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# On the potential theory of some systems of coupled PDEs

Abderrahim Aslimani, Imad El Ghazi,
Mohamed El Kadiri, Sabah Haddad

*Abstract.* In this paper we study some potential theoretical properties of solutions and super-solutions of some PDE systems (S) of type $L_1u = -\mu_1 v$, $L_2 v = -\mu_2 u$, on a domain $D$ of $\mathbb{R}^d$, where $\mu_1$ and $\mu_2$ are suitable measures on $D$, and $L_1$, $L_2$ are two second order linear differential elliptic operators on $D$ with coefficients of class $\mathcal{C}^\infty$. We also obtain the integral representation of the nonnegative solutions and supersolutions of the system (S) by means of the Green kernels and Martin boundaries associated with $L_1$ and $L_2$, and a convergence property for increasing sequences of solutions of (S).

*Keywords:* harmonic function; superharmonic function; potential; elliptic linear differential operator; kernel; coupled PDEs system; Kato measure

*Classification:* 31B05, 31B10, 31B35

## 1. Introduction

Let $D$ be a domain of $\mathbb{R}^d$ $(d \geq 1)$, $L_1$, $L_2$ two second order elliptic or parabolic linear differential operators with coefficients of class $\mathcal{C}^\infty$, and $\mu_1$ and $\mu_2$ two suitable measures on $D$. We suppose that $D$ is Green domain for $L_1$ and $L_2$. The potential theory of a system (S) of PDE of type

$$\begin{cases} L_1 u = -\mu_1 v, \\ L_2 v = -\mu_2 u, \end{cases}$$

has been studied by many authors.

If $L_1 = L_2 = \Delta$, the Laplace operators, $\mu_2 = 0$ and $\mu_1 = \lambda$ the Lebesgue measure, the solutions of the corresponding system (S) are the pairs $(u, v)$ where $u$ is a biharmonic function, that is, a solution of the biharmonic equation $\Delta^2 u = 0$, and $v = -\Delta u$. More generally, if $L_1$, $L_2$ are two elliptic linear second order differential operators on $D$, $\mu_2 = 0$ and $\mu_1 = \lambda$, the system (S) is equivalent to biharmonic equation $L_1 L_2 u = 0$. The corresponding equations with limit conditions were extensively studied by many authors [15], [16], [17], [18], [27], [28].

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In the 1970's Smyrnelis [25], [26] developed an axiomatic theory for the solutions of these systems when $\mu_1 = 0$ or $\mu_2 = 0$, called axiomatic theory of biharmonic spaces. Just after, Bouleau [4], [5] studied the corresponding probabilistic potential theory.

In the case where $\mu_1 \neq 0$, $\mu_2 \neq 0$ and $L_1, L_2$ are elliptic, a probabilistic potential theory of systems of type (S) has been studied by Chen and Zhao [9]. The axiomatic theory of biharmonic spaces of Smyrnelis do not apply in this case. However, the theory of balayage developed by Bliedtner and Hansen [2] can be considered as the axiomatic theory which apply in this setting. Many potential theoretical problems in this theory remain still open.

This work was initiated by the papers [12], [13] of the third named author on the biharmonic Martin boundary and the integral representation of nonnegative biharmonic functions in a general axiomatic setting which apply to a system of type (S) with $\mu_1 = 0$ or $\mu_2 = 0$.

Our main purpose in this paper is to extend some potential theoretical methods to the systems of type (S). In particular we will define and study the notions of superharmonic pairs and potentials associated with (S) on the domain $D$. We also obtain the integral representation of the potentials and the nonnegative harmonic pairs. The results of this article can be extended to the systems of type (S) associated with operators in the more general class of second order linear elliptic differential operators leading to harmonic spaces and studied by R.-M. Hervé in [22]. We have considered the operators with coefficients of class $\mathcal{C}^\infty$ for simplicity and in order to use the distribution theory.

**Notations.** Let $A$ be a subset of $\mathbb{R}^d$, $d \geq 1$. We denote by $\overline{A}$ the closure of $A$ in the Alexandroff compactification $\overline{\mathbb{R}^d}$ of $\mathbb{R}^d$, and by $\partial A$ the boundary of $A$ in $\overline{\mathbb{R}^d}$. By function on $A$ we mean a function on $A$ with values in $[-\infty, +\infty]$. Let $U$ be an open subset of $\mathbb{R}^d$. The set of nonnegative Borel functions on $U$ is denoted by $\mathcal{B}_+(U)$. If $f$ is a function on $U$, we denote by $\widehat{f}$ the l.s.c. regularization of $f$. Recall that $\widehat{f}$ is defined by $\widehat{f}(x) = \liminf_{y\to x} f(y)$ for all $x \in U$ and that $\widehat{f}$ is the greater l.s.c. function such that $\widehat{f} \leq f$ on $U$.

The natural order on the set of pairs of functions on $A$ is defined by

$$(f, g) \geq (h, k) \iff f(x) \geq h(x) \text{ and } g(x) \geq k(x)\ \forall x \in A,$$

and we simply write $(f, g) \geq 0$ for $(f, g) \geq (0, 0)$.

## 2. Potential theory associated with a second order elliptic operator

For the convenience of reader who is not familiar with potential theory, we give here some recalls on potential theory of an elliptic operator of second order on a domain of $\mathbb{R}^d$.

Let $D$ be a domain of $\mathbb{R}^d$, $d \geq 1$, and

$$L = \sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

a second order linear differential operator with locally Lipschitz coefficients on $D$, of type elliptic or simply elliptic, that is, for all $x \in D$ the quadratic form $\sum_{i,j} a_{ij}(x)\xi_i\xi_j$ is positive definite on $\mathbb{R}^d$. In this section we recall some results on potential theory of operator $L$ that will be used in the sequel. For more details on this theory the reader is referred to [22, Chapter VII] and the bibliography therein.

**Definition 2.1.** A real function $u$ on an open subset $\omega$ of $D$ is said to be $L$-harmonic (or simply harmonic if there is no risk of confusion) in $\omega$ if $u$ is of class $\mathcal{C}^2$ and $Lu = 0$ on $\omega$.

When the coefficients $a_{ij}$ and $b_i$ of $L$ are of class $\mathcal{C}^\infty$, then a continuous real function on $D$ is $L$-harmonic if and only if $u$ is a solution of $Lu = 0$ in the distribution sense, that is

$$\sum_{i,j}\int_D a_{ij}u(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}(x)\,d\lambda(x) - \sum_i\int_D b_i(x)u(x)\frac{\partial\varphi}{\partial x_i}(x)\,d\lambda(x) \\ + \int_D c(x)u(x)\varphi(x)\,d\lambda(x) = 0$$

for all $\varphi \in \mathcal{D}(D)$, where $\mathcal{D}(D)$ is the space of functions of class $\mathcal{C}^\infty$ of compact supports, that is, which vanish outside a compact of $D$, and $\lambda$ denotes the Lebesgue measure on $\mathbb{R}^d$.

