Kybernetika

Krzysztof Andrzej Krakowski; Fátima Silva Leite

Geometry of the rolling ellipsoid

Kybernetika, Vol. 52 (2016), No. 2, 209–223

Persistent URL: `http://dml.cz/dmlcz/145771`

KYBERNETIKA — VOLUME *52* (2016), NUMBER 2, PAGES 209–223

GEOMETRY OF THE ROLLING ELLIPSOID

Krzysztof A. Krakowski and Fátima Silva Leite

We study rolling maps of the Euclidean ellipsoid, rolling upon its affine tangent space at a point. Driven by the geometry of rolling maps, we find a simple formula for the angular velocity of the rolling ellipsoid along any piecewise smooth curve in terms of the Gauss map. This result is then generalised to rolling any smooth hyper-surface. On the way, we derive a formula for the Gaussian curvature of an ellipsoid which has an elementary proof and has been previously known only for dimension two.

Keywords: ellipsoid, rolling maps, Gaussian curvature, geodesics, hypersurface

Classification: 53B21, 53A05, 58E10, 70B10, 35B06

1. INTRODUCTION

There is a considerable area of research in robotics devoting attention to replicating capability of humans, such as bipedal locomotion or contact sensing. Both activities involve complicated physical interactions between two surfaces. To overcome these difficulties it is necessary and legitimate to resort to approximated models. Concerning human locomotion, ellipsoids have been recently used to model the feet [11], and in this context the foot-floor relationship involves rolling motions of the ellipsoid over the tangent plane, with non-holonomic constraints of no-slip and no-twist. Dexterous manipulation is an important problem in the study of multi-fingered robotic hands [1, 22]. The contour of fingertips may also be simplified by idealised geometries such as half ellipsoids. Typical robotic fingers roll on the boundary of an object, so that the velocity of the contact points on each of the objects remains the same. This is a rolling motion with the non-holonomic constraint of no-slip.

Another possible application involving interpolation on ellipsoids, comes from the fact that the ellipsoid is the earth model for the World Geodetic System (WGS84), which is currently the reference system used by the Global Positioning System (GPS). Given a set of GPS points on the earth surface, along with the time at which the coordinates were collected and, additionally, the speed and the heading of a vehicle at those coordinates, one may want to find an interpolating (smooth) path for the position, speed and heading of the vehicle at a time between the tabulated points. This sort of problems arises, for instance, in route planning for marine navigation or in control of autonomous surface

DOI: 10.14736/kyb-2016-2-0209

vehicles, where important aspects such as safety, energy saving and collision avoidance have to be taken into consideration. We refer to [29] and references wherein where applications in Earth navigation can be seen from many different perspectives. These problems can be formulated as interpolation problems on the ellipsoid, where the data consists of a discrete set of time-labeled points (positions) along with derivatives (speed and heading) at the prescribed points. Such interpolation problems on non-flat manifolds can be solved, at least theoretically, using rolling techniques, so that the data on the manifold is projected to data on a flat space, typically the affine tangent space at a point. If one knows how to solve the kinematic equations of the rolling ellipsoid, at least for some simple curves, this approach provides an algorithm to solve the interpolation problem explicitly. Illustration of this algorithms has been done for the ellipsoid equipped with a non-Euclidean metric in [16] and for other Riemannian manifolds in [10]. The same idea has been proposed in [5] to solve multiclass classification problems in the context of pattern recognition.

Having these applications in mind, the main objective here is to study the geometry of the rolling ellipsoid and present a geometric view on the invariants of rolling motions along geodesics, in terms of angular velocity. The remaining part of this paper is organised in the following way. Differential geometric preliminaries are covered in Section 2. The notion of rolling maps for Riemannian manifolds is revisited in Section 2.1. In Section 2.2 we review the differential geometric properties of the ellipsoid and the geometric shape related tensors: the second fundamental form and the Weingarten operator. Section 3 contains the main results. An elementary proof of the formula for the Gaussian curvature of an ellipsoid, generalised to dimension higher than two, is given in Proposition 3.1. A remarkable simple formula for the angular velocity in the body coordinate system in terms of the Gauss map along the rolling curve is given in Theorem 3.3. A more general situation of a Riemannian hyper-surface rolling upon its affine tangent space at a point is considered in Section 3.2. In this setting, we obtain a simple relationship between the angular velocity and the Gauss map. In Section 3.3 we derive a Lax form equation for the rolling ellipsoid along reparametrised geodesic. Section 4 outlines further directions for this research and concludes the paper.

2. PRELIMINARIES

To formulate our results it will be necessary to review the definition of a rolling map and some geometric properties of the n-dimensional ellipsoid $\boldsymbol{\mathcal{E}}^n$. In this section we outline some notions of differential geometry that will be useful later, in particular the concept of *second fundamental form* and the related notion of *Weingarten map*.

Consider an n-dimensional Riemannian manifold $\mathbf{M}$ isometrically embedded in Euclidean space $\mathbb{R}^m$, with $1 < n < m$. The embedding defines a Riemannian metric g on $\mathbf{M}$ that is inherited from the ambient Euclidean metric on $\mathbb{R}^m$ in the following way. For any two vectors U and V in the tangent space $\mathbf{T}_p\mathbf{M}$ at a point $p \in \mathbf{M}$ the metric g determines the inner product $\langle U, V\rangle := g(p)(U, V)$ that is a restriction of the standard Euclidean '*dot*' product to $\mathbf{T}_p\mathbf{M}$.

Let $\mathfrak{X}(\mathbf{M})$ denote the space of smooth vector fields on $\mathbf{M}$. For any $X, Y \in \mathfrak{X}(\mathbf{M})$, the Riemannian (or Levi–Civita) connection $\nabla_X Y : \mathfrak{X}(\mathbf{M}) \times \mathfrak{X}(\mathbf{M}) \to \mathfrak{X}(\mathbf{M})$, defines differentiation of the vector field Y in the direction X.

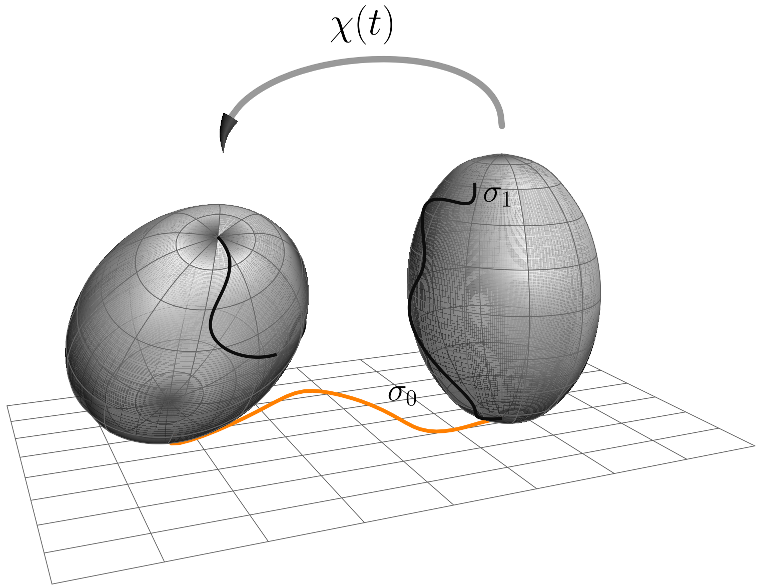

Fig. 1. the rolling map $\boldsymbol{\chi}\colon I \to \mathbb{SE}(3)$ of an ellipsoid $\boldsymbol{\mathcal{E}}^2 \hookrightarrow \mathbb{R}^3$ rolling on a plane along a rolling curve $\sigma_1 \colon I \to \boldsymbol{\mathcal{E}}^2$; the point of contact traces the development curve σ_0 in the plane.

