\alpha=1/m$ using a PPP $(\Psi_m)_1$ with covariance function C/m.

3. Strong Brillinger-mixing property of stationary α-DPPs with applications to shot-noise processes

An α-DPP(C) $\Psi_\alpha \sim P_\alpha$ (with an admissible $\alpha \geqslant -1$ and covariance function C) turns out to be (strictly) *stationary* if $C(x,y) = C(o, y-x) = C_0(y-x)$ for all $x, y \in \mathbb{R}^d$, where $C_0(x)$ is a complex-valued, non-negative-definite function on $\mathbb{R}^d$, i.e. $\sum_{i,j=1}^{n} z_i C_0(x_i - x_j)\overline{z_j} \geqslant 0$ for all $x_1, \dots, x_n \in \mathbb{R}^d$, $z_1, \dots, z_n \in \mathbb{C}$, and $n \geqslant 1$ implying that $C_0(o) \geqslant 0$, $C_0(-x) = \overline{C_0(x)}$ and $|C_0(x)| \leqslant C_0(o)$ for $x \in \mathbb{R}^d$. Note that the stationarity of $\Psi_\alpha \sim P_\alpha$ implies that $|C(x,y)|^2 = |C(o, y-x)|^2$, which does not necessarily mean that $C(x,y) = C(o, y-x)$, as the Ginibre kernel $C(x,y) = \pi^{-2}\exp\{2x\overline{y} - |x|^2 - |y|^2\}$ (regarding x, y as complex numbers) reveals, see e.g. [13].

For stationary α-DPPs with covariance function $C_0(x)$ we write α-DPP(C_0) for short. We assume in addition that $C_0(x)$ is continuous at the origin o (and hence everywhere) such that $C_0(o) > 0$ and $\|C_0\|_1 := \int_{\mathbb{R}^d} |C_0(x)|\,\mathrm{d}x < \infty$ implying $\|C_0\|_2^2 := \int_{\mathbb{R}^d} |C_0(x)|^2\,\mathrm{d}x \leqslant C_0(o)\|C_0\|_1$.

Under these conditions, a famous result by Bochner & A. Ya. Khintchine, see [18] for references, ensures that there exists a finite (spectral) measure μ on $[\mathbb{R}^d, \mathcal{B}^d]$ having even bounded, continuous and (square) integrable Lebesgue density φ on $\mathbb{R}^d$ such that

$$C_0(x) = \int_{\mathbb{R}^d} \mathrm{e}^{2\pi \mathrm{i}\langle x,y\rangle}\varphi(y)\mathrm{d}y \qquad \text{and} \qquad \varphi(y) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i}\langle x,y\rangle} C_0(x)\mathrm{d}x,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^d$. These Fourier representations show immediately that $C_0(o) = \int_{\mathbb{R}^d}\varphi(y)\mathrm{d}y$ and $\sup_{x\in\mathbb{R}^d}\varphi(x) \leqslant \|C_0\|_1$. A remarkable result proved in [18] for DPPs is as follows: Provided that the covariance function C_0 is continuous and square integrable on $\mathbb{R}^d$, there exists a stationary DPP(C_0) if the corresponding spectral density φ satisfies the condition $0 \leqslant \varphi(y) \leqslant 1$ for all $y \in \mathbb{R}^d$. A suitable way to find pairs (C_0, φ) is to consider the characteristic function C_0 of a random vector having a bounded (symmetric or unimodal) probability density φ, see e.g. [22].

We give two examples: Let S be a positive definite $d \times d$ matrix with $\det(S) > 0$.