For every open subset $\omega$ of $D$, we denote by $\mathcal{H}_L(\omega)$ the set of $L$-harmonic functions on $\omega$. It is clear that the set $\mathcal{H}_L(\omega)$ is a $\mathbb{R}$-vector space. Moreover, the mapping $\mathcal{H} : \mathcal{O} \ni \omega \mapsto \mathcal{H}_L(\omega)$ is a sheaf on $D$ of vector spaces of continuous functions, where $\mathcal{O}$ is the set of open subsets of $D$.

**Definition 2.2.** A relatively compact open subset $\omega$ of $D$ is said to be $L$-regular or simply regular if for any real continuous function $f$ on $\partial\omega$, there exists a unique function $h = H_\omega(f) \in \mathcal{H}_L(\omega)$ such that $\lim_{x\in\omega, x\to\zeta} h(x) = f(\zeta)$ for all $\zeta \in \partial\omega$.

By [22, p. 561], every open ball $B \subset \overline{B} \subset D$ is $L$-regular.

Let $\omega$ be a regular relatively compact open subset of $D$. By the minimum principle, if $f \in \mathcal{C}_+(\partial\omega)$ then $H_\omega(f) \geq 0$. It follows from the definition that for all $x \in \omega$, the map $f \in \mathcal{C}(\partial\omega) \mapsto \mathbb{R}$ is a positive linear form on $\mathcal{C}(\partial\omega)$, hence it is a nonnegative Radon measure on $\partial\omega$; we denote it by $\mu_x^\omega$. We then have

$$H_\omega(f) = \int f\,d\mu_x^\omega$$

for all $x \in \omega$ and $f \in \mathcal{C}(\partial\omega)$.

The following results are consequences of [22, Theorem 34.1] and [8, p. 15] and [22, p. 426]:

**Theorem 2.3** (Harnack principle)**.** *Let $K$ be a compact subset of $D$ and $x_0 \in K$. Then there exists a constant $C > 0$, depending only on $K$, such that for any nonnegative $L$-harmonic function $h$ on $D$, one has $\sup_{x\in K} h \le Ch(x_0)$.*

**Corollary 2.4.** *For any $x_0 \in D$, the set $H_L(x_0) = \{h \in \mathcal{H}_L^+(D) : h(x_0) = 1\}$ is equicontinuous at $x_0$.*

**Theorem 2.5.** *Let $\omega$ be a sub-domain of $D$, and $(h_n)$ an increasing sequence of $L$-harmonic functions on $\omega$. Then the function $h = \sup_n h_n$ is $L$-harmonic in $\omega$ if (and only if) it is finite at some point of $\omega$.*

**Definition 2.6.** Let $\omega$ be an open subset of $D$. A function $u : \omega \to \overline{\mathbb{R}}$ is said to be $L$-hyperharmonic on $\omega$ if

(i) $u$ is l.s.c. and $u > -\infty$ on $\omega$;
(ii) for every open ball $B \subset \overline{B} \subset \omega$, $\int_{\partial B}^{*} u\, d\mu_x^\omega \le u(x)$ $\forall x \in B$.

The function $u$ is said to be $L$-superharmonic on $\omega$ if $u$ is $L$-hyperharmonic and not identical to $+\infty$ in each connected component of $\omega$; $u$ is $L$-hypoharmonic (resp. $L$-subharmonic) if $-u$ is $L$-hyperharmonic (resp. $L$-superharmonic).

It is clear from the definition that a $L$-superharmonic function $u$ on an open subset $\omega$ of $D$ is finite on a dense subset of $\omega$. The set of $L$-hyperharmonic (resp. $L$-superharmonic) functions on $D$ is a convex cone. It is denoted by $\mathcal{H}_L^*(D)$ (resp. $\mathcal{S}_L(D)$) or simply $\mathcal{H}^*(D)$ (resp. $\mathcal{S}(D)$) if here is no risk of confusion.

Let $u$ be a $L$-superharmonic on $D$. It follows easily from the above theorem that for any regular relatively compact open subset $\omega$ of $D$, the function $x \mapsto \int u\, d\mu_x^\omega$ is $L$-harmonic in $\omega$. We also denote it by $H_\omega(u)$. If $(u_n)$ is an increasing sequence of $L$-superharmonic functions in a domain $\omega \subset D$, then $\sup_n u_n$ is $L$-harmonic or identical to $+\infty$ in $\omega$.

**Definition 2.7.** A subset $A$ of $D$ is said to be locally $L$-polar (resp. $L$-polar), or simply locally polar (resp. polar), if for every $x \in A$ there exists an open neighborhood $\omega$ of $x$ and a function $u$ $L$-superharmonic on $\omega$ (resp. $D$) such that $A \cap \omega \subset \{x \in \omega : u(x) = +\infty\}$ (resp. $A \subset \{x \in D : u(x) = +\infty\}$).

**Properties.**

1. Any $L$-polar subset $A$ of $D$ is locally polar.
2. Any subset of a locally $L$-polar (resp. polar) set is locally $L$-polar (resp. polar).
3. Any locally $L$-polar subset of $D$ has an empty interior.
4. Let $(A_n)$ be a sequence of $L$-polar sets of $D$, then $\cup_n A_n$ is $L$-polar.

**Theorem 2.8** ([8, p. 47])**.** *Suppose that there exists a $L$-superharmonic $u > 0$ on $D$ which is not $L$-harmonic. Then every locally polar set is polar.*

We say that a property $P(x)$ relative to $x \in D$ holds quasi-everywhere (q.e) if it holds outside a polar set. For more details on polar sets we refer to [8], [11], [22].

