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Abstract. In this short article we answer the question posed in Ghadermazi M., Karamzadeh O.A.S., Namdari M., *On the functionally countable subalgebra of* $C(X)$, Rend. Sem. Mat. Univ. Padova **129** (2013), 47–69. It is shown that $C_c(X)$ is isomorphic to some ring of continuous functions if and only if $\upsilon_0 X$ is functionally countable. For a strongly zero-dimensional space X, this is equivalent to say that X is functionally countable. Hence for every P-space it is equivalent to pseudo-$\aleph_0$-compactness.

Keywords: zero-dimensional space; strongly zero-dimensional space; $\mathbb{N}$-compact space; Banaschewski compactification; character; ring homomorphism; functionally countable subring; functional separability

Classification: Primary 54C30, 54D35, 46E25; Secondary 54D60, 54C40

1. Introduction

For topological spaces X and E, the space X is called E-completely regular provided that it can be topologically embedded into the product space E^κ, for some cardinal number κ. If we consider the particular case where $E = \mathbb{N}$ (i.e., the set of all natural numbers with the discrete topology), then one can verify that X is zero-dimensional and Hausdorff (i.e., a T_2-space with a base consisting of clopen sets) if and only if X is $\mathbb{N}$-completely regular. We also recall that a topological space X is E-compact if it is embeddable as a closed subset into the product space E^τ for some cardinal number τ. The notions of E-completely regular and E-compact spaces were introduced by Mrówka and Engelking in [7]. Mrówka continued to investigate the properties of such spaces in [17], [19]. For a special case $E = \mathbb{N}$, see [15], [16]. He also wrote a survey on E-compact spaces in [18]. The reader is referred to [22] for terminology and notions about E-compactness. The following theorem is needed in the sequel, see e.g., [18, Theorem 4.14].

Theorem 1.1. *For every E-completely regular space X there exists a space $\upsilon_E(X)$ such that:*

(a) *$\upsilon_E(X)$ is E-compact and it contains X as a dense subspace;*
(b) *every continuous function $f : X \to Y$, where Y is an arbitrary E-compact space, admits a continuous extension $f^* : \upsilon_E(X) \to Y$.*

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Any E-compact space Z containing X as a dense subspace with the properties (a) and (b) of Theorem 1.1, is homeomorphic with $\upsilon_E(X)$. As a special case, for every zero-dimensional space X there exists an $\mathbb{N}$-compact space $\upsilon_0 X$ such that every continuous function $f : X \to Y$, with Y an $\mathbb{N}$-compact space, has a unique extension $f^* : \upsilon_0 X \to Y$. Also, we could replace an arbitrary $\mathbb{N}$-compact space Y with the fixed discrete space $\mathbb{Z}$ (i.e., the set of all integer numbers), and have the following characterization of $\mathbb{N}$-compactification of a zero-dimensional space, see e.g., [22, 5.4(d)].

Theorem 1.2. *An $\mathbb{N}$-compact extension T of a zero-dimensional space X is homeomorphic with $\upsilon_0 X$ if and only if for each continuous function $f : X \to \mathbb{Z}$, there exists $F : T \to \mathbb{Z}$ such that $F|_X = f$.*

A topological space X is called strongly zero-dimensional if X is a nonempty completely regular Hausdorff space and every finite cozero-set cover $\{U_i\}_{i=1}^k$ of the space X has a finite open refinement $\{V_i\}_{i=1}^m$ such that $V_i \cap V_j = \emptyset$, whenever $i \neq j$. Equivalently, a nonempty completely regular Hausdorff space X is strongly zero-dimensional if and only if for every pair A, B of completely separated subsets of the space X, there exists a clopen set U in X such that $A \subseteq U \subseteq X \setminus B$, see e.g., [10].

It is well-known that every strongly zero-dimensional realcompact space is $\mathbb{N}$-compact. It is also easy to see that every countable subset of $\mathbb{R}$ is Lindelöf and zero-dimensional and hence is strongly zero-dimensional. Therefore every countable subset of $\mathbb{R}$ is $\mathbb{N}$-compact. So we have the following trivial lemma.