Gaussian distribution:

$$C_0(x) = \exp\{-2\pi^2\langle x, Sx\rangle\}, \quad \varphi(y) = \frac{\exp\{-\frac{1}{2}\langle y, S^{-1}y\rangle\}}{\sqrt{(2\pi)^d\det(S)}};$$

Cauchy distribution:

$$C_0(x) = \exp\{-2\pi\sqrt{\langle x, Sx\rangle}\}, \quad \varphi(y) = \frac{\Gamma(\frac{1}{2}(d+1))/\pi^{d+1}}{\det(S)(1+\langle y, S^{-1}y\rangle)^{(d+1)/2}}.$$

The conditions on $\det(S)$ to ensure the existence of a DPP(C_0) can be easily found. Each rotation-invariant characteristic function has a canonical Schoenberg representation which for $d = 2$ is as follows: $C_0(x) = \int_0^\infty J_0(2\pi\|x\|t)\,\mathrm{d}F(t)$ with some distribution function F on $[0,\infty)$ and the Bessel function $J_0(y) = 2\pi^{-1}\int_0^1 \cos(ys)/\sqrt{1-s^2}\,\mathrm{d}s$.

Next we give the second-order characteristics for a stationary α-DPP(C_0) with intensity $\lambda = C_0(o)$:

(i) *(reduced) second product density* $\varrho_\alpha(x)$:

$$\varrho_\alpha(x) := \lambda^{-1}\varrho_\alpha^{(2)}(x,o) = C_0(o) + \alpha\frac{|C_0(x)|^2}{C_0(o)} \geqslant 0 \quad \text{for } x \in \mathbb{R}^d,\ \alpha \geqslant -1, \tag{3.1}$$

(ii) *pair correlation function* $g_\alpha(r)$: If the α-DPP(C_0) is motion-invariant, i.e. $c(\|x\|) = C_0(x)$ for some function $c(\cdot)$ on $[0,\infty)$, we define

$$g_\alpha(r) := \lambda^{-2}\varrho_\alpha^{(2)}(x,o) = 1 + \alpha\frac{|c(r)|^2}{C_0(o)^2} \geqslant 0 \quad \text{for } r = \|x\| \geqslant 0,\ \alpha \geqslant -1. \tag{3.2}$$

(iii) *Ripley's and multiparameter K-function*: $K(r) := \mathcal{K}(B_d(r))$ and $K(\mathbf{r}) := \mathcal{K}(B_d(\mathbf{r}))$, where

$$\begin{aligned} \mathcal{K}(B) &:= \lambda^{-1}\alpha_{\mathrm{red}}^{(2)}(B) \\ &= |B| + \frac{\alpha}{C_0(o)^2}\int_B |C_0(x)|^2\,\mathrm{d}x \geqslant 0 \quad \text{for } B \in \mathcal{B}_b^d,\ \alpha \geqslant -1. \end{aligned} \tag{3.3}$$

Theorem 2. *For an admissible $\alpha \geqslant -1$, let $\Psi_\alpha \sim P_\alpha$ be a stationary α-DPP(C_0) defined by the product densities* (2.2) *with an absolutely integrable covariance function C_0 and intensity $\lambda = C_0(o)$. Then $\Psi_\alpha \sim P_\alpha$ is strongly Brillinger-mixing. More precisely, the corresponding total variations* (1.4) *expressed by the cumulant densities* (2.5) *possess the bounds*

$$\|\gamma_{\mathrm{red}}^{(k)}\|_{TV} \leqslant \frac{|\alpha|^{k-1}}{C_0(o)}(k-1)!\|C_0\|_1^{k-2}\|C_0\|_2^2 \leqslant (k-1)!(|\alpha|\|C_0\|_1)^{k-1} \quad \textit{for } k \geqslant 2. \tag{3.4}$$