**Theorem 2.9** (cf. [22, Property (C), p. 521, and Corollary of Theorem 36.2]). *Let $(u_i)_{i\in I}$ be a locally uniformly bounded family of $L$-hyperharmonic functions on an open subset $\omega$ of $D$. Then the function $\widehat{\inf}_{i\in I} u_i = \widehat{\inf_{i\in I} u_i}$ is $L$-hyperharmonic in $\omega$ and $\widehat{\inf}_{i\in I} u_i = \inf_{i\in I} u_i$ q.e.*

**Theorem 2.10** (Removable singularities principle [11, Theorem, p. 60]). *Let $E$ be a closed $L$-polar subset of $D$, and $u$ a $L$-superharmonic function on $D\setminus E$ such that $u$ is locally lower bounded in a neighborhood of any point of $E$. Then there exists a unique $L$-superharmonic function $s$ on $D$ such that $s = u$ on $D \setminus E$.*

The following proposition is an easy corollary of the above theorem:

**Proposition 2.11.** *Let $u$ and $v$ be two $L$-hyperharmonic functions on $D$. If $u = v$ q.e. then $u = v$ on $D$.*

The following proposition follows easily from [11, pp. 38–39]:

**Proposition 2.12.** *Let $u$ be a nonnegative $L$-superharmonic function $u$ on $D$. Then the following statements are equivalent.*

1. *For any $L$-subharmonic function $v$ on $D$, if $v \le u$, then $v \le 0$ in $D$.*
2. *For any $L$-superharmonic function $v$ on $D$, if $v + u \ge 0$, then $v \ge 0$ in $D$.*
3. *For any $L$-harmonic function $h$ on $D$, if $h \le u$, then $h \le 0$ in $D$.*

**Definition 2.13.** A $L$-superharmonic function on $D$ is said to be a $L$-potential (or simply a potential) on $D$ if it verifies one of the equivalent conditions of the above proposition.

**Theorem 2.14** (Riesz decomposition [10, Theorem 2.2.2]). *Let $u$ be a nonnegative $L$-superharmonic function on $D$. Then there exists a unique $L$-potential $p$ and a unique nonnegative $L$-harmonic function $h$ on $D$ such that $u = p + h$.*

The following theorem is a combination of properties of solutions of operator $L$ in [22, Chapter VII] and [22, Proposition 18.1 and Theorem 18.2]:

**Theorem 2.15.** *Assume that there exists a $L$-potential $p > 0$ on $D$. Then there exists a function $G : D \times D \to \overline{\mathbb{R}}_+$ with the following properties.*

1. *$G$ is l.s.c. on $D\times D$ and continuous on $D\times D\setminus\delta$, where $\delta = \{(x,x); x \in D\}$ is the diagonal of $D \times D$.*
2. *For all $y \in D$, the function $G(.,y)$ is $L$-harmonic on $D \setminus \{y\}$.*
3. *For any $L$-potential $p$ in $D$ there exists a unique (Radon) measure $\mu$ on $D$ such that $p = G\mu := \int_D G(.,y)\, d\mu(y)$.*

The following proposition is an easy consequence of Theorems 2.14 and 2.15(3).

**Proposition 2.16.** *Let $u$ be a $L$-superharmonic function on $D$. If $Lu = 0$ in the distribution sense, then $u$ is $L$-harmonic.*

**Definition 2.17.** A domain $D$ of $\mathbb{R}^d$ on which there exists a $L$-potential $> 0$ is called a $L$-Green domain (or simply a Green domain). The corresponding function $G$ in Theorem 2.15 in the above theorem is called a $L$-Green kernel of $L$ in $D$.

If $G$ and $G'$ are two $L$-Green kernels, there exists a continuous $\varphi > 0$ on $D$ such that $G'(.,y) = \varphi(y)G(.,y)$ for all $y \in D$. Since the coefficients of $L$ are supposed to be of class $\mathcal{C}^\infty$, there exists a unique $L$-Green kernel $G$ such that for all $y \in D$, we have $LG(.,y) = -\epsilon_y$ in the distribution sense, where $\epsilon_y$ is the Dirac measure at $y$; this function $G$ is then called the Green kernel of $L$.

*Remark* 2.18.

1. Any subdomain of a $L$-Green domain is a $L$-Green domain.
2. Any bounded domain $D$ is a $L$-Green kernel for any elliptic operator of second order with Lipschitz coefficients.

*Remark* 2.19. The domain $D$ is not always a $L$-Green domain. For example let $L = \Delta$, the Laplace operator, $d = 2$ and $D = \mathbb{R}^2$. Then $D$ is not a $L$-Green domain.

Let us recall here that a kernel on a measurable space $(E, \mathcal{E})$ is a function $V : E \times \mathcal{E} \to \overline{\mathbb{R}}_+$ such that

1. for any $A \in \mathcal{E}$, the function $E \ni x \mapsto V(x, A)$ is measurable;
2. for any $x \in E$, the function $\mathcal{E} \ni A \mapsto V(x, A)$ is a measure on $(E, \mathcal{E})$.

The potential $Vf$ or $V(f)$ of a nonnegative $\mathcal{E}$-measurable function $f$ on $E$ is defined by

$$Vf(x) = \int_E f(y)V(x, dy), \ \forall x \in E.$$

Let us denote by $\mathcal{E}_+$ the convex cone of nonnegative $\mathcal{E}$-measurable functions on $E$ with value in $\overline{\mathbb{R}}$. We can define equivalently a kernel on $(E, \mathcal{E})$ as being a function $V : \mathcal{E}_+ \to \mathcal{E}_+$ such that

1. $V(0) = 0$;
2. for any sequence $(f_n)$ in $\mathcal{E}_+$,

$$V(\sum_n f_n) = \sum_n V(f_n).$$

If $\|V1\|_\infty < 1$, then $I - V$ is a kernel on $(E, \mathcal{E})$, and its restriction to the cone $b\mathcal{E}_+$ of bounded non-negative $\mathcal{E}$-measurable functions can be extended to an invertible bounded positive linear operator on the linear space $b\mathcal{E} = b\mathcal{E}_+ - b\mathcal{E}_+$. The operator $(I - V)^{-1}$ can be extended in a unique way to a kernel on $(E, \mathcal{E})$, denoted again by $(I - V)^{-1}$. Then we have

$$(I - V)^{-1} = \sum_k V^k.$$

Suppose now that $D$ is $L$-Green domain and let $G$ be the $L$-Green kernel of $D$. For any nonnegative Radon measure $\mu$ on $D$ we denote by $G^\mu$ the kernel defined on $D$ by

$$G^\mu f(x) = \int_D G(x, y)f(y)\, d\mu(y), \ x \in D,$$

for all nonnegative Borel measurable function $f$ on $D$, and by $G^\mu$ the function $G^\mu 1$. Then we have $LG^\mu f = -f\mu$ in the distribution sense. The function $G^\mu$ is called the potential of $\mu$. A nonnegative Radon measure on $D$ is said to be Kato measure for $L$ if its potential $G^\mu$ is finite and continuous on $D$.