We now present a definition and some results concerning rolling maps.

2.1. Rolling maps

In the context of this paper, rolling maps describe how a manifold rolls over another manifold of the same dimension without slipping and without twisting. These nonholonomic constraints arise in connection with the rubber rolling model that has been investigated, for instance, in [2] and [3].

A differential geometric definition of a *rolling map* for manifolds embedded in Euclidean space is due to Sharpe [24, Appendix B]. The following definition is its generalisation from Euclidean to a Riemannian ambient space, cf. [9]. The existence and uniqueness of rolling maps have been studied in [21] and then in [24].

Definition 2.1. Let $\mathbf{M}_0$ and $\mathbf{M}_1$ be two n-manifolds isometrically embedded in an m-dimensional Riemannian manifold $\mathbf{M}$ and let $\sigma_1 \colon I \to \mathbf{M}_1$ be a piecewise smooth curve in $\mathbf{M}_1$, assuming $1 < n < m$. Let $\mathsf{Isom}(\mathbf{M})$ be the (connected) Lie group of isometries of $\mathbf{M}$. A *rolling map* of $\mathbf{M}_1$ on $\mathbf{M}_0$ along σ_1, without slipping or twisting, is a piecewise smooth map $\boldsymbol{\chi}\colon I \to \mathsf{Isom}(\mathbf{M})$ satisfying the following conditions:

Rolling for all $t \in I$,

(a) $\boldsymbol{\chi}(t)(\sigma_1(t)) \in \mathbf{M}_0$;

(b) $\mathbf{T}_{\boldsymbol{\chi}(t)(\sigma_1(t))}(\boldsymbol{\chi}(t)(\mathbf{M}_1)) = \mathbf{T}_{\boldsymbol{\chi}(t)(\sigma_1(t))}\mathbf{M}_0$.

The curve $\sigma_0\colon I \to \mathbf{M}_0$ defined by $\sigma_0(t) := \boldsymbol{\chi}(t)(\sigma_1(t))$ is called the *development curve* of σ_1, and σ_1 is the rolling curve.

No-slip $\dot\sigma_0(t) = \boldsymbol{\chi}_*(t)(\dot\sigma_1(t))$, for almost all $t \in I$, where $\boldsymbol{\chi}_*$ is the push-forward of $\boldsymbol{\chi}$.

No-twist two complementary conditions, for almost all $t \in I$,

tangential $\big(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}^{-1}(t)\big)_*(\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0) \subset \mathbf{T}^{\perp}_{\sigma_0(t)}\mathbf{M}_0$,

normal $\big(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}^{-1}(t)\big)_*(\mathbf{T}^{\perp}_{\sigma_0(t)}\mathbf{M}_0) \subset \mathbf{T}_{\sigma_0(t)}\mathbf{M}_0$,

where $\mathbf{T}^{\perp}_p\mathbf{M}_0$ denotes the *normal* space at $p \in \mathbf{M}_0$.

To give a simple interpretation of Definition 2.1 suppose that $\mathbf{M} = \mathbb{R}^3$. Then the group of isometries of $\mathbf{M}$ is the Euclidean group of motions $\mathbb{SE}(3) = \mathbb{SO}(3) \ltimes \mathbb{R}^3$, the semi-direct product of the special orthogonal group $\mathbb{SO}(3)$ and $\mathbb{R}^3$. Let (R, s) be an element of $\mathbb{SE}(3)$, where $R \in \mathbb{SO}(3)$ and $s \in \mathbb{R}^3$. Then, the matrix $\mathbf{\Omega} = \big(\dot{\boldsymbol{\chi}} \circ \boldsymbol{\chi}^{-1}\big)_* = \dot{\boldsymbol{\chi}}_* \circ \boldsymbol{\chi}_*^{-1} = \dot R\, R^{\mathrm{T}}$ corresponds to the "spatial angular velocity". More common is to use the equivalent $\omega \in \mathbb{R}^3$ and the cross product, with the standard identification

$$\mathbf{\Omega} = \begin{pmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{pmatrix}, \quad \text{where} \quad \mathbf{\Omega}\, v = \omega \times v, \quad \text{for any} \quad v \in \mathbb{R}^3.$$

Hence, for a two dimensional compact surface the "no-slip" condition becomes $\omega \times \big(\sigma_0 - s\big) = -\dot s$ and the "no-twist" conditions mean that ω has no component normal to $\mathbf{T}_{\sigma_0(t)}\mathbf{M}_0$, or equivalently, it is tangent to $\mathbf{M}_0$ at the point of contact.

It is known that rolling maps are *symmetric* and *transitive*. To be more precise, if $\mathbf{M}_1$ rolls upon $\mathbf{M}_0$ with rolling map $\boldsymbol{\chi}$, along σ_1, then $\mathbf{M}_0$ rolls upon $\mathbf{M}_1$ with rolling map $\boldsymbol{\chi}^{-1}$ along the development curve σ_0. If $\mathbf{M}_1$ rolls upon $\mathbf{M}_2$ with rolling map $\boldsymbol{\chi}_1$, rolling curve σ_1 and development curve σ_2, and $\mathbf{M}_2$ rolls upon $\mathbf{M}_3$ with rolling map $\boldsymbol{\chi}_2$, rolling curve σ_2 and development curve σ_3, then $\mathbf{M}_1$ rolls upon $\mathbf{M}_3$ with rolling map $\boldsymbol{\chi}_2 \circ \boldsymbol{\chi}_1$, along rolling curve σ_1 and development curve σ_3. Rolling maps preserve *parallel transport* and *covariant differentiation*, cf. [25]. As a consequence, the geodesic curvatures of rolling and development curves coincide. In particular, geodesics in $\mathbf{M}_0$ are mapped to geodesics in $\mathbf{M}_1$. This is a one-to-one relationship.

Proposition 2.2 below contains differential equations for rolling maps of smooth manifolds of co-dimension 1. These equations are a direct consequence of [24, Lemma 2.3, Appendix B]. First, recall the two operators of differential geometry, the *second fundamental form* $\mathbf{II}\colon \mathfrak{X}(\mathbf{M}) \times \mathfrak{X}(\mathbf{M}) \to \mathfrak{X}(\mathbf{M})^{\perp}$ and its dual, the *Weingarten map* $\mathbf{\Xi}\colon \mathfrak{X}(\mathbf{M}) \times \mathfrak{X}(\mathbf{M})^{\perp} \to \mathfrak{X}(\mathbf{M})$. Intuitively, the second fundamental form is the normal component of the covariant derivative $\overline{\nabla}_X Y$ in the ambient space. It turns out, somewhat surprising, that $\mathbf{II}(X, Y)$ is symmetric and depends only on the values of the vector fields $X, Y \in \mathfrak{X}(\mathbf{M})$. Both operators are tensor fields and are related through

$$\langle X, \mathbf{\Xi}(Y, \Lambda)\rangle = -\langle \mathbf{II}(X, Y), \Lambda\rangle, \quad \text{for all} \quad X, Y \in \mathfrak{X}(\mathbf{M}) \quad \text{and} \quad \Lambda \in \mathfrak{X}(\mathbf{M})^{\perp}. \tag{1}$$

The case of co-dimension one is particularly simple because $\mathfrak{X}(\mathbf{M})^\perp$, the space of normal vector fields on $\mathbf{M}$, is a field spanned by the collection of unique unit normal vectors and then the *"no-twist"* constraints in Definition 2.1 can be expressed as follows.