Lemma 1.3. *Let X be a zero-dimensional Hausdorff space. For each continuous function $f : X \to \mathbb{R}$ with countable image, there exists an extension $f^* : \upsilon_0 X \to \mathbb{R}$ such that the image of f^* is equal to the image of f.*

For an arbitrary completely regular Hausdorff space X, we denote by $C_c(X)$ the set of all continuous real-valued functions on X with countable image. The set $C_c(X)$ forms a subring of $C(X)$ (i.e., the set of all continuous real valued functions on X) with pointwise addition and multiplication. Ghadermazi, Karamzadeh and Namdari showed in [9] that for a completely regular Hausdorff space X there exists a zero-dimensional space Y such that $C_c(X) \cong C_c(Y)$. In view of this fact, in the present article we restrict our attention to zero-dimensional spaces. In the same article, the authors required to find an example of a zero-dimensional space X for which $C_c(X)$ is not isomorphic to any $C(Y)$. They remarked that for an uncountable discrete space X, the ring $C_c(X)$ is not isomorphic to any ring of continuous functions. They also remained an unsettled question to determine completely regular Hausdorff spaces X for which $C_c(X)$ is isomorphic to some ring of continuous functions. The reader may consult [9] for all prerequisites and unfamiliar notions for the functionally countable algebras.

In the present article, we give a complete answer to the aforementioned question and by virtue of it, we will show that not only for discrete spaces but for various kinds of zero-dimensional spaces X the algebra $C_c(X)$ is not isomorphic to any ring of continuous functions.

We remind the reader that $\beta_0 X$ is the unique (up to homeomorphism) zero-dimensional compact space which contains X as a dense subset such that every continuous two-valued function $f : X \to \{0,1\}$ has a unique extension to $\beta_0 X$.

Bhattacharjee, Knox and McGovern have found that the maximal ideal space of $C_c(X)$ is homeomorphic with $\beta_0 X$, see [4]. They remarked that the proof of this fact can be modeled after [11, Theorem 5.1].

We conclude this section with the following proposition. We also remind the reader that a subset $Y \subseteq X$ is C_c-embedded in X if for each $f \in C_c(Y)$, there exists $F \in C_c(X)$ such that $F|_Y = f$.

Proposition 1.4. *An $\mathbb{N}$-compact extension T of a zero-dimensional space X is homeomorphic with $\upsilon_0 X$ if and only if X is C_c-embedded in T.*

PROOF: By Lemma 1.3, the necessity is trivial. For sufficiency, it is enough to show that for each continuous function $f : X \to \mathbb{N}$, there exists a continuous function $F : T \to \mathbb{N}$ such that $F|_X = f$. Since $f \in C_c(X)$, there exists $h \in C_c(T)$ such that $h|_X = f$. The subset $h(T)$ of $\mathbb{R}$ is countable. Hence there exists an increasing sequence $r_1 < r_2 < \cdots < r_n < \cdots$ such that for each $n \in \mathbb{N}$, $r_n \notin h(T)$ and

$$r_n < n < r_{n+1}.$$

Define $W_1 = F^{-1}(-\infty, r_2)$ and for each $n > 2$, $W_n = F^{-1}(r_n, r_{n+1})$. Each W_n is clopen and $T = \bigcup_{n\in\mathbb{N}} W_n$. Define the map $k : T \to \mathbb{N}$ to be such that for each $n \in \mathbb{N}$, $k|_{W_n} = n$. Clearly k is continuous and $k|_X = f$. So we are done. □

2. When is $C_c(X) \cong C(Y)$?

We recall that a completely regular Hausdorff topological space X is functionally countable if each continuous real-valued function on X has countable image. It was mentioned in [14] that X is functionally countable if and only if every second countable continuous image of X is countable. Moreover, X is functionally countable if and only if every metrizable image of X is countable. All functionally countable spaces are zero-dimensional and pseudo-$\aleph_1$-compact space (i.e., every discrete family of non empty open sets is at most countable). In the literature of rings of continuous functions, functional countability appeared in [1], [2], [3], [13], [21], [23]. To achieve our main theorem we need the following proposition. Before, we recall that a character on $C_c(X)$ is a non zero algebra homomorphism from $C_c(X)$ onto $\mathbb{R}$. For example, the evaluation δ_x at a point $x \in X$, which is defined by $\delta_x(f) = f(x)$, for all $f \in C_c(X)$, is a character on $C_c(X)$. The following proposition is basic for the rest of this section. We remind the reader that there are several proofs for determining all the characters on $C(X)$, whenever X is a realcompact space, see e.g., [8], [5]. We adapt the latest proof which appeared in [5] to determine all the characters on $C_c(X)$, whenever X is an $\mathbb{N}$-compact space. The referee noted that the procedure of the proof of the following proposition is somehow similar to the proof of [12, Proposition 2.7].