Following [8, p. 47], we have:

**Theorem 2.20.** *Suppose that $D$ is a $L$-Green domain. Then there exists up to a homeomorphism a compact topological space $\widehat{D}$ containing $D$ and a function $K : D \times \Delta \longrightarrow \mathbb{R}$, where $\Delta = \widehat{D} \setminus D$ such that*

1. *the induced topology on $D$ is identical to the Euclidean topology;*
2. *$D$ is dense in $\widehat{D}$;*
3. *for each $Y \in \Delta$, the function $K(.,Y)$ is $L$-harmonic $> 0$ on $D$ and the functions $K(x,.)$, $x \in D$, are continuous on $\Delta$ and separate the points of $\Delta$;*
4. *for each $L$-harmonic $\geq 0$ there exists a measure $\mu$ on $\Delta$ such that $h = K\mu := \int_\Delta K(.,Y)\, d\mu(Y)$;*
5. *for each $L$-harmonic $\geq 0$ there exists a unique measure $\mu$ on $\Delta$, carried by $\Delta_1$ such that $h = K\mu := \int_\Delta K(.,Y)\, d\mu(Y)$, where $\Delta_1$ is the $G_\delta$-set of minimal elements of $\Delta$.*

The set $\Delta$ is called the Martin boundary of $D$ (with respect to $L$) and $\Delta_1$ is the minimal Martin boundary of $D$. The kernel (function) $K$ is called the Martin kernel of $D$. Let us recall here that a point $Y \in \Delta$ is said to be minimal if the $L$-harmonic function $K(.,Y)$ is on an extreme generator of the cone $\mathcal{H}_L^+ D)$ of nonnegative $L$-harmonic functions on $D$. For further results on classical and axiomatic potential theory we refer to [1], [21], [22].

## 3. Harmonic and superharmonic pairs

In all the rest of this paper $D$ is a domain of $\mathbb{R}^d$, $d \geq 1$, and $L_1$, $L_2$ are two second order elliptic linear differential operators on $D$ with coefficients of class $\mathcal{C}^\infty$. We suppose that $D$ is a Green domain for $L_1$ and $L_2$ with $L_1$-Green and $L_2$-Green kernels $G_1$ and $G_2$ respectively, normalized in such a way that, for all $y \in D$ we have $L_1 G_1(.,y) = L_2 G_2(.,y) = -\epsilon_y$ in the distribution sense.

We also fix $\mu_1$ and $\mu_2$, two Kato measures for $L_1$ and $L_2$ respectively, such that

$$\|G_1^{\mu_1}\|_\infty \|G_2^{\mu_2}\|_\infty < 1, \tag{3.1}$$

and consider the following PDE's system (S)

$$\begin{cases} L_1 u = -\mu_1 v, \\ L_2 v = -\mu_2 u. \end{cases}$$

A solution of the system $(S)$ is a pair $(u, v)$ of continuous functions on $D$ such that $L_1 u = -\mu_1 v$ and $L_2 v = -\mu_2 u$ in the distribution sense. The condition (3.1) insure the existence of positive solutions of the system (S) in the case where there exist positive bounded and continuous functions $h$ and $k$ on $D$ such that $L_1 h = 0$ and $L_2 k = 0$.

If $L_1 = L_2 = \Delta$ (the Laplace operator), the measure $\mu_2 = 0$ and $\mu_1 = \lambda$ the Lebesgue measure, the solutions of the corresponding system (S) are the pairs $(u, v)$ where $u$ is a biharmonic function, that is, a solution of the biharmonic equation $\Delta^2 u = 0$, and $v = -\Delta u$.

In all the rest of this paper we denote by $V_1$ and $V_2$ the (Borel) kernels on $D$ defined by

$$V_1 = \sum_{k=0}^{+\infty} (G_1^{\mu_1} G_2^{\mu_2})^k$$

and

$$V_2 = \sum_{k=0}^{+\infty} (G_2^{\mu_2} G_1^{\mu_1})^k.$$

It follows from the hypothesis on the measures $\mu_1$ and $\mu_2$ that for any bounded $f \in \mathcal{B}_+(D)$, the functions $V_i f$, $i = 1, 2$, are finite and continuous on $\Omega$.

In this section we give some definitions, notations and properties of solutions (harmonic pairs) and supersolutions (superharmonic pairs) of the system $(S)$.

**Definition 3.1.** A pair $(u, v)$ of continuous functions on $D$ is said to be harmonic if $u$ and $v$ are solutions of (S) in the distribution sense.

Let $B$ be a ball $B \subset \overline{B} \subset D$ and $f$ defined on a set containing $\partial B$. If $f_{|\partial B}$ is finite and continuous, we denote by $H_B^i(f)$, $i = 1, 2$, the solution of the Dirichlet problem relative to the operator $L_i$ on $B$ for the data $f$ on $\partial B$, and if $f_{|\partial B}$ is l.s.c. $\geq 0$ we define $H_B(f)$ by $H_B(f) = \sup\{H_B(\varphi) : \varphi \in \mathcal{C}(\partial B)\}$. We also denote by $K_B^{i,\mu_i}$ the Borel kernel defined on $B$ by

$$K_B^{i,\mu_i} f = \int_B G_B^i(., y) f(y)\, d\mu_i(y), \ \forall f \in \mathcal{B}_+(B),$$

where $G_B^i$ is the Green kernel on $B$ relative to $L_i$. We also denote by $V_B^i$, $i = 1, 2$, the Borel kernels defined on $B$ by

$$V_B^1 = \sum_{k=0}^{+\infty} (K_B^{1,\mu_1} K_B^{2,\mu_2})^k \ \text{ and } \ V_B^2 = \sum_{k=0}^{+\infty} (K_B^{2,\mu_2} K_B^{1,\mu_1})^k.$$

*Remark* 3.2. If the function $f \in \mathcal{B}_+(B)$ is bounded, then the functions $V_B^{i,\mu_i} f$, $i = 1, 2$, are continuous on $B$.

**Theorem 3.3.** *A pair $(u, v)$ of continuous functions on $D$ is harmonic if and only if for any open ball $B \subset \overline{B} \subset D$ we have $H_B^1(u) + K_B^{1,\mu_1}(v) = u$ and $H_B^2(v) + K_B^{2,\mu_2}(u) = v$ in $B$.*