P r o o f. From (2.6) with $j = k$, $x_k = o$ and $C(x,y) = C_0(y-x)$ we see that $c_\alpha^{(k)}(x_1,\ldots,x_{k-1},o)$ takes on the form

$$\alpha^{k-1}\sum_{\pi\in\mathcal{P}(\{1,\ldots,k-1\})} C_0(x_{\pi(1)})C_0(x_{\pi(2)}-x_{\pi(1)})\ldots C_0(x_{\pi(k-1)}-x_{\pi(k-2)})C_0(-x_{\pi(k-1)}),$$

whence together with $|C_0(x_1)C_0(-x_{k-1})| \leqslant \frac{1}{2}|C_0(x_1)|^2 + \frac{1}{2}|C_0(x_{k-1})|^2$ and $\|C_0\|_2^2 \leqslant \lambda\|C_0\|_1$ it follows that

$$\begin{aligned}
\lambda\|\gamma_{\mathrm{red}}^{(k)}\|_{TV} &= \int_{(\mathbb{R}^d)^{k-1}} |c_\alpha^{(k)}(x_1,\ldots,x_{k-1},o)|\,\mathrm{d}(x_1,\ldots,x_{k-1})\\
&\leqslant \frac{1}{2}|\alpha|^{k-1}(k-1)!\int_{(\mathbb{R}^d)^{k-1}} \big(|C_0(x_1)|^2 + |C_0(x_{k-1})|^2\big)\\
&\quad\times\prod_{j=2}^{k-1}|C_0(x_j - x_{j-1})|\,\mathrm{d}(x_1,\ldots,x_{k-1})\\
&= |\alpha|^{k-1}(k-1)!\int_{\mathbb{R}^d}|C_0(x)|^2\,\mathrm{d}x\bigg(\int_{\mathbb{R}^d}|C_0(x)|\,\mathrm{d}x\bigg)^{k-2} \leqslant \lambda(k-1)!|\alpha|^{k-1}\|C_0\|_1^{k-1}.
\end{aligned}$$

The last line is equivalent to the asserted estimate (3.4). Thus, Theorem 2 is proved. □

Corollary 1. *Under the assumptions of Theorem* 2, *the stationary* α-$\mathrm{DPP}(C_0)$ $\Psi_\alpha \sim P_\alpha$ *has a trivial tail-σ-algebra* $\mathcal{F}_\infty^{\Psi_\alpha} := \bigcap\limits_{r>0}\mathcal{F}_r^{\Psi_\alpha}$, *where* $\mathcal{F}_r^{\Psi_\alpha} := \sigma\big(\{\Psi_\alpha(B\cap B_d(r)^c) = n\}\colon n\in\mathbb{N},\ B\in\mathcal{B}_b^d\big)$.

The proof of Corollary 1, which holds true for any strongly Brillinger-mixing PP, can be found in [8].

To the best of the author's knowledge, strongly Brillinger-mixing PPs (without using this name) have been first considered in [12] in order to study rates of convergence in the central limit theorem (CLT for short) for so-called shot-noise processes driven by stationary independently marked Brillinger-mixing PPs on $\mathbb{R}^d$.

Let $\Psi_\alpha = \sum\limits_{i\geqslant 1}\delta_{X_i} \sim P_\alpha$ be a stationary α-$\mathrm{DPP}(C_0)$ with intensity $\lambda = C_0(o)$ satisfying the assumptions of Theorem 2. Further, let $\{M_i\colon i\geqslant 1\}$ be a sequence (independent of Ψ_α) on $[\Omega,\mathcal{F},\mathsf{P}]$ of independent copies of a generic (or typical) random mark M_0 taking values in a Polish mark space $[\mathbb{M},\mathcal{M}]$ having the mark distribution Q on $\mathcal{M}$. A *shot-noise process* (SNP for short) driven by the α-$\mathrm{DPP}(C_0)$ and a mark distribution Q with a measurable *response function* $f_n\colon \mathbb{R}^d\times\mathbb{M}\to\mathbb{R}^1$ is defined by the random sum

$$S(f_n,\Psi_\alpha) := \sum_{i\geqslant 1} f_n(X_i,M_i) \quad \text{for } n\geqslant 1. \tag{3.5}$$

Here, the mark M_i can be interpreted as a random effect, e.g. the strength of a shot, connected by the response function f_n with the location X_i. Formula (3.5) expresses the total effect generated by the atoms X_i of Ψ_α in some window set $W_n\subset\mathbb{R}^d$.