Proposition 2.1. *Let X be an $\mathbb{N}$-compact space and Φ be a character on $C_c(X)$. There exists a unique $x \in X$ such that $\Phi(f) = f(x)$, for all $f \in C_c(X)$.*

Proof: (Uniqueness) Since for each two distinct elements x, y in X there exists a clopen set U such that $x \in U$ and $y \notin U$, the uniqueness of the point x is trivial.

(Existence) For every $f \in C_c(X)$, $f(X)$ is a countable subset of $\mathbb{R}$ and therefore is $\mathbb{N}$-compact. The space X is $\mathbb{N}$-compact and since $C_c(X)$ separates points from closed sets, X can be embedded as a closed subset of the product space $\prod_{f\in C_c(X)} f(X)$. Thus, we can identify each point of X with the point $(f(x))_{f\in C_c(X)}$ of the product space. For every $f \in C_c(X)$ consider the projection $\pi_f \in C_c(X)$ where

$$\pi_f(x) = f(x),$$

for $x = (f(x))_{f\in C_c(X)}$. Note that for each $f \in C_c(X)$, $\Phi(f) \in f(X)$. To see this, we have $\Phi(f - \Phi(f)) = 0$ and therefore $f - \Phi(f)$ is a nonunit in $C_c(X)$ and hence there exists $t \in X$ such that $f(t) = \Phi(f)$.

Consider the point

$$z = (\Phi(f))_{f\in C_c(X)} \in \prod_{f\in C_c(X)} f(X).$$

First we claim that $z \in X$. Otherwise, since X is a closed set in $\prod_{f\in C_c(X)} f(X)$, there would exist $\epsilon > 0$ and a nonempty finite subset $J \subseteq C_c(X)$ such that the set

$$\Omega = \bigcap_{g\in J} \{(x_f)_{f\in C_c(X)} : |x_g - \Phi(\pi_g)| < \epsilon\}$$

is empty. Define

$$k = \sum_{g\in J} (\pi_g - \Phi(\pi_g))^2 \in C_c(X),$$

and observe that $\Phi(k) = 0$. Hence, k is a nonunit in $C_c(X)$ and thus $k(z_k) = 0$ for some $z_k \in X$. Then, $|\pi_g(z_k) - \Phi(\pi_g)| = 0$ for all $g \in J$ and so $z_k \in \Omega = \emptyset$, which is a contradiction. We derive that $z \in X$, as desired. Now pick $f \in C_c(X)$ and $\epsilon > 0$. Since $z \in X$ and f is continuous on X, there exists $\delta > 0$ and a nonempty finite subset $J \subseteq C_c(X)$ such that, for $x \in X$,

$$(*) \qquad |\pi_g(x) - \Phi(\pi_g)| < \delta \quad \forall g \in J \implies |f(x) - f(z)| < \epsilon.$$

Now define

$$h = (f - \Phi(f))^2 + \sum_{g\in J} (\pi_g - \Phi(\pi_g))^2 \in C_c(X).$$

Clearly $\Phi(h) = 0$ and hence there exists $z_h \in X$ such that $f(z_h) = \Phi(f)$ and $\pi_g(z_h) = \Phi(\pi_g)$ for all $g \in J$. These equalities together with $(*)$ yield that

$$|\Phi(f) - f(z)| = |f(z_h) - f(z)| < \epsilon.$$

Therefore $\Phi(f) = f(z)$ and the proof is complete. □

We recall that a maximal ideal M in $C_c(X)$ is real if $\frac{C_c(X)}{M}$ is isomorphic with the field $\mathbb{R}$.