Proof: Let us first suppose that the pair $(u,v)$ is harmonic in $D$. Then we have $L_1u=-\mu_1u$ and $L_2v=-\mu_2v$ in $D$ in the distribution sense. Let $B$ and $B'$ be two open balls of $D$ such that $B\subset\overline{B}\subset B'\subset\overline{B}'\subset D$. Since $u$ and $v$ are bounded on $B'$ and $\mu_1,\mu_2$ are Kato measures, then the functions $K^{1,\mu_1}_{B'}u$ and $K^{2,\mu_2}_{B'}v$ are continuous on $B'$ and we have $L_1(u-K^{1,\mu_1}_{B'}v)=L_2(v-K^{2,\mu_2}_{B'}u)=0$ in the distribution sense. Hence the functions $u-K^{1,\mu_1}_{B'}v$ and $v-K^{2,\mu_2}_{B'}u$ are respectively $L_1$-harmonic and $L_2$-harmonic in $B'$. So we have $H^1_B(u-K^{1,\mu_1}_{B'}v)=u-K^{1,\mu_1}_{B'}v$ and $H^2_B(v-K^{2,\mu_2}_{B'}u)=v-K^{2,\mu_2}_{B'}u$ in $B$. Hence $H^1_B(u)=u-(K^{1,\mu_1}_{B'}v-H^1_B(K^{1,\mu_1}_{B'}v))=u-K^{1,\mu_1}_{B}v$ and $H^2_B(v)=u-(K^{2,\mu_2}_{B'}u-H^2_B(K^{2,\mu_2}_{B'}u))=u-K^{2,\mu_2}_{B}u$ in $B$, which gives $H^1_B(u)+K^{1,\mu_1}_B(v)=u$ and $H^2_B(v)+K^{2,\mu_2}_B(u)=v$ in $B$. Conversely, suppose that $H^1_B(u)+K^{1,\mu_1}_B(v)=u$ and $H^2_B(v)+K^{2,\mu_2}_B(u)=v$ in any open ball $B\subset\overline{B}\subset D$. Since $H^1_B(u)$ and $H^2_B(v)$ are $L_1$-harmonic and $L_2$-harmonic respectively, then we obviously have $L_1u=L_1K^{1,\mu_1}_B(v)=-\mu_1v$ and $L_2v=L_2K^{2,\mu_2}_B(v)=-\mu_2u$ on $B$ in the distribution sense. As $B$ is arbitrary we conclude that $L_1u=-\mu_1v$ and $L_2v=-\mu_2u$ on $D$ in the distribution sense, that is, the pair $(u,v)$ is harmonic on $D$. $\square$

**Definition 3.4.** A pair $(u,v)$ of locally integrable functions on $D$ is said to be superharmonic if $u$ and $v$ are l.s.c. $>-\infty$ on $D$ and satisfy the inequalities

$$\begin{cases} L_1u\le-\mu_1v\\ L_2v\le-\mu_2u,\end{cases}$$

in distribution sense.

It follows clearly from the definition that a pair $(u,v)$ of locally integrable functions on $D$ is harmonic if and only if the pairs $(u,v)$ and $(-u,-v)$ are superharmonic.

**Proposition 3.5.** *Let $(u,v)$ be a nonnegative superharmonic pair on $D$. Then $u$ is $L_1$-superharmonic and $v$ is $L_2$-superharmonic on $D$.*

Proof: The proposition is an obvious consequence of the definition of superharmonic pairs. $\square$

The following theorem can be proved in the same way as Theorem 3.3.

**Theorem 3.6.** *A pair $(u,v)$ of locally integrable l.s.c. functions on $D$ is superharmonic if and only if for any open ball $B\subset\overline{B}\subset D$ we have $H^1_B(u)+K^{1,\mu_1}_B(v)\le u$ and $H^2_B(u)+K^{2,\mu_2}_B(v)\le u$ on $B$.*

We denote by $\mathcal{S}(D)$ the set of superharmonic pairs in $D$. Let $(u_j,v_j),(u,v)\in\mathcal{S}(D)$, $j=1,2$, and $\alpha\in\mathbb{R}_+$, then we define

$$(u_1,v_1)+(u_2,v_2)=(u_1+u_2,v_2+v_2),$$

and

$$\alpha(u,v)=(\alpha u,\alpha v)\ if\ \alpha>0\ and\ 0(u,v)=(0,0).$$

**Corollary 3.7.** *The set $\mathcal{S}(D)$ is a convex cone.*

We denote by $\mathcal{S}_+(D)$ the set of nonnegative superharmonic pairs on $D$.

**Corollary 3.8.** *$\mathcal{S}_+(D)$ is a convex cone.*

**Corollary 3.9.** *Let $(u_1, v_1)$ and $(u_2, v_2)$ are two superharmonic pairs on $D$. Then the pair $(u_1 \wedge u_2, v_1 \wedge v_2)$ is superharmonic in $D$.*

**Corollary 3.10.** *Let $((u_n, v_n))$ be an increasing sequence of nonnegative superharmonic pairs on an open subset $\omega$ of $D$ and suppose that $\sup_n u_n(x_0) + \sup_n v(x_0) < +\infty$ for some point $x_0 \in D$. Then the pair $(u, v) = \sup_n(u_n, v_n)$ is superharmonic on $\omega$.*

PROOF: In view of Proposition 3.5 the functions $u_n$, resp. $v_n$, $n \in \mathbb{N}$, are $L_1$-superharmonic, resp. $L_2$ superharmonic, and hence the function $u$, resp. $v$, is $L_1$-superharmonic, resp. $L_2$-superharmonic on $D$, since it is finite at $x_0$. It follows that $u$ and $v$ are l.s.c. and locally integrable. Let $B$ be an open ball of $\mathbb{R}^d$ such that $\overline{B} \subset D$, then we have $H_B^1(u_n) + K_B^{1,\mu_1}(v_n) \leq u$ and $H_B^2(u_n) + K_B^{2,\mu_2}(v_n) \leq u$ on $B$ for any integer $n$. By letting $n \to +\infty$, we therefore have $H_B^1(u_n) + K_B^{1,\mu_1}(v_n) \leq u$ and $H_B^2(u_n) + K_B^{2,\mu_2}(v_n) \leq u$ on $B$. It follows from Theorem 3.6 that the pair $(u, v)$ is superharmonic on $D$. □

**Theorem 3.11.** *Let $(u_1, u_2)$ be a pair of Borel functions on $D$ such that*

$$G_j^{\mu_j}|u_k| \not\equiv +\infty, \; j \neq k, \; j, k = 1, 2.$$

*Then the following statements are equivalent.*

1. *The pair $(u_1, u_2)$ is superharmonic.*
2. *The function $u_1 - G_1^{\mu_1}(u_2)$, resp. $u_2 - G_2^{\mu_2}(u_1)$, defined on the subset of $D$ where the difference is well defined has an extension to $L_1$-superharmonic, resp. $L_2$-superharmonic, function on $D$.*

PROOF: Let us first remark that the hypothesis of the theorem implies that for any $j, k = 1, 2$, $j \neq k$, the function $G_j^{\mu_j}|u_k|$ is $L_j$-superharmonic on $D$, and so are the functions $G_j^{\mu_j} u_k^+$ and $G_j^{\mu_j} u_k^-$. It follows then that these functions are finite $L_j$-q.e.