Proposition 2.2. In terms of angular velocity in the *body coordinate system* the rolling equations of an n-manifold $\mathbf{N}$, isometrically embedded in $\mathbb{R}^{n+1}$, on its affine tangent space are given by

$$\big(\chi^{-1}\circ\dot{\chi}\big)_* V = -\mathbf{II}(\dot{\sigma}_1, V) \quad \text{and} \quad \big(\chi^{-1}\circ\dot{\chi}\big)_* \Lambda = -\Xi(\dot{\sigma}_1, \Lambda),$$

for all $V \in \mathbf{T}_{\sigma_1(t)}\mathbf{N}$ and $\Lambda \in \mathbf{T}^\perp_{\sigma_1(t)}\mathbf{N}$.

In this paper we are concerned with an n-ellipsoid embedded in $\mathbb{R}^{n+1}$, cf. Figure 1. Here, the Euclidean group $\mathbb{SE}(n+1) = \mathbb{SO}(n+1) \ltimes \mathbb{R}^{n+1}$ is the group of isometries acting on $\boldsymbol{\mathcal{E}}^n$, and both, $\big(\dot{\chi}\circ\chi^{-1}\big)_*$ and $\big(\chi^{-1}\circ\dot{\chi}\big)_*$ are curves in the Lie algebra $\mathfrak{so}(n+1) = \mathbf{T}_e\mathbb{SO}(n+1)$ that can be identified with $(n+1)\times(n+1)$-skew-symmetric matrices.

2.2. Differential geometric properties of ellipsoids

To complete the preliminaries we give formulae for the Weingarten map and the second fundamental form for the ellipsoid. These will be needed later on when we apply Proposition 2.2 to our case. The detailed derivations can be found in [17]. From now on we always suppose that $\boldsymbol{\mathcal{E}}^n \hookrightarrow \mathbb{R}^{n+1}$, where $n > 1$.

Given a positive definite diagonal matrix $\mathbf{D} = \operatorname{diag}(d_1, d_2, \dots, d_{n+1}) \succ 0$ define an ellipsoid $\boldsymbol{\mathcal{E}}^n$ by

$$\boldsymbol{\mathcal{E}}^n := \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \langle x, \mathbf{D}^{-2}x\rangle = 1 \right\}.$$

The Weingarten map Ξ_Λ at $p \in \boldsymbol{\mathcal{E}}^n$ is found in a standard way by differentiation of a unit normal vector $\Lambda = \mathbf{D}^{-2}p/\big|\mathbf{D}^{-2}p\big| \in \mathbf{T}^\perp_p\boldsymbol{\mathcal{E}}^n$ moving along a curve in $\boldsymbol{\mathcal{E}}^n$

$$\Xi_\Lambda(X) = -\mathbf{D}^{-2}\left(\frac{X}{\big|\mathbf{D}^{-2}p\big|} - \frac{p}{\big|\mathbf{D}^{-2}p\big|^3}\langle \mathbf{D}^{-2}X, \mathbf{D}^{-2}p\rangle\right), \quad \text{for any} \quad X \in \mathbf{T}_p\boldsymbol{\mathcal{E}}^n.$$

The second fundamental form can now be calculated using formula (1). Hence

$$\mathbf{II}(X,Y) = \frac{\langle \mathbf{D}^{-1}X, \mathbf{D}^{-1}Y\rangle}{\big|\mathbf{D}^{-2}p\big|^2}\,\mathbf{D}^{-2}p, \quad \text{for any} \quad X, Y \in \mathbf{T}_p\boldsymbol{\mathcal{E}}^n. \tag{2}$$

We are now ready to present our main results.

3. MAIN RESULTS

We begin this section by deriving one of the fundamental geometric quantity of the n-ellipsoid. It is well known that $\boldsymbol{\mathcal{E}}^n$ is a smooth Riemannian manifold with positive Gaussian curvature. Its precise value is given by the following proposition.

Proposition 3.1. The Gaussian curvature $K(p)$ of $\mathcal{E}^n$, isometrically embedded in $\mathbb{R}^{n+1}$, is given by

$$K(p) = \frac{\det(\mathbf{D}^{-2})}{\left|\mathbf{D}^{-2}p\right|^{n+2}} > 0.$$

Proof. To simplify calculations we make use of the following relations of the *scalar second fundamental form* h and the *shape operator* s, cf. [18, page 140]: $\mathbf{II}(X,Y) = h(X,Y)\Lambda$ and $h(X,Y) = \langle X, sY\rangle$, for any tangent vectors X, Y and a normal unit vector Λ. From (2) it follows that

$$h(X,Y) = \frac{\langle \mathbf{D}^{-1}X, \mathbf{D}^{-1}Y\rangle}{\left|\mathbf{D}^{-2}p\right|} = \langle X, sY\rangle, \quad \text{hence} \quad sY = \frac{\mathbf{D}^{-2}Y}{\left|\mathbf{D}^{-2}p\right|}.$$

The shape operator s is positive definite and self adjoint. Suppose that v_i are its eigenvectors and λ_i are the corresponding eigenvalues so that $sv_i = \lambda_i v_i$, for $i = 1, 2, \ldots, n$. Then, with the *generalised cross product* to $\mathbb{R}^{n+1}$, cf. [26, page 84], one has

$$v_1 \times v_2 \times \cdots \times v_n = \alpha\Lambda, \quad \text{where} \quad \Lambda \in \mathbf{T}_p^{\perp}\mathcal{E}^n.$$

On the one hand

$$\big(sv_1\big) \times \big(sv_2\big) \times \cdots \times \big(sv_n\big) = \big(\lambda_1\lambda_2\cdots\lambda_n\big)\, v_1 \times v_2 \times \cdots \times v_n = \big(\alpha\Lambda\big) \prod_{i=1}^{n} \lambda_i,$$

while on the other hand, by the properties of linear mapping acting on the cross product (cf. [17]), there is

$$\big(sv_1\big) \times \big(sv_2\big) \times \cdots \times \big(sv_n\big) = \frac{\mathbf{D}^{-2}v_1}{\left|\mathbf{D}^{-2}p\right|} \times \frac{\mathbf{D}^{-2}v_2}{\left|\mathbf{D}^{-2}p\right|} \times \cdots \times \frac{\mathbf{D}^{-2}v_n}{\left|\mathbf{D}^{-2}p\right|} = \frac{\det(\mathbf{D}^{-2})}{\left|\mathbf{D}^{-2}p\right|^{n}}\, \mathbf{D}^2\big(\alpha\Lambda\big).$$

Since the Gaussian curvature is defined as the product of the eigenvalues of s, therefore

$$K(p) = \prod_{i=1}^{n} \lambda_i = \frac{\det(\mathbf{D}^{-2})}{\left|\mathbf{D}^{-2}p\right|^{n}} \langle \mathbf{D}^2\Lambda, \Lambda\rangle = \frac{\det(\mathbf{D}^{-2})}{\left|\mathbf{D}^{-2}p\right|^{n}} \frac{\langle p, \mathbf{D}^{-2}p\rangle}{\left|\mathbf{D}^{-2}p\right|^{2}} = \frac{\det(\mathbf{D}^{-2})}{\left|\mathbf{D}^{-2}p\right|^{n+2}}.$$

What was to show. □

We shall study now a relationship between the Gauss map and the angular velocity of the rolling n-ellipsoid upon its affine tangent space at a point.