For example, M_i could be the radius of a ball centered at X_i and $f_n(X_i, M_i) = |(X_i + B_d(M_i)) \cap W_n|$, where the sequence $(W_n)_{n\in\mathbb{N}}$ consists of increasing, convex and compact sets in $\mathbb{R}^d$ with inball radius $r(W_n) \underset{n\to\infty}{\longrightarrow} \infty$, see, e.g. [7].

The cumulants $\mathrm{Cum}_k(S(f_n, \Psi_\alpha)) := \mathrm{i}^{-k} \lim\limits_{t\to 0}(\mathrm{d}^k/\mathrm{d}t^k) \log \mathsf{E} \exp\{\mathrm{i}tS(f_n, \Psi_\alpha)\}$ of any order $k \geqslant 1$ can be calculated by means of generalized versions of the Campbell theorem, see [3], and expressed and estimated in terms of the cumulant densities (2.5) (with $C(x_i, x_j)$ replaced by $C_0(x_j - x_i)$) and the moment quantities $u_k^{(n)}(x) := \int_{\mathbb{M}} f_n(x, m)^k Q(\mathrm{d}m)$ and $v_k^{(n)}(x) := \int_{\mathbb{M}} |f_n(x, m)|^k Q(\mathrm{d}m)$ for $k \in \mathbb{N}$.

For example, the mean $\mathsf{E}(S(f_n, \Psi_\alpha))$ equals $C_0(o) \int_{\mathbb{R}^d} u_1^{(n)}(x)\,\mathrm{d}x$ and the variance of (3.5) takes on the form

$$\mathrm{Var}\big(S(f_n, \Psi_\alpha)\big) = C_0(o) \int_{\mathbb{R}^d} u_2^{(n)}(x)\,\mathrm{d}x + \alpha \int_{(\mathbb{R}^d)^2} u_1^{(n)}(x) u_1^{(n)}(x + y)|C_0(y)|^2\,\mathrm{d}(x, y).$$

Now we are able to formulate the results on rates of convergence in the CLT for the standardized SNP $Z(f_n, \Psi_\alpha) := \big(S(f_n, \Psi_\alpha) - \mathsf{E}(S(f_n, \Psi_\alpha))\big)/\sqrt{\mathrm{Var}(S(f_n, \Psi_\alpha))}$ driven by an α-DPP(C_0).

Theorem 3. *Let $(A_n)_{n\in\mathbb{N}}$ be an unboundedly increasing sequence as $n \to \infty$ such that for $a, b > 0$ and all $k \geqslant 2$,*

$$\mathrm{Var}\big(S(f_n, \Psi_\alpha)\big)/A_n \geqslant a, \quad \sup_{x\in\mathbb{R}^d} v_k^{(n)}(x) \leqslant b^k k!, \quad \int_{\mathbb{R}^d} v_k^{(n)}(x)\,\mathrm{d}x \leqslant A_n b^k k!. \tag{3.6}$$

Then

$$\sup_{x\in\mathbb{R}^d} \big|\mathsf{P}(Z(f_n, \Psi_\alpha) \leqslant x) - \Phi(x)\big| \leqslant \mathrm{const}(a, b, \alpha, C_0) A_n^{-1/2}, \tag{3.7}$$

where $\Phi(x) := (2\pi)^{-1/2} \int_{-\infty}^x \mathrm{e}^{-y^2/2}\,\mathrm{d}y$ denotes the standard normal distribution function. Further, large deviations relations in the sense of H. Cramér hold: There is a positive constant ε depending on a, b, α, C_0 such that in the interval $0 \leqslant x \leqslant \varepsilon\sqrt{A_n}$ the asymptotic relation