1 ⇒ 2 Suppose that the pair $(u_1, u_2)$ is superharmonic on $D$ and satisfies the condition of the theorem. Then the function $G_1^{\mu_1}(u_2) = G_1^{\mu_1}(u_2^+) - G_1^{\mu_1}(u_2^-)$ is a difference of two $L_1$-superharmonic functions on $D$, hence it is locally integrable and we have $L_1(u_1 - G_1^{\mu_1} u_2) = L_1 u_1 + \mu_1 u_2 \leq 0$ in the distribution sense on $D$. Then $u_1 - G_1^{\mu_1} u_2$ is Lebesgue-a.e. equal to a $L_1$-superharmonic function $t_1$ on $D$. It follows then that $u_1 + G_1^{\mu_1}(u_2^-) = t_1 + G_1^{\mu_1}(u_2^+)$ Lebesgue-a.e. on $D$. The second function is obviously $L_1$-superharmonic on $D$. On the other hand, the function $u_1 + G_1^{\mu_1}(u_2^-)$ is l.s.c. and we have $L_1(u_1 + G_1^{\mu_1}(u_2^-)) = L_1 u_1 - \mu_1 u_2^- = L_1 u_1 + \mu_1 u_2 - \mu_1 u_2^+ \leq 0$ on $D$, which proves that the function $u_1 + G_1^{\mu_1}(u_2^-)$ is $L_1$-superharmonic on $D$. Then we have $u_1 + G_1^{\mu_1}(u_2^-) = t_1 + G_1^{\mu_1}(u_2^+)$ everywhere, since two $L_1$-superharmonic equal Lebesgue-a.e. are equal everywhere. Hence the

function $u_1 - G_1(u_2)$ defined quasi-everywhere on $D$ has an extension to the $L_1$-superharmonic function $t_1$ on $D$. By a similar argument the same result also holds for the function $u_2 - G_2^{\mu_2}(u_1)$.

$2 \Rightarrow 1$ Suppose that $u_1 - G_1^{\mu_1}(u_2)$ and $u_2 - G_2^{\mu_2}(u_1)$ can be extended respectively to $L_1$-superharmonic and $L_2$-superharmonic functions $s$ and $t$ on $D$, then we have $u_1 = G_1^{\mu_1}(u_2) + s$ and $u_2 = G_2^{\mu_2}(u_1) + t$ on $D$. Hence $L_1 u_1 = -\mu_1 u_2 + L_1 s \leq -\mu_1 u_2$ and $L_2 u_2 = -\mu_2 u_1 + L_2 t \leq -\mu_2 u_1$ in the distribution sense on $D$, that is, the pair $(u_1, u_2)$ is superharmonic on $D$. $\square$

**Corollary 3.12.** *Let $(u_1, u_2)$ be a pair of Borel functions on $D$ such that*

$$G_j^{\mu_j}|u_k| \not\equiv +\infty, \ j \neq k, \ j,k = 1,2.$$

*Then the following statements are equivalent.*

1. *The pair $(u_1, u_2)$ is harmonic on $D$.*
2. *The functions $G_1^{\mu_1}|u_2|$, $G_2^{\mu_2}|u_1|$ are finite and the functions $u_1 - G_1^{\mu_1}u_2$, $u_2 - G_2^{\mu_2}u_1$ are respectively $L_1$-harmonic and $L_2$-harmonic on $D$.*

**Theorem 3.13.** *Let $t_i$, $i = 1,2$, be two $L_i$-harmonic functions on $D$ such that $G_j^{\mu_j} t_i$ is finite and $G_k^{\mu_k} G_j^{\mu_j} t_i$ is bounded on $D$, $j \neq k$, $j,k \in \{1,2\}$. Then the pairs of functions $(V_1 t_1, V_2 G_2^{\mu_2} t_1)$ and $(V_2 G_1^{\mu_1} t_2, V_2 t_2)$ are harmonic on $D$.*

Proof: It is clear that the functions $V_1 t_1$ and $V_2 G_1^{\mu_1} t_2$ are continuous on $D$. On the other hand, we have $V_1 t_1 = \sum_{n=0}^{+\infty} (G_1^{\mu_1} G_2^{\mu_2})^n (t_1)$, hence $L_1 V_1 t_1 = -\mu_1 G_2^{\mu_2} V_1 t_1$ (in the distribution sense) on $D$. We also have $L_2(V_2 G_1^{\mu_2} t_1) = -\mu_2 V_1 t_1$ on $D$. Hence the first pair $(V_1 t_1, V_2 G_1^{\mu_1} t_1)$ is harmonic on $D$. The same holds for the pair $(V_2 G_1^{\mu_1} t_2, V_2 t_2)$. $\square$

## 4. Nonnegative superharmonic pairs of functions

**Proposition 4.1.** *Let $h_i$ be a nonnegative $L_i$-harmonic functions on $D$, $i = 1,2$. If the function $V_1 h_1 + V_1 G_2^{\mu_2} h_1$, resp. $V_2 G_1^{\mu_1} h_1 + V_2 h_2$, is finite and continuous on $D$, then the pair $(V_1 h_1, V_2 G_2^{\mu_2} h_1)$, resp. $(V_1 G_1^{\mu_1} h_2, V_2 h_2)$, is harmonic on $D$.*

Proof: The functions $u = V_1 h_1 + V_1 G_1^{\mu_1} h_2$ and $v = V_2 G_2^{\mu_2} h_1 + V_2 h_2$ are clearly superharmonic on $D$, and we have $L_1 u = -\mu_1 v$ and $L_2 v = -\mu_2 u$ in the distribution sense. Then, by definition, the pair $(u, v)$ is harmonic. Since $h_1$ and $h_2$ are arbitrary, then by taking $h_2 = 0$ (resp. $h_1 = 0$) we obtain the result. $\square$

**Corollary 4.2.** *Let $h_i$ be a nonnegative $L_i$-harmonic functions on $D$, $i = 1,2$. If the function $h_1$, resp. $h_2$, is bounded on $D$, then the pair $(V_1 h_1, V_2 G_2^{\mu_2} h_1)$, resp. $(V_1 G_1^{\mu_1} h_2, V_2 h_2)$, is harmonic on $D$.*

Proof: Suppose for example that $h_1$ is bounded on $D$. Then by the hypothesis on measures $\mu_1$ and $\mu_2$, the functions $V_1 h_1$ and $V_1 G_2^{\mu_2} h_1$ are finite and continuous on $D$. Hence the pair $(V_1 h_1, V_1 G_2^{\mu_2} h_1)$ is harmonic on $D$ by the above proposition. $\square$

**Theorem 4.3.** *Let $(u,v)$ be a nonnegative superharmonic pair on $D$. If the functions $G_1^{\mu_1}v$ and $G_2^{\mu_2}u$ are finite and continuous on $D$, and if $L_1u=-\mu_1v$ and $L_2v=-\mu_2u$ in the distribution sense, then the pair $(u,v)$ is harmonic on $D$.*