3.1. Rolling an ellipsoid

Throughout this paper, we shall use the following convention. For any two vectors $X, Y \in \mathbb{R}^m$, with $m > 1$, let the tensor product $X \otimes Y$ denote the $m \times m$-matrix with entries $(X \otimes Y)_{ij} = X^i Y^j$. If X, Y are represented as column vectors, then $X \otimes Y = X\,Y^{\mathrm{T}}$. Furthermore, we let $X \wedge Y = X \otimes Y - Y \otimes X$, so that $X \wedge Y$ is a skew-symmetric $m \times m$-matrix whose rank is zero or two.

Given a rolling curve $\sigma_1 \colon I \to \mathcal{E}^n$, the following lemma establishes a relationship between the angular velocity of a rolling ellipsoid and the rolling curve σ_1.

Lemma 3.2. Let χ be the rolling map of an ellipsoid rolling upon its affine tangent space at a point. Then

$$\big(\chi^{-1}\circ\dot{\chi}\big)_* = \frac{1}{\big|\mathbf{D}^{-2}\sigma_1\big|^2}\,(\mathbf{D}^{-2}\dot{\sigma}_1)\wedge(\mathbf{D}^{-2}\sigma_1).$$

P r o o f. The above equality follows from Proposition 2.2. To prove the lemma let us first rewrite the Weingarten map and the second fundamental form in a more convenient way:

$$\begin{aligned}\mathbf{\Xi}_\Lambda(X) &= -\frac{1}{\big|\mathbf{D}^{-2}p\big|}(\mathbf{D}^{-2}X-\Lambda\langle\mathbf{D}^{-2}X,\Lambda\rangle) \quad \text{and}\\ \mathbf{II}(X,Y) &= \frac{1}{\big|\mathbf{D}^{-2}p\big|}\langle\mathbf{D}^{-2}X,Y\rangle\,\Lambda.\end{aligned}$$

Any vector $U\in\mathbf{T}_p\mathbb{R}^{n+1}$ can be uniquely written as $U=V+\alpha\Lambda$, where $V\in\mathbf{T}_p\boldsymbol{\mathcal{E}}^n$. Hence

$$\begin{aligned}\big(\chi^{-1}\circ\dot{\chi}\big)_*U &= \big(\chi^{-1}\circ\dot{\chi}\big)_*\big(V+\alpha\Lambda\big) = -\mathbf{II}(\dot{\sigma}_1,V)-\alpha\mathbf{\Xi}(\dot{\sigma}_1,\Lambda)\\ &= \frac{1}{\big|\mathbf{D}^{-2}p\big|}\Big(-\langle\mathbf{D}^{-2}\dot{\sigma}_1,V\rangle\,\Lambda+\alpha\mathbf{D}^{-2}\dot{\sigma}_1-\alpha\Lambda\langle\mathbf{D}^{-2}\dot{\sigma}_1,\Lambda\rangle\Big)\\ &= \frac{1}{\big|\mathbf{D}^{-2}p\big|}\Big(\langle\Lambda,U\rangle\,\mathbf{D}^{-2}\dot{\sigma}_1-\langle\mathbf{D}^{-2}\dot{\sigma}_1,U\rangle\,\Lambda\Big).\end{aligned}$$

Since a covector in Euclidean space is identified with the transpose of a vector, after expanding Λ to its full expression, one arrives at

$$\big(\chi^{-1}\circ\dot{\chi}\big)_* = \frac{\mathbf{D}^{-2}\dot{\sigma}_1}{\big|\mathbf{D}^{-2}p\big|^2}(\mathbf{D}^{-2}p)^{\mathrm{T}}-\frac{\mathbf{D}^{-2}p}{\big|\mathbf{D}^{-2}p\big|^2}(\mathbf{D}^{-2}\dot{\sigma}_1)^{\mathrm{T}},$$

where $p=\sigma_1(t)$. The result now follows. □

Consider the *Gauss map.* Given a smooth curve $\sigma_1\colon I\to\boldsymbol{\mathcal{E}}^n$, the smooth unit normal vector along σ_1 given by

$$\eta := \rho\,\mathbf{D}^{-2}\sigma_1, \quad \text{where} \quad \rho=\frac{1}{\big|\mathbf{D}^{-2}\sigma_1\big|}$$

defines a smooth map $\eta\colon I\to\mathbf{S}^n$ called the *Gauss map.* There is the following simple formula relating the Gauss map along a rolling curve in $\boldsymbol{\mathcal{E}}^n$ to a curve in $\mathfrak{so}(n+1)$. This follows immediately from Lemma 3.2.

Theorem 3.3. The angular velocity in the body coordinate system of an ellipsoid rolling upon its affine tangent space satisfies

$$\big(\chi^{-1}\circ\dot{\chi}\big)_* = \dot{\eta}\wedge\eta.$$

Remarkably, the above relationship between the angular velocity and the Gauss map holds in a more general situation. The authors are not aware of this relation written explicitly anywhere in the literature therefore it is given here as Theorem 3.4 with a short proof.

3.2. Rolling a Hypersurface

Theorem 3.3 can be generalised to an n-dimensional Riemannian manifold $\mathbf{M}$ embedded in $\mathbb{R}^{n+1}$, a hypersurface. This generalisation expresses the fact that the Gauss map contains enough information to describe the angular velocity of a rolling hypersurface.

Let $\sigma_1 \colon I \to \mathbf{M} \hookrightarrow \mathbb{R}^{n+1}$ be a piecewise differentiable rolling curve in $\mathbf{M}$ and let η be the Gauss map along σ_1.

Theorem 3.4. Suppose that $\boldsymbol{\chi} \colon I \to \mathbb{SE}(n+1)$ is the rolling map of $\mathbf{M}$ rolling upon its affine tangent space $\mathbf{M}_0$ at a point, without slip or twist. Then

$$(\dot{\boldsymbol{\chi}} \circ \boldsymbol{\chi}^{-1})_* = \eta \wedge \dot{\eta}, \tag{3}$$

for almost all t.

Proof. Let $\sigma_0 \colon I \to \mathbf{M}_0$ be the development curve. If $\boldsymbol{\chi}$ is the rolling map then (cf. [24, p. 380])

$$(\dot{\boldsymbol{\chi}} \circ \boldsymbol{\chi}^{-1})_* V = \mathbf{II}(\dot{\sigma}_0, V) \quad \text{and} \quad (\dot{\boldsymbol{\chi}} \circ \boldsymbol{\chi}^{-1})_* \Lambda = \boldsymbol{\Xi}(\dot{\sigma}_0, \Lambda), \tag{4}$$

for any $V \in \mathbf{T}_{\sigma_0}\mathbf{M}_0$ and $\Lambda \in \mathbf{T}^{\perp}_{\sigma_0}\mathbf{M}_0$. The proof amounts to showing that $\eta \wedge \dot{\eta}$ satisfies (4). Equality (3) follows then by comparing the range of the two operators.