$$\frac{\mathsf{P}(Z(f_n, \Psi_\alpha) \geqslant x)}{1 - \Phi(x)} = \exp\Big\{\sum_{k=3}^{\infty} \Lambda_k^{(n)} x^k\Big\}\Big(1 + \mathcal{O}\Big(\frac{1 + x}{\sqrt{A_n}}\Big)\Big) \quad \text{as } n \to \infty \tag{3.8}$$

holds, where the so-called Cramér series in the exponent converges absolutely for $x \in [0, \varepsilon\sqrt{A_n}]$ with coefficients $\Lambda_k^{(n)}$ depending on the cumulants $\mathrm{Cum}_j(Z(f_n, \Psi_\alpha))$, $j = 3, \ldots, k$, see [12], p. 724 for an explicit formula. The corresponding relation is also valid for the ratio $\mathsf{P}(Z(f_n, \Psi_\alpha) \leqslant -x)/\Phi(-x)$.

Proof. The proof of (3.7) and (3.8) is based on sharp bounds of the cumulants $\mathrm{Cum}_k(S(f_n,\Psi_\alpha))$. To obain them we have only to mimic the proof of Theorem 5 in [12]. From (3.6) we immediately find that $aA_n \leqslant \mathrm{Var}(S(f_n,\Psi_\alpha)) \leqslant 2b^2(C_0(o)+|\alpha|\|C_0\|_2^2)A_n$. Further, we find a constant B depending on a,b,α,C_0 such that $|\mathrm{Cum}_k(S(f_n,\Psi_\alpha))| \leqslant A_n B^k k!$ for $k \geqslant 2$, which implies $|\mathrm{Cum}_k(Z(f_n,\Psi_\alpha))| \leqslant aB^2 k!/\Delta_n^{k-2}$ with $\Delta_n = a\sqrt{A_n}/B$. Now we are in a position to apply a famous lemma due to Statulevičius, see [26], which provides the optimal Berry-Esseen bound (3.7) as well as the large deviations relation (3.8) completing the proof of Theorem 3. □

Corollary 2. *In the particular case of an unmarked stationary α-DPP(C_0) satisfying assumptions of Theorem 2 with response function $f_n(x,m) = \mathbf{1}_{W_n}(x)$, assumptions (3.6) are fulfilled (due to (3.4)) for $A_n = |W_n|$. Hence, the Berry-Esseen bounds (3.7) and (3.8) are valid for the total number of atoms $S(f_n,\Psi_\alpha) = \Psi_\alpha(W_n)$ in an increasing domain W_n as defined above.*

A similar result can be obtained for an integrated SNP

$$S(f_n,\Psi_\alpha) = \int_{W_n} \sum_{i\geqslant 1} g(y - X_i, M_i)\,\mathrm{d}y$$

with response function $f_n(x,m) = \int_{W_n} g(y-x,m)\,\mathrm{d}y$ for some weight function $g\colon \mathbb{R}^d \times \mathbb{M} \to \mathbb{R}^1$ satisfying $\int_{\mathbb{M}} \bigl(\int_{\mathbb{R}^d} |g(x,m)|\,\mathrm{d}x\bigr)^k Q(\mathrm{d}m) \leqslant b^k k!$ for $k \geqslant 2$.

4. Some applications to statistical second-order analysis of stationary α-DPPs

At the beginning of this final section, we generalize a CLT proved in [7] for stationary Poisson processes on $\mathbb{R}^d$ with the aim to establish a goodness-of-fit tests for the K-functions of stationary α-DPPs defined by (3.3). For this purpose, let $\Psi_\alpha = \sum\limits_{i\geqslant 1} \delta_{X_i} \sim P_\alpha$ be a stationary α-DPP(C_0) with intensity $\lambda = C_0(o)$ satisfying the assumptions of Theorem 2. We assume that Ψ_α can be observed in an expanding observation window W_n as defined in Section 3. We introduce the empirical set function