Proof: It is clear that the functions $u$ and $v$ are respectively nonnegative $L_1$-superharmonic and $L_2$-superharmonic functions on $D$. Let $h_1$ and $h_2$ be the harmonic parts of $u$ and $v$ respectively. Then, by hypothesis we have $u=h_1+G_1^{\mu_1}v$ and $v=h_2+G_2^{\mu_2}u$ and hence $u$ and $v$ are continuous on $D$, and consequently the pair $(u,v)$ is harmonic on $D$. □

**Theorem 4.4.** *A nonnegative superharmonic pair $(u,v)$ on $D$ is harmonic on $D$ if and only if the following conditions hold:*

1. *$L_1u=-\mu_1v$ and $L_2v=-\mu_2u$ in the distribution sense;*
2. *$G_1^{\mu_1}v$ and $G_2^{\mu_2}u$ are finite and continuous on $D$.*

Proof: Suppose first that the pair $(u,v)$ is harmonic on $D$. Then we have $L_1u=-\mu_1v$, $L_2v=-\mu_2u$ (in the distribution sense) and the functions $u$ and $v$ are respectively $L_1$-superharmonic and $L_2$-superharmonic on $D$. Let $h_1$ and $h_2$ be the harmonic parts of $u$ and $v$ in the Riesz decomposition of $u$ and $v$ respectively. Then we necessarily have $u=h_1+G_1^{\mu_1}v$ and $v=h_2+G_2^{\mu_2}u$. Since the functions $u,v,h_1,h_2$ are continuous, we deduce that $G_1^{\mu_1}v$ and $G_2^{\mu_2}u$ are continuous on $D$. Conversely, if the condition 2 holds, then $u$ and $v$ are continuous on $D$, since they are respectively $L_1$-superharmonic and $L_2$-superharmonic, and since the continuous functions $G_1^{\mu_1}v$ and $G_2^{\mu_2}u$ are their potential parts in the Riesz decomposition of $L_1$-superharmonic, resp. $L_2$-superharmonic, functions. Hence it follows that the pair $(u,v)$ is continuous and hence harmonic on $D$ if the condition 1 holds. □

**Proposition 4.5.** *Let $((u_n,v_n))$ be an increasing sequence of nonnegative harmonic pairs on $D$, and suppose that $\sup_n u_n+\sup_n v_n$ is finite at some point $x_0\in D$. Then the pair $(u,v)=\sup_n(u_n,v_n)$ is harmonic if and only if the functions $G_1^{\mu_1}v$ and $G_2^{\mu_2}u$ are finite and continuous on $D$.*

Proof: By hypothesis, the pair $(u,v)$ is superharmonic on $D$ by Corollary 3.10. For every $n$ let $h_n$, resp. $k_n$, be the $L_1$-harmonic, resp. $L_2$-harmonic, part of $u$, resp. $v$, in the Riesz decomposition of $L_1$-superharmonic functions, resp. $L_2$-superharmonic functions. Then we have $u_n=h_n+G_1^{\mu_1}v_n$ and $v_n=k_n+G_2^{\mu_2}v_n$ on $D$. The sequences $(h_n)$ and $(k_n)$ are increasing (because $h_n$, resp. $k_n$ is the greatest $L_1$-harmonic, resp. $L_2$-harmonic minorant of $u_n$ resp. $v_n$) and the function $h=\sup_n h_n$, resp. $k=\sup k_n$, is $L_1$-harmonic, resp. $L_2$-harmonic, on $D$. By letting $n\to+\infty$, we have $u=h+G_1^{\mu_1}v$ and $v=k+G_2^{\mu_2}u$. From these equalities we deduce that $L_1u=-\mu_1v$ and $L_2v=-\mu_2u$ in the distribution sense. It follows by Theorem 4.8 that the pair $(u,v)$ is harmonic on $D$ if and only if $G_1^{\mu_1}v$ and $G_2^{\mu_2}u$ are finite and continuous on $D$. □

**Proposition 4.6.** *Let $t_i$ be a nonnegative $L_i$-superharmonic functions on $D$, $i=1,2$. If the function $V_1t_1+V_1G_1^{\mu_1}t_2+V_2G_2^{\mu_2}t_1+V_2t_2$ is finite at some point of $D$, then the pairs $(V_1t_1,V_2G_2^{\mu_2}t_1)$ and $(V_1G_1^{\mu_1}t_2,V_2t_2)$ are superharmonic on $D$.*

Proof: By the definition of the kernel $V_1$, the function $V_1t_1$, finite at some point of $D$, is $L_1$-superharmonic, hence it is locally integrable and l.s.c. on $D$. On the other hand, we easily have in the distribution sense, $L_1(V_1t_1) = -L_1t_1 - \mu_1 V_2 G_2^{\mu_2} t_1 \leq -\mu_1 V_2 G_2^{\mu_2} t_1$ and $L_2(V_2 G_2^{\mu_2} t_1) = -\mu_2 t_2 - \mu_2 V_1 t_1 \leq -\mu_2 V_1 t_1$ on $D$, by the definition of the kernels $V_1$ and $V_2$. Hence the pair $(V_1t_1, V_2G_2^{\mu_2}t_1)$ is superharmonic on $D$. In the same way we prove that $(V_1G_1^{\mu_1}t_2, V_2t_2)$ is superharmonic on $D$. $\square$

**Corollary 4.7.** *Let $(u, v)$ be a nonnegative superharmonic pair on $D$. If $u$ and $v$ are bounded, then for $i = 1, 2$ there exists a unique nonnegative $L_i$-superharmonic function $t_i$ on $D$ such that $u = t_1 + G_1^{\mu_1} v$ and $v = t_2 + G_2^{\mu_2} u$.*

**Theorem 4.8.** *Let $(u, v)$ be a nonnegative superharmonic pair on $D$. Then for each $i = 1, 2$, there exists a unique nonnegative $L_i$-superharmonic function $t_i$ on $D$ such that $u = t_1 + G_1^{\mu_1} v = V_1t_1 + V_1G_1^{\mu_1}t_2$ and $v = t_2 + G_2^{\mu_2} u = V_2G_2^{\mu_2}t_1 + V_2t_2$.*