Suppose $V \in \mathbf{T}_{\sigma_0}\mathbf{M}_0$ and $\Lambda \in \mathbf{T}^{\perp}_{\sigma_0}\mathbf{M}_0$ are any two vectors in the tangent and normal space, respectively. Then

$$\begin{aligned}(\eta \wedge \dot{\eta})V &= \eta\,\langle \dot{\eta}, V\rangle - \dot{\eta}\,\langle \eta, V\rangle = \eta\,\langle \dot{\eta}, V\rangle \quad \text{and}\\ (\eta \wedge \dot{\eta})\Lambda &= \eta\,\langle \dot{\eta}, \Lambda\rangle - \dot{\eta}\,\langle \eta, \Lambda\rangle = -\dot{\eta}\,\langle \eta, \Lambda\rangle.\end{aligned}$$

Note that the second equality is precisely an expression defining the Weingarten map $\boldsymbol{\Xi}$, cf. [6, Section 2.2]. Since any normal vector $\Lambda \in \mathbf{T}^{\perp}_{\sigma_0}\mathbf{M}_0$ is equal to $\Lambda = \alpha\,\eta$, for some $\alpha \in \mathbb{R}$, from the above expressions it follows that

$$\langle (\eta \wedge \dot{\eta})V, \Lambda\rangle = \alpha\,\langle \dot{\eta}, V\rangle = -\langle (\eta \wedge \dot{\eta})\Lambda, V\rangle.$$

This shows that $\eta \wedge \dot{\eta}$ satisfies the Weingarten equation (1) and therefore it satisfies (4).

Finally, since the normal space $\mathbf{T}^{\perp}_{\sigma_0}\mathbf{M}_0$ is spanned by the normal vector η, then $(\dot{\boldsymbol{\chi}} \circ \boldsymbol{\chi}^{-1})_*$ is rank two. Thus the ranks of the two linear operators coincide therefore the equality (3) holds, wherever they are defined, thus proving the claim. □

We now move to a brief analysis of the case when an ellipsoid is rolling along a geodesic.

3.3. Rolling an ellipsoid along a geodesic

We want to characterise rolling maps in the special case of rolling along geodesics. Inevitably, such analysis interwinds with mechanical interpretation. This is because geodesics are the curves along which points travel in the absence of external forces. The history of geodesics on ellipsoids goes back to the XIX century and C. G. J. Jacobi, who first showed that the problem of geodesics on a triaxial ellipsoid can be reduced by quadratures. Here we confine ourselves to kinematics and study differential equations governing the angular velocity expressed as $\dot{\eta} \wedge \eta$, where η is the Gauss map along the rolling curve. This section draws on works of Knörrer [14], where the classical C. Neumann problem is linked to geodesics on quadrics and on Moser's spectral analysis [20].

3.3.1. The Carl Neumann problem

Consider a point on the sphere moving in a quadratic potential $\langle \gamma, \mathbf{A}\gamma \rangle$, where $\mathbf{A}$ is a symmetric matrix. The equation governing the motion of this point is

$$\ddot{\gamma} = -\mathbf{A}\gamma + u\gamma, \quad \text{where} \quad u = \langle \gamma, \mathbf{A}\gamma \rangle - \langle \dot{\gamma}, \dot{\gamma} \rangle. \tag{5}$$

Knörrer attributes finding algebraic integrals of the Neumann problem (5) to Karen Uhlenbeck's informal preprint [27], whose copy eluded the authors.

Let $\mathbb{C}(z)$ denote the set of complex rational functions that are not everywhere zero.

Lemma 3.5. (K. Uhlenbeck [27]) For any $x, y \in \mathbb{R}^{n+1}$, let $\Phi_z(x, y) \in \mathbb{C}(z)$ be the rational function

$$\Phi_z(x, y) = \langle x, \big(\mathbf{A} - z\,\mathbb{I}\big)^{-1} y \rangle^2 + \big(1 - \langle x, \big(\mathbf{A} - z\,\mathbb{I}\big)^{-1} x \rangle\big) \, \langle y, \big(\mathbf{A} - z\,\mathbb{I}\big)^{-1} y \rangle, \tag{6}$$

where $\mathbb{I}$ is the identity matrix. If $\xi(t)$ is a solution to the Neumann problem (5) then the coefficients of $\Phi_z(\dot{\xi}, \xi) \in \mathbb{C}(z)$ are independent of t.

3.3.2. Isospectral analysis

As above, denote by η the unit-normal vector field along the curve $\gamma\colon I \to \boldsymbol{\mathcal{E}}^n$. Normality of η induces $\langle \eta, \eta \rangle = 1$ and $\langle \dot{\eta}, \eta \rangle = 0$. It will be convenient to introduce two quantities, the skew-symmetric $\boldsymbol{\vartheta}$ and the symmetric $\boldsymbol{\zeta}$ defined by

$$\boldsymbol{\vartheta} := \dot{\eta} \wedge \eta \quad \text{and} \quad \boldsymbol{\zeta} := \eta \otimes \eta.$$

It has been observed by Uhlenbeck, as noted by T. Ratiu [23], that the Neumann problem is equivalent to the following

$$\dot{\boldsymbol{\zeta}} = \big[\boldsymbol{\vartheta}, \boldsymbol{\zeta}\big] \quad \text{and} \quad \dot{\boldsymbol{\vartheta}} = \big[\boldsymbol{\zeta}, \mathbf{A}\big]. \tag{7}$$

From the results of previous sections it follows that $\boldsymbol{\vartheta}$ coincides with the angular velocity of the rolling ellipsoid upon its affine tangent space at a point. We can now look at $\boldsymbol{\vartheta}$ as a curve in the Lie algebra $\boldsymbol{\vartheta}\colon I \to \mathfrak{so}(n+1)$. The aim of this section is to derive the Lax equation, in terms of $\boldsymbol{\vartheta}$ and $\boldsymbol{\zeta}$, that governs rolling $\boldsymbol{\mathcal{E}}^n$ along a geodesic.

In his paper [14], Knörrer shows that the Gauss map η along a reparametrised geodesic solves the Neumann problem. Based on Knörrer's result we now reformulate (6) in terms of $\boldsymbol{\vartheta}$ and $\boldsymbol{\zeta}$.