$$(4.1)\quad (\widehat{\lambda^2\mathcal{K}})_n(B) := \sum_{i,j\geqslant 1}^{\neq} \frac{\mathbf{1}_{W_n}(X_i)\mathbf{1}_{W_n}(X_j)\mathbf{1}_B(X_j - X_i)}{|(W_n - X_i)\cap(W_n - X_j)|} \quad \text{for } B \in \mathcal{B}_b^d \cap [-R,R]^d,$$

where the sum $\sum^{\neq}$ runs over pairwise distinct indices. It turns out that (4.1) is an unbiased and strongly consistent estimator (since Ψ_α is ergodic thanks to Corollary 1) for $\lambda^2\mathcal{K}(B) := \lambda\alpha_{\mathrm{red}}^{(2)}(B) = C_0(o)^2|B| + \alpha\int_B |C_0(x)|^2\,\mathrm{d}x$, see (3.3). Estimators of this type are known as *edge-corrected*, *Horvitz-Thompson* or *Ohser-Stoyan* estimators. For a detailed discussion of such estimators and K-functions we refer to the monograph [14], Chapt. 4.3.

In what follows, we consider estimator (4.1) only on the family $\mathcal{B}^d_{\mathrm{sym}}(R)$ of centrally symmetric sets B, i.e. $B = -B$ contained in $[-R, R]^d$, in particular d-balls $B_d(r)$ for $r \in [0, R]$ and cuboids $B_d(\mathbf{r}) = \bigtimes_{i=1}^{d} = [-r_i, r_i]$ for $\mathbf{r} = (r_1, \ldots, r_d) \in [0, R]^d$.

Furthermore, we formulate a multivariate CLT for the sequence random vectors $(X_n(B_1), \ldots, X_n(B_k))$ for $B_1, \ldots, B_k \in \mathcal{B}^d_{\mathrm{sym}}(R)$, where

$$X_n(B_i) = \sqrt{|W_n|}\big((\widehat{\lambda^2\mathcal{K}})_n(B_i) - \lambda^2\mathcal{K}(B_i)\big) \quad \text{for } i = 1, \ldots, k. \tag{4.2}$$

Theorem 4. *For an admissible* $\alpha \geqslant -1$, *let* $\Psi_\alpha = \sum_{i\geqslant 1} \delta_{X_i} \sim P_\alpha$ *be a stationary* α-DPP(C_0) *with intensity* $\lambda = C_0(o)$ *satisfying the assumptions of Theorem 2. Then* $(X_n(B_1), \ldots, X_n(B_k))$ *converges in distribution to a mean zero Gaussian vector* $(X(B_1), \ldots, X(B_k))$ *with covariance matrix* $(\mathsf{E}(X(B_i)X(B_j))^k_{i,j=1}$, *where for any* $B, B' \in \mathcal{B}^d_{\mathrm{sym}}(R)$ *we have*

$$\begin{aligned}
\mathsf{E}(X(B)X(B')) &= 2\lambda^2\mathcal{K}(B \cap B') \\
&\quad + 4\alpha\lambda \int_B \int_{B'} \big(|C_0(x)|^2 + |C_0(y)|^2 + |C_0(x-y)|^2\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 4\alpha\lambda^2|B||B'| \int_{\mathbb{R}^d} |C_0(u)|^2\mathrm{d}u + 8\alpha^2 \int_B \int_{B'} \mathrm{Re}\big(C_0(x)C_0(y)\overline{C_0(x+y)}\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 4\lambda^3|B \cap B'| + 2\alpha^2 \int_B \int_{B'} \int_{\mathbb{R}^d} |C_0(u)|^2|C_0(u+x+y)|^2\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 4\alpha^2\lambda \int_B \int_{B'} \int_{\mathbb{R}^d} \mathrm{Re}\big(C_0(x)C_0(u)\overline{C_0(x+u)} + C_0(y)C_0(u)\overline{C_0(y+u)}\big)\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 4\alpha^3 \int_B \int_{B'} \int_{\mathbb{R}^d} \mathrm{Re}\big(C_0(x)C_0(y)C_0(u)\overline{C_0(x+y+u)}\big)\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 2\alpha^3 \int_B \int_{B'} \int_{\mathbb{R}^d} \mathrm{Re}\big(C_0(u)C_0(u+x+y)\overline{C_0(u+x)C_0(u+y)}\big)\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