Theorem 2.2. *Let X be a zero-dimensional space. Then X is $\mathbb{N}$-compact if and only if every real maximal ideal M in $C_c(X)$ is fixed (i.e., there exists $p \in X$ such that $M = \{f \in C_c(X) : f(p) = 0\}$).*

Proof: Let X be $\mathbb{N}$-compact (i.e., $\upsilon_0 X = X$) and M be a real maximal ideal in $C_c(X)$. Then M is the kernel of some character on $C_c(X)$. Hence by Proposition 2.1, the ideal M must be fixed. Conversely, assume that every real maximal ideal of $C_c(X)$ is fixed and $\upsilon_0 X \neq X$. Consider a point $p \in \upsilon_0 X \setminus X$. By Lemma 1.3, we have the ring isomorphism Φ from $C_c(X)$ to $C_c(\upsilon_0(X))$, which maps each $f \in C_c(X)$ to its unique extension on $\upsilon_0 X$. Clearly the ideal

$$M_c^p = \{f \in C_c(\upsilon_0 X) : f(p) = 0\}$$

is a fixed maximal ideal of $C_c(\upsilon_0 X)$. Therefore the ideal $\Phi^{-1}(M_c^p)$ is a real maximal ideal of $C_c(X)$. But we have

$$\Phi^{-1}(M_c^p) = \{f \in C_c(X) : f^*(p) = 0\},$$

which clearly is not a fixed ideal of $C_c(X)$ and this is a contradiction. □

We remind the reader that an ideal I of $C_c(X)$ is a contraction of an ideal J of $C(X)$ provided that $I = J \cap C_c(X)$.

Remark 2.3. For a zero-dimensional space X, Theorem 2.2 shows that every real maximal ideal of $C_c(X)$ is a contraction of a unique fixed maximal ideal in $C(X)$ if and only if X is $\mathbb{N}$-compact. P. Nyikos gave an example of a realcompact and zero-dimensional space which is not $\mathbb{N}$-compact, see [20]. Therefore we infer that there exists a zero-dimensional space X for which the subring $C_c(X)$ has a real maximal ideal that is not a contraction of a real maximal ideal of the ring $C(X)$.

Theorem 2.4. *Let X be an $\mathbb{N}$-compact space. Then $C_c(X)$ is isomorphic to some ring of continuous functions if and only if X is functionally countable.*

Proof: Suppose that there exists a topological space Y such that $C_c(X) \cong C(Y)$. We denote the maximal ideal spaces of $C_c(X)$ and $C(Y)$ by $\mathcal{M}_c(X)$ and $\mathcal{M}(Y)$, respectively. Since $C_c(X)$ and $C(Y)$ are isomorphic, $\mathcal{M}_c(X)$ must be homeomorphic with $\mathcal{M}(Y)$. The Gelfand-Kolmogoroff theorem states that $\mathcal{M}(Y)$ is homeomorphic with the Stone-Čech compactification of Y, denoted by βY. On the other hand $\mathcal{M}_c(X)$ is homeomorphic with the Banaschewski compactification of X, see [11], [4]. Hence $\beta_0 X$ is homeomorphic with βY. Therefore Y must be strongly zero-dimensional. Without loss of generality we can assume that Y is also realcompact. Now let $\Phi : C(Y) \to C_c(X)$ be our ring isomorphism. For each $x \in X$, define the character $\Phi_x : C(Y) \to \mathbb{R}$ such that for each $f \in C(Y)$,

$\Phi_x(f) = \Phi(f)(x)$. Since Y is real compact, by [10, 10.5(c)], there exists a unique point $\pi(x) \in Y$ such that

$$\Phi_x(f) = f(\pi(x)),$$

for each $f \in C(Y)$. The mapping π from X into Y, thus defined, evidently satisfies

$$\Phi(f) = f \circ \pi,$$

for each $f \in C(Y)$. We need to show that π is a homeomorphism.

The map π is one to one. For $p \neq q \in X$ there exists a clopen set U such that $p \in U$ and $q \notin U$. Consider the characteristic function χ_U. There exists $f \in C(Y)$ such that $\Phi(f) = f \circ \pi = \chi_U$. Hence $f(\pi(p)) = 1$ and $f(\pi(q)) = 0$. Therefore $\pi(p) \neq \pi(q)$.

The map π is continuous. Suppose that V is a clopen subset of Y. Consider the characteristic function $\chi_V \in C(Y)$. Since the function $\Phi(\chi_V) = \chi_V \circ \pi$ is continuous and two valued, the subset

$$(\chi_V \circ \pi)^{-1}(1) = \pi^{-1}(V),$$

is open in X. Therefore the map π is continuous.