Consider the following quadratic form, Frobenius inner product, on the space of compatible matrices

$$\langle\!\langle \mathbf{A}, \mathbf{B} \rangle\!\rangle := \frac{1}{2}\operatorname{trace}\big(\mathbf{A}^{\mathrm{T}}\mathbf{B}\big) = \frac{1}{2}\operatorname{trace}\big(\mathbf{A}\mathbf{B}^{\mathrm{T}}\big). \tag{8}$$

It is ease to see that for any $X, Y, Z, W \in \mathbb{R}^m$ the following relation between Frobenius inner product (8) and the inner (dot) Euclidean product $\langle ., . \rangle$ holds

$$\langle\!\langle X \wedge Y, Z \wedge W \rangle\!\rangle = \langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle. \tag{9}$$

Proposition 3.6. Let $\gamma \colon I \to \boldsymbol{\mathcal{E}}^n$ be a geodesic parameterised by

$$\ddot{\gamma}(t) = -\mathbf{D}^{-2}\gamma(t) + v(t)\,\dot{\gamma}(t).$$

Then, the coefficients of $\Psi_z(\boldsymbol{\zeta}, \boldsymbol{\vartheta})$ given by

$$\Psi_z(\boldsymbol{\zeta}, \boldsymbol{\vartheta}) := -\frac{1}{2}\,\langle\!\langle \boldsymbol{\vartheta}\big(\mathbf{D}^{-2} - z\,\mathbb{I}\big)^{-1}, \big(\mathbf{D}^{-2} - z\,\mathbb{I}\big)^{-1}\boldsymbol{\vartheta} \rangle\!\rangle + \langle\!\langle \boldsymbol{\zeta}, \big(\mathbf{D}^{-2} - z\,\mathbb{I}\big)^{-1} \rangle\!\rangle$$

are independent of t.

Proof. For the proof one could simply show that, under the hypothesis, $\Psi_z(y \otimes y, x \wedge y) = 2\,\Phi_z(x, y)$. But it will be more instructive to prove the result by referring back to the original version of Neumann's problem (5). To shorten the notation, denote the resolvent of $\mathbf{D}^{-2}$ by $\mathbf{R}_z = \big(z\,\mathbb{I} - \mathbf{D}^{-2}\big)^{-1}$. Knörrer's result asserts that η is a solution to Neumann's problem

$$\ddot{\eta} = -\mathbf{D}^{-2}\eta + u\eta = \mathbf{R}_z^{-1}\eta + (u - z)\eta.$$

Applying '$\wedge\eta$' to both sides of the above equality yields $\dot{\boldsymbol{\vartheta}} = \ddot{\eta} \wedge \eta = \big(\mathbf{R}_z^{-1}\eta\big) \wedge \eta$. To prove the claim, it is enough to show that the derivative of $\Psi_z(\boldsymbol{\zeta}, \boldsymbol{\vartheta})$ with respect to t is zero. By (9), and because $\mathbf{R}_z$ is symmetric,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\!\langle \boldsymbol{\vartheta}\mathbf{R}_z, \mathbf{R}_z\boldsymbol{\vartheta} \rangle\!\rangle = 2\,\langle\!\langle \mathbf{R}_z\dot{\boldsymbol{\vartheta}}\mathbf{R}_z, \boldsymbol{\vartheta} \rangle\!\rangle = 2\,\langle\!\langle \eta \wedge (\mathbf{R}_z\eta), \dot{\eta} \wedge \eta \rangle\!\rangle = -2\,\langle \dot{\eta}, \mathbf{R}_z\eta\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\langle \eta, \mathbf{R}_z\eta\rangle.$$

Since $\langle \eta, \mathbf{R}_z\eta\rangle = \operatorname{trace}(\eta\eta^{\mathrm{T}}\mathbf{R}_z) = 2\,\langle\!\langle \eta \otimes \eta, \mathbf{R}_z \rangle\!\rangle$, the result now follows. □

Consider the matrix

$$L(\boldsymbol{\zeta}, \boldsymbol{\vartheta}) := P_{\boldsymbol{\zeta}}(\mathbf{D}^{-2} + \boldsymbol{\vartheta}^2)P_{\boldsymbol{\zeta}}, \tag{10}$$

where $P_{\boldsymbol{\zeta}} = \mathbb{I} - \boldsymbol{\zeta}$ is a projection operator. We will show that L is an isospectral operator. Based on Moser's analysis [20, p. 149] we will verify that

$$\frac{\det\big(z\mathbb{I} - L(\boldsymbol{\zeta}, \boldsymbol{\vartheta})\big)}{\det\big(z\mathbb{I} - \mathbf{D}^{-2}\big)} = -2z\,\Psi_z(\boldsymbol{\zeta}, \boldsymbol{\vartheta}), \tag{11}$$

showing that the zeros of Ψ_z, where Ψ_z is thought to be a function of z, are the $m-1$ nontrivial eigenvalues of L. The trivial zero eigenvalue corresponds to $\boldsymbol{\zeta}$. Formula (11) is a particular case of the Weinstein-Aronszajn determinant, cf. [12].

To verify (11), we follow the theory of perturbation of a symmetric bilinear operator of rank three, cf. [20, p. 154]. The matrix L can be written in terms of η and $\dot\eta$ as

$$L = \mathbf{D}^{-2} - \eta \otimes \big(\mathbf{D}^{-2}\eta\big) - \big(\mathbf{D}^{-2}\eta\big) \otimes \eta + \langle \eta, \mathbf{D}^{-2}\eta\rangle\, \eta \otimes \eta - \dot\eta \otimes \dot\eta.$$

The above expression is the generic form of a perturbation of a symmetric operator A in a finite dimensional vector space V,

$$L = A + \sum_{i=1}^{r} x_i \otimes \xi_i,$$

where $\{\, x_i \,\}$ and $\{\, \xi_i \,\}$ are two sets of linearly independent vectors in V, and r is the rank of the perturbation. In this case the spectrum of L is determined by the formula

$$\frac{\det\big(z\mathbb{I} - L\big)}{\det\big(z\mathbb{I} - A\big)} = \det\big(\mathbb{I} - W\big), \quad \text{where} \quad W_{ij} = \langle \mathbf{R}_z x_i, \xi_j\rangle \quad \text{and} \quad \mathbf{R}_z = \big(z\,\mathbb{I} - A\big)^{-1}.$$

The task therefore is to calculate $\det\big(\mathbb{I} - W\big)$. We assume here that the eigenvalues of $\mathbf{D}$, the semi-axes of $\boldsymbol{\mathcal{E}}^n$, are all different and that $x_1 = \eta$, $x_2 = \mathbf{D}^{-2}\eta$ and $x_3 = \dot\eta$ are three linearly independent vectors in $\mathbb{R}^{n+1}$. If $\mathbf{B}$ is the 3×3-matrix

$$\mathbf{B} = \begin{pmatrix} \langle \mathbf{D}^{-2}\eta, \eta\rangle & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{then} \quad L = \mathbf{D}^{-2} + \sum_{i,j=1}^{3} x_i \otimes (B_{ij} x_j).$$

The 3×3-matrix W is given by

$$W_{ij} = \langle \mathbf{R}_z x_i, \xi_j\rangle = \langle \mathbf{R}_z x_i, \sum_{k=1}^{3} B_{jk} x_k\rangle = \sum_{k=1}^{3} B_{jk} \langle \mathbf{R}_z x_i, x_k\rangle = \sum_{k=1}^{3} B_{jk}\, Q_z(x_i, x_k),$$

where Q_z is a symmetric bilinear form $Q_z(x, y) = \langle \mathbf{R}_z x, y\rangle$. Thus,

$$W = \begin{pmatrix} Q_z(\eta,\eta) & Q_z(\mathbf{D}^{-2}\eta,\eta) & Q_z(\dot\eta,\eta) \\ Q_z(\mathbf{D}^{-2}\eta,\eta) & Q_z(\mathbf{D}^{-2}\eta,\mathbf{D}^{-2}\eta) & Q_z(\mathbf{D}^{-2}\eta,\dot\eta) \\ Q_z(\dot\eta,\eta) & Q_z(\mathbf{D}^{-2}\eta,\dot\eta) & Q_z(\dot\eta,\dot\eta) \end{pmatrix} \mathbf{B}^{\mathrm{T}}.$$