The proof of Theorem 4 is based on the Brillinger-mixing property for α-DPP(C_0) and a CLT for empirical higher-order moment measures of stationary (not necessarily strongly) Brillinger-mixing PPs first proved by E. Jolivet in [15], see also [16], [17] for similar results. The covariance $\mathsf{E}(X(B)X(B'))$ could be derived from a general formula for the covariance of two second-order statistics in [6], p. 97. Theorem 4 can be extended to a functional CLT for the random processes $Y_n(\mathbf{r}) := X_n(B_d(\mathbf{r}))$ in the Skorokhod space $\mathbf{D}[0, R]^d$ of càdlàg functions on the cube $[0, R]^d$. It is a hard and tedious work to verify the tightness conditions in detail but the continuity of $C_0(x)$ on $[0,R]^d$ is indeed sufficient. The limiting Gaussian process $Y(\mathbf{r}) := X(B_d(\mathbf{r}))$ is (P-a.s.) continuous with a covariance function $\gamma(\mathbf{s},\mathbf{t}) := \mathsf{E}(Y(\mathbf{s})Y(\mathbf{t}))$ completely determined by α and C_0. A simulation of $Y(\mathbf{r})$ on $[0,R]^d$ seems to be possible and would provide the quantiles of sup- and L^2-norms for testing the goodness-of-fit in analogy to a Poisson process, see [7]. This could be a challenge for future research.

To conclude, we mention two possible goodness-of-fit tests for stationary (and isotropic) α-DPPs(C_0) $\Psi_\alpha = \sum\limits_{i\geqslant 1} \delta_{X_i}$ with intensity $\lambda = C_0(o)$ satisfying the assumptions of Theorem 2. For this purpose we define kernel-type estimators for $\lambda\varrho_\alpha(x)$ and $\lambda^2 g_\alpha(r)$, see (3.1) and (3.2),

$$(4.3)\quad \widehat{(\lambda\varrho_\alpha)}_n(x) := \frac{1}{b_n^d} \sum_{i,j\geqslant 1}^{\neq} \frac{\mathbf{1}_{W_n}(X_i)\mathbf{1}_{W_n}(X_j)K_d((X_j - X_i - x)/b_n)}{|(W_n - X_i)\cap(W_n - X_j)|}, \quad x \neq o,$$

$$(4.4)\quad \widehat{(\lambda^2 g_\alpha)}_n(r) := \frac{1}{d\omega_d b_n} \sum_{i,j\geqslant 1}^{\neq} \frac{\mathbf{1}_{W_n}(X_i)\mathbf{1}_{W_n}(X_j)K_1((\|X_j - X_i\| - r)/b_n)}{\|X_j - X_i\|^{d-1}|(W_n - X_i)\cap(W_n - X_j)|}, \quad r > 0,$$

where $\omega_d := |B_d(1)|$, W_n is expanding as defined in Section 3, K_s is a symmetric, bounded and boundedly supported function such that $\int_{\mathbb{R}^s} K_s(x)\mathrm{d}x = 1$, and $b_n \downarrow 0$ such that $b_n^s|W_n| \underset{n\to\infty}{\longrightarrow} \infty$ for $s = 1, d$.

For Brillinger-mixing PPs, various CLTs have been proved in [9], [10] for both the estimators (4.3) and (4.4). These results permit us to establish asymptotic χ^2-tests to check hypotheses about the second product densities and pair correlation functions. In our situation, these tests can be reformulated for testing hypotheses about the function $|C_0(x)|$ or its isotropic counterpart $|c(r)| = |C_0(x)|$ for $r = \|x\|$. The following limit theorem (formulated as Theorems 3.3 and 4.1 in [10] in a more general setting) provide the basis for these tests.