The image $\pi(X)$ is dense in Y. For if $y \in Y \setminus \mathrm{cl}_Y \pi(X)$, there exists a clopen set V such that $y \in V$ and $V \cap \mathrm{cl}_Y \pi(X) = \emptyset$. Evidently Φ takes the characteristic function χ_V to zero and this is a contradiction, for Φ is one to one.

Now we show that the map $\pi : X \to \pi(X)$ is open. If W is a clopen subset of X, for the characteristic function $\chi_W \in C_c(X)$, there exists $f \in C(Y)$ such that

$$\Phi(f) = f \circ \pi = \chi_W.$$

Note that $f(\pi(W)) = \{1\}$ and $f(\pi(X \setminus W)) = \{0\}$. Hence $\pi(W)$ and $\pi(X \setminus W)$ are two closed and disjoint subsets of $\pi(X)$ and their union is $\pi(X)$. Therefore $\pi(W)$ is open in $\pi(X)$. Thus X is homeomorphic with $\pi(X)$.

We claim that Y is functionally countable. Consider the ring homomorphism

$$\Phi^{-1} : C_c(X) \to C(Y).$$

For each $y \in Y$, define the character $\Phi_y^{-1} : C_c(X) \to \mathbb{R}$ by

$$\Phi_y^{-1}(g) = \Phi^{-1}(g)(y).$$

Since X is $\mathbb{N}$-compact, there exists a unique point $\sigma(y) \in X$ such that

$$\Phi_y^{-1}(g) = \Phi^{-1}(g)(y) = g(\sigma(y)).$$

Hence the function $\sigma : Y \to X$, thus defined, satisfies $\Phi^{-1}(g) = g \circ \sigma$, for all $g \in C_c(X)$. Since Φ^{-1} is onto, for each $f \in C(Y)$ there exists $g \in C_c(X)$ such that $\Phi^{-1}(g) = g \circ \sigma = f$. Note that g has a countable image and hence $f \in C_c(Y)$.

If we show that the image $\pi(X)$ is C_c-embedded in Y, then by Proposition 1.4, we observe that $\pi(X) = Y$. To see this, let $h \in C_c(\pi(X))$. Then $h \circ \pi \in C_c(X)$. There exists $T \in C_c(Y) = C(Y)$ (for Y is functionally countable) such that

$$\Phi(T) = T \circ \pi = h \circ \pi.$$

The restriction of T to $\pi(X)$ is h. Hence by Proposition 1.4, $\pi(X) = Y$. Therefore X is homeomorphic with Y. For completing the proof we observe that X and Y are homeomorphic, Y is functionally countable and hence X is functionally countable. □

By Lemma 1.3, one can deduce that for a zero-dimensional space X we have $C_c(X) \cong C_c(\upsilon_0 X)$. Therefore the following result is immediate.

Corollary 2.5. *For a zero-dimensional space X, the ring $C_c(X)$ is isomorphic to some ring of continuous functions if and only if $\upsilon_0 X$ is functionally countable.*

Remark 2.6. It is well-known that a realcompact strongly zero-dimensional space is $\mathbb{N}$-compact. Also since each continuous real valued function on a completely regular Hausdorff space X has an extension to υX with the same image, we have $\upsilon_0 X = \upsilon X$.

Theorem 2.4 together with Remark 2.6, imply the following corollary.

Corollary 2.7. *Let X be a strongly zero-dimensional space. Then $C_c(X)$ is isomorphic to some ring of continuous functions if and only if X is functionally countable.*

As an application of Corollary 2.7, consider the space of irrational numbers, denoted by $\mathbb{P}$. The ring $C_c(\mathbb{P})$ is not isomorphic to any ring of continuous functions. Notice that for showing this fact, all ring theoretic methods which were chosen in [9] are not applicable here.

Remark 2.8. In [13], it is shown that a P-space is functionally countable if and only if X is pseudo-$\aleph_1$-compact. Since every P-space is strongly zero-dimensional, Corollary 2.7 shows that for an arbitrary P-space X, the ring $C_c(X)$ is isomorphic to some ring of continuous functions if and only if X is pseudo-$\aleph_1$-compact.

Remark 2.9. With regard to Theorem 2.4 and Corollary 2.7, for a strongly zero-dimensional space X, $\upsilon_0 X$ is functionally countable if and only if X is functionally countable. The interested reader is encouraged to find an example of a zero-dimensional non functionally countable space X, for which $\upsilon_0 X$ is functionally countable.

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