It is easily checked that Q_z satisfies the property $Q_z(\mathbf{D}^{-2}x, y) = z\, Q_z(x, y) - \langle x, y\rangle$, for any $x, y \in \mathbb{R}^{n+1}$. Henceforth, the entries in the above matrix transform to

$$Q_z(\mathbf{D}^{-2}\eta, \eta) = z\, Q_z(\eta, \eta) - 1, \qquad Q_z(\mathbf{D}^{-2}\eta, \dot\eta) = z\, Q_z(\dot\eta, \eta)$$

and

$$Q_z(\mathbf{D}^{-2}\eta, \mathbf{D}^{-2}\eta) = z\big(z\, Q_z(\eta, \eta) - 1\big) - \langle \mathbf{D}^{-2}\eta, \eta\rangle.$$

Denote $\alpha := \langle \mathbf{D}^{-2}\eta, \eta\rangle$. Then,

$$\det(\mathbb{I} - W) = \det\begin{pmatrix} (z-\alpha)\, Q_z(\eta,\eta) & Q_z(\eta,\eta) & Q_z(\dot\eta,\eta) \\ z\big((z-\alpha)\, Q_z(\eta,\eta) - 1\big) & z\, Q_z(\eta,\eta) & z\, Q_z(\dot\eta,\eta) \\ (z-\alpha)\, Q_z(\dot\eta,\eta) & Q_z(\dot\eta,\eta) & 1 + Q_z(\dot\eta,\dot\eta) \end{pmatrix}$$

$$= z \det\begin{pmatrix} 0 & Q_z(\eta,\eta) & Q_z(\dot\eta,\eta) \\ -1 & Q_z(\eta,\eta) & Q_z(\dot\eta,\eta) \\ 0 & Q_z(\dot\eta,\eta) & 1 + Q_z(\dot\eta,\dot\eta) \end{pmatrix} = z \det\begin{pmatrix} Q_z(\eta,\eta) & Q_z(\dot\eta,\eta) \\ Q_z(\dot\eta,\eta) & 1 + Q_z(\dot\eta,\dot\eta) \end{pmatrix}.$$

It is now easy to see that

$$\Psi_z(\boldsymbol{\zeta}, \boldsymbol{\vartheta}) = \frac{1}{2}\left(Q_z^2(\dot\eta,\eta) - Q_z(\eta,\eta) - Q_z(\eta,\eta)\, Q_z(\dot\eta,\dot\eta)\right),$$

which verifies equality (11).

Matrix L in (10) is a symmetric matrix depending on $\boldsymbol{\zeta}$ and $\boldsymbol{\vartheta}$ with the property that its eigenvalues remain fixed as the ellipsoid rolls along a geodesic. This is a direct consequence of Proposition 3.6. Matrices with this property are called *isospectral*. Because the eigenvalues are constant with time, the corresponding eigenvectors satisfy simple ordinary differential equations. These equations can be written in the Lax form.

Corollary 3.7. The Lax representation of the geodesic flow for the rolling ellipsoid is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} L = \big[\boldsymbol{\vartheta}, L\big]. \tag{12}$$

P r o o f. This statement requires a calculation. One easily checks, with the bracket properties (7), that

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{\boldsymbol{\zeta}} = \big[\boldsymbol{\vartheta}, P_{\boldsymbol{\zeta}}\big].$$

For $N_{\boldsymbol{\vartheta}} = \mathbf{D}^{-2} + \boldsymbol{\vartheta}^2$ one also has

$$P_{\boldsymbol{\zeta}}\left(\frac{\mathrm{d}}{\mathrm{d}t} N_{\boldsymbol{\vartheta}} - \big[\boldsymbol{\vartheta}, N_{\boldsymbol{\vartheta}}\big]\right) P_{\boldsymbol{\zeta}} = P_{\boldsymbol{\zeta}}\left(\big[\boldsymbol{\zeta}, \mathbf{D}^{-2}\big]\boldsymbol{\vartheta} + \boldsymbol{\vartheta}\big[\boldsymbol{\zeta}, \mathbf{D}^{-2}\big] - \big[\boldsymbol{\vartheta}, \mathbf{D}^{-2}\big]\right) P_{\boldsymbol{\zeta}}$$
$$= P_{\boldsymbol{\zeta}}\left(\mathbf{D}^{-2} P_{\boldsymbol{\zeta}} \boldsymbol{\vartheta} - \boldsymbol{\vartheta} P_{\boldsymbol{\zeta}} \mathbf{D}^{-2}\right) P_{\boldsymbol{\zeta}} = 0,$$

because $P_{\boldsymbol{\zeta}}\, \boldsymbol{\vartheta}\, P_{\boldsymbol{\zeta}} = 0$. The Lie bracket is a derivation, in the sense that

$$\big[\boldsymbol{\vartheta}, L\big] = \big[\boldsymbol{\vartheta}, P_{\boldsymbol{\zeta}}\, N_{\boldsymbol{\vartheta}}\, P_{\boldsymbol{\zeta}}\big] = \big[\boldsymbol{\vartheta}, P_{\boldsymbol{\zeta}}\big]\, N_{\boldsymbol{\vartheta}}\, P_{\boldsymbol{\zeta}} + P_{\boldsymbol{\zeta}}\, \big[\boldsymbol{\vartheta}, N_{\boldsymbol{\vartheta}}\big]\, P_{\boldsymbol{\zeta}} + P_{\boldsymbol{\zeta}}\, N_{\boldsymbol{\vartheta}}\, \big[\boldsymbol{\vartheta}, P_{\boldsymbol{\zeta}}\big].$$

From the above calculations, the Lax form (12) is now easily verified. □

4. CONCLUSION AND FURTHER QUESTIONS

In this paper we have shown that the kinematic equations for the rolling motion of an ellipsoid can be quantified in terms of the Gauss map. Differential equations for the angular velocity hold in a more general situation and remain valid for any hypersurface rolling upon its affine tangent space at a point. Since the rolling maps are transitive, these results can be applied to describe rolling of one hypersurface upon another hypersurface. We have examined the case of ellipsoid rolling along a geodesic and derived the Lax equation for the geodesic flow in terms of the angular velocity. In contrast with previous studies our approach uses the Riemannian geometric formalism rather than Hamiltonian mechanics.

A possible extension of this study is to consider the pseudo-Riemannian case. Rolling maps in these spaces have been studied, for instance in [7], and [15]. Connecting this knowledge with the studies of geodesics on ellipsoids, or on more general quadrics, could bring some new insights into integrals of motions for Lorentzian metrics.

We plan to consider, in future work, more general settings and a geometric interpretation of the corresponding integrals of motions.

ACKNOWLEDGMENTS

The authors wish to thank professor Velimir Jurdjevic for helpful discussions and professor Yuri Fedorov, for pointing us to [8] and [4]. Eventually, we were led to papers [13, 14] by Knörrer and then to the theory of isospectral methods related to Jacobi geodesic problems in [19] and [28] in the Hamiltonian framework.