Theorem 5. *In addition to the above assumptions, let* $b_n^{d+2}|W_n| \underset{n\to\infty}{\longrightarrow} 0$ *and let* $|C_0(\cdot)|^2$ *be Lipschitz-continuous at* $x_1,\ldots,x_q \neq o$ $(x_i \neq \pm x_j,\ i \neq j)$ *and in the isotropic case, let* $b_n^3|W_n| \underset{n\to\infty}{\longrightarrow} 0$ *and let* $|c(\cdot)|^2$ *be Lipschitz-continuous at* $r_1,\ldots,r_q > 0$ $(r_i \neq r_j,\ i\neq j)$. *Further, let* $\|K_s\|_2^2 = \int_{\mathbb{R}^s} K_s^2(x)\mathrm{d}x$ *for* $s = 1, d$. *Then each of the test statistics*

$$\frac{4b_n^d|W_n|}{\|K_d\|_2^2} \sum_{i=1}^q \Big(\sqrt{\widehat{(\lambda\varrho_\alpha)}_n(x_i)} - \sqrt{\lambda^2 + \alpha|C_0(x_i)|^2}\Big)^2$$

and

$$\frac{2d\omega_d b_n|W_n|}{\|K_1\|_2^2} \sum_{i=1}^q r_i^{d-1}\Big(\sqrt{\widehat{(\lambda^2 g_\alpha)}_n(r_i)} - \sqrt{\lambda^2 + \alpha|c(r_i)|^2}\Big)^2$$

converges in distribution to a chi-square-distributed random variable χ_q^2 *with* q *degrees of freedom. Furthermore,* $\widehat{\lambda_n} := \Psi_\alpha(W_n)/|W_n| \xrightarrow[n\to\infty]{\mathsf{P}\text{-a.s.}} \lambda = C_0(o)$ *(by Corollary* 1*) and* $\sqrt{|W_n|}(\widehat{\lambda_n} - C_0(o))$ *is asymptotically normally distributed with mean zero and variance* $\sigma^2 = C_0(o) + \alpha \int_{\mathbb{R}^d} |C_0(x)|^2 \, \mathrm{d}x$ *(by Corollary* 2*), see also* [25] *for* $\alpha = -1$*. The variance* σ^2 *can be consistently estimated, see* [11]*, so that a hypothetical intensity* $\lambda = C_0(o)$ *can be checked by an asymptotically standard normally distributed test statistic.*

A detailed proof of Theorem 5 is omitted. We only remark that the assumptions of Theorem 5 imply all the conditions formulated in Theorems 3.3 and 4.1 in [10]. In particular, condition $\gamma(u,\infty)$ can be easily verified (with arguments used in the proof of Theorem 2) due to our assumption $\|C_0\|_1 < \infty$. In [1], the weaker assumption $\|C_0\|_2 < \infty$ is shown to be also sufficient to obtain asymptotic normality of (4.3) and (4.4) at least in the special case of DPPs, however, under slightly different conditions on b_n and W_n.

Remark 3. The essential parts of this paper have been presented at the 18th SGSIA-Workshop in Lingen, March 22–27, 2015. Meanwhile, the arXiv paper [1] appeared, concerning the usual (not strong) Brillinger-mixing property of DPPs ($\alpha = -1$) with a variety of statistical applications. In contrast to paper [1] which is focused on DDPs ($\alpha = -1$) satisfying $\|C_0\|_2 < \infty$, the present paper deals with a larger class of α-DPPs ($\alpha \geqslant -1$) under the stronger condition $\|C_0\|_1 < \infty$. The latter condition leads to stronger results with rates of convergence, see Theorem 3, but on the other hand, it excludes some interesting covariance functions C_0. In summary, a precise comparison of both papers reveals that there is only a minor overlap due to different methods, generality and aims. In conclusion, the two papers supplement each other.

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