The work was partially supported by FCT, under project PTDC/EEA-CRO/122812/2010.

(Received May 7, 2015)

REFERENCES

[1] A. Bicchi, R. Sorrentino, and C. Piaggio: Dexterous manipulation through rolling. In: ICRA'95, IEEE Int. Conf. on Robotics and Automation 1995, pp. 452–457. DOI:10.1109/robot.1995.525325

[2] A. V. Borisov and I. S. Mamaev: Isomorphism and Hamilton representation of some nonholonomic systems. Sibirsk. Mat. Zh. *48* (2007), 1, 33–45. DOI:10.1007/s11202-007-0004-6

[3] A. V. Borisov and I. S. Mamaev: Rolling of a non-homogeneous ball over a sphere without slipping and twisting. Regular and Chaotic Dynamics *12* (2007), 2, 153–159. DOI:10.1134/s1560354707020037

[4] A. V. Borisov and I. S. Mamaev: Isomorphisms of geodesic flows on quadrics. Regular and Chaotic Dynamics *14* (2009), 4 – 5, 455–465. DOI:10.1134/s1560354709040030

[5] R. Caseiro, P. Martins, J. F. Henriques, F. Silva Leite, and J. Batista: Rolling Riemannian manifolds to solve the multi-class classification problem. In: CVPR 2013, pp. 41–48. DOI:10.1109/cvpr.2013.13

[6] I. Chavel: Riemannian Geometry — A Modern Introduction. Second edition. Cambridge Studies in Advanced Mathematics, No. 98. Cambridge University Press, Cambridge 2006. DOI:10.1017/cbo9780511616822

[7] P. Crouch and F. Silva Leite: Rolling maps for pseudo-Riemannian manifolds. In: Proc. 51th IEEE Conference on Decision and Control, (Hawaii 2012). DOI:10.1109/cdc.2012.6426140

[8] Y. N. Fedorov and B. Jovanović: Nonholonomic LR systems as generalized Chaplygin systems with an invariant measure and flows on homogeneous spaces. J. Nonlinear Science *14* (2004), 4, 341–381. DOI:10.1007/s00332-004-0603-3

[9] K. Hüper, K. A. Krakowski., and F. Silva Leite: Rolling Maps in a Riemannian Framework. Textos de Matemática *43*, Department of Mathematics, University of Coimbra 2011, pp. 15–30.

[10] K. Hüper and F. Silva Leite: On the geometry of rolling and interpolation curves on S^n, SO_n and Graßmann manifolds. J. Dynam. Control Systems *13* (2007), 4, 467–502. DOI:10.1007/s10883-007-9027-3

[11] N. M. Justin Carpentier J.-P. L. Andrea Del Prete: An analytical model of rolling contact and its application to the modeling of bipedal locomotion. In: Proc. IMA Conference on Mathematics of Robotics 2015, pp. 452–457.

[12] T. Kato: Perturbation Theory for Linear Operators. Springer-Verlag, Classics in Mathematics *132*, 1995. DOI:10.1007/978-3-642-66282-9

[13] H. Knörrer: Geodesics on the ellipsoid. Inventiones Mathematicae *59* (1980), 119–144. DOI:10.1007/bf01390041

[14] H. Knörrer: Geodesics on quadrics and a mechanical problem of C. Neumann. J. für die reine und angewandte Mathematik *334* (1982), 69–78. DOI:10.1515/crll.1982.334.69

[15] A. Korolko and F. Silva Leite: Kinematics for rolling a Lorentzian sphere. In: Proc. 50th IEEE Conference on Decision and Control and European Control Conference (IEEE CDC-ECC 2011), Orlando 2011, pp. 6522–6528. DOI:10.1109/cdc.2011.6160592

[16] K. Krakowski and F. Silva Leite: An algorithm based on rolling to generate smooth interpolating curves on ellipsoids. Kybernetika *50* (2014), 4, 544–562. DOI:10.14736/kyb-2014-4-0544

[17] K. A. Krakowski and F. Silva Leite: Why controllability of rolling may fail: a few illustrative examples. In: Pré-Publicações do Departamento de Matemática, no. 12-26. University of Coimbra 2012, pp. 1–30.

[18] J. M. Lee, J: Riemannian Manifolds: An Introduction to Curvature. Springer-Verlag, Graduate Texts in Mathematics *176*, New York 1997.

[19] J. Moser: Three integrable Hamiltonian systems connected with isospectral deformations. Advances Math. *16* (1975), 2, 197–220. DOI:10.1016/0001-8708(75)90151-6

[20] J. Moser: Geometry of quadrics and spectral theory. In: The Chern Symposium 1979 (W.-Y. Hsiang, S. Kobayashi, I. Singer, J. Wolf, H.-H. Wu, and A. Weinstein, eds.), Springer, New York 1980, pp. 147–188. DOI:10.1007/978-1-4613-8109-9_7

[21] K. Nomizu: Kinematics and differential geometry of submanifolds. Tôhoku Math. J. *30* (1978), 623–637. DOI:10.2748/tmj/1178229921

[22] A. M. Okamura, N. Smaby, and M. R. Cutkosky: : An overview of dexterous manipulation. In: ICRA'00, IEEE Int Conf. on Robotics and Automation 2000, pp. 255–262. DOI:10.1109/robot.2000.844067

[23] T. Raţiu: The C. Neumann problem as a completely integrable system on an adjoint orbit. Trans. Amer. Math. Soc. *264* (1981), 2, 321–329. DOI:10.1090/s0002-9947-1981-0603766-3

[24] R. W. Sharpe: Differential Geometry: Cartan's Generalization of Klein's Erlangen Program. Springer-Verlag, Graduate Texts in Mathematics *166*, New York 1997.

[25] F. Silva Leite and K. A. Krakowski: Covariant differentiation under rolling maps. In: Pré-Publicações do Departamento de Matemática, No. 08-22, University of Coimbra 2008, pp. 1–8.

[26] M. Spivak: Calculus on Manifolds. Mathematics Monograph Series, Addison-Wesley, New York 1965.

[27] K. Uhlenbeck: Minimal 2-spheres and tori in S^k. Preprint, 1975.

[28] A. P. Veselov: A few things I learnt from Jürgen Moser. Regular and Chaotic Dynamics *13* (2008), 6, 515–524. DOI:10.1134/s1560354708060038

[29] A. Weintrit and T. Neumann, eds.: Methods and Algorithms in Navigation: Marine Navigation and Safety of Sea Transportation. CRC Press, 2011. DOI:10.1201/b11344

Krzysztof Andrzej Krakowski, Wydział Matematyczno-Przyrodniczy Uniwersytet Kardynała Stefana Wyszyńskiego w Warszawie, Poland and Institute of Systems and Robotics, University of Coimbra–Pólo II, 3030-290 Coimbra. Portugal.
e-mail: k.krakowski@uksw.edu.pl

Fátima Silva Leite, Department of Mathematics, University of Coimbra Apartado 3008, 3001-454 Coimbra and Institute of Systems and Robotics, University of Coimbra–Pólo II, 3030-290 Coimbra. Portugal.
e-mail: fleite@mat.uc.pt