Proof: According to Proposition 3.5, the function $u$ is $L_1$-superharmonic $\geq 0$ on $D$. Hence $\nu = -L_1 u$, in the distribution sense, is a nonnegative Radon measure on $D$. We have $\mu_1 v \leq \nu$, and, consequently, $G_1^{\mu_1} v \leq G_1^{\nu} = u$. Let $n$ be an integer. Since $\mu_1$ is Kato measure, the function $G_1^{\mu_1}(v \wedge n)$ is finite and continuous, so that $u - G_1^{\mu_1}(v \wedge n)$ is l.s.c. locally integrable, and we have $L_1(u - G_1^{\mu_1}(v \wedge n)) = -\nu + \mu_1(v \wedge n) \leq 0$ in the distribution sense, hence the function $t_1^n = u - G_1^{\mu_1}(v \wedge n)$ is $L_1$-superharmonic, and $u = t_1^n + G_1^{\mu_1}(v \wedge n)$. The sequence of functions $(t_1^n)$ is decreasing, then the nonnegative function $t_1 = \widehat{\inf_n t_1^n}$ is $L_1$-superharmonic and verify $u = t_1 + G_1^{\mu_1} v$. In the same way, there exists a nonnegative $L_2$-superharmonic function $t_2$ such that $v = t_2 + G_2^{\mu_2} u$. Hence $u = t_1 + G_1^{\mu_1} t_2 + G_1^{\mu_1} G_2^{\mu_2} u$, and therefore $(I - G_1^{\mu_1} G_2^{\mu_2}) u = t_1 + G_1^{\mu_1} t_2$. By applying the operator $V_1$ to the members of the last equality we obtain $u = V_1t_1 + V_1G_1^{\mu_1}t_2$. Similarly we have $v = G_2^{\mu_2} v + t_2 = V_2 G_2^{\mu_2} t_2 + V_2 t_2$.

Let us now prove the uniqueness of the pair $(t_1, t_2)$. Suppose that $t_1'$ and $t_2'$ are as above. Then we have $u = t_1 + G_1^{\mu_1} v = t_1' + G_1^{\mu_1} v$ and $v = t_2 + G_1^{\mu_2} v = t_2' + G_2^{\mu_2} u$. Hence $t_1 = t_1'$ and $t_2 = t_2'$ q.e, and therefore $t_1 = t_1'$ and $t_2 = t_2'$ by Proposition 2.11. This ends the proof. $\square$

**Lemma 4.9.** *Let $t_i$ be a nonnegative $L_i$-superharmonic functions on $D$, $i = 1, 2$. If the pair $(V_1t_1 + V_1G_1^{\mu_1}t_2, V_2G_2^{\mu_2}t_1 + V_2t_2)$ is harmonic, then $t_i$ is $L_i$-harmonic on $D$, $i = 1, 2$.*

Proof: Indeed we have $L_1(V_1t_1 + V_1G_1^{\mu_1}t_2) = L_1t_1 - \mu_1(V_2G_2^{\mu_2}t_1 + V_2t_2)$ and $L_2(V_2G_2^{\mu_2}t_1 + V_2t_2) = L_2t_2 - \mu_2(V_1t_1 + V_1G_1^{\mu_1}t_2)$ in the distribution sense. Hence if the pair $(V_1t_1 + V_1G_1^{\mu_1}t_2, V_2G_2^{\mu_2}t_1 + V_2t_2)$ is harmonic, then for each $i = 1, 2$, we have $L_it_i = 0$ and thus $t_i$ is $L_i$-harmonic on $D$ by Proposition 2.16. $\square$

**Theorem 4.10.** *Let $(u, v)$ be a nonnegative harmonic pair on $D$. Then for each $i = 1, 2$, there exists a unique nonnegative $L_i$-harmonic function $t_i$ on $D$ such that $(u, v) = (V_1t_1 + V_1G_1^{\mu_1}t_2, V_2G_2^{\mu_2}t_1 + V_2t_2)$.*

Proof: The theorem follows easily from Theorem 4.8 and Lemma 4.9. $\square$

**Proposition 4.11.** *Let $(u_j, v_j)$, $j = 1, 2$, be two nonnegative superharmonic pairs. If the pair $(u_1+v_1, u_2+v_2)$ is harmonic then the pairs $(u_j, v_j)$ are harmonic, $j = 1, 2$.*

Proof: According to Theorem 4.10 we have $(u_j, v_j) = (V_1 t_1^j + V_1 G_1^{\mu_1} t_2^j, V_2 G_2^{\mu_2} t_1^j + V_2 t_2^j)$ for $j = 1, 2$, where $t_i^j$ is $L_i$-harmonic, $i = 1, 2$. If the pair $(u_1 + v_1, u_2 + v_2)$ is harmonic, then, according to Lemma 4.9, for each $i = 1, 2$, the $L_i$-superharmonic functions $t_i^1 + t_i^2$ are $L_i$-harmonic, hence $t_i^j$ is $L_i$-harmonic for all $j$ and $i$. Hence $(u_j, v_j)$ is harmonic by Corollary 3.9. □

We denote by $\mathcal{H}_+(D)$ the set of nonnegative harmonic pairs on $D$.

**Corollary 4.12.** *The set $\mathcal{H}_+(D)$ of nonnegative harmonic pairs on $D$ is a band in the cone $\mathcal{S}_+(D)$.*

**Lemma 4.13.** *Let $\omega$ be a connected relatively compact open subset of $D$. Then there exists a harmonic pair $(h, k)$ on $\omega$ such that $\inf_{x\in\omega} h(x), \inf_{x\in\omega} k(x) > 0$ on $\omega$.*

Proof: Since for each $i = 1, 2$, $D$ is a $L_i$-Green domain, then by [22, Théorème 16.1], there exists a $L_i$-harmonic $h_i > 0$ on $D$. The restrictions of $h_1$ and $h_2$ to $\omega$ are bounded. The measures $\mu_1$ and $\mu_2$ are of Kato type, hence by Corollary 4.2 the pair of the restrictions to $\omega$ of the functions $V_1^\omega h + V_1^\omega G_1^{\omega,\mu_1} k$ and $G_2^{\omega,\mu_2} h + V_2^\omega k$ has the required property of the lemma. □

For a pair $(f, g)$ of functions on an open subset of $\mathbb{R}^d$, the pair

$$(\liminf_{x\to z} f(x), \liminf_{x\to z} g(x)) \in \overline{\mathbb{R}}^2$$

is denoted by $\liminf_{x\to z}(f, g)(x)$.

**Theorem 4.14.** *Let $\omega$ be a relatively compact open subset of $D$ and let $(u, v)$ a superharmonic pair on $\omega$ such that $\liminf_{\omega\ni x\to z}(u, v)(x) \geq 0$ for all $z \in \partial\omega$. Then $(u, v) \geq 0$ on $\omega$.*

Proof: We may assume that $\omega$ is connected. By the above lemma there exists a harmonic pair $(h, k)$ on an open neighborhood of $\overline{\omega}$ such that $\inf_{x\in\omega} h(x) > 0$ and $\inf_{x\in\omega} k(x) > 0$ on $\omega$. Suppose for example that $u$ is not nonnegative, then the function $u/h$ attains its minimum $\alpha < 0$ at some point $x_0$ of $\omega$. Let $\beta = \inf \frac{v}{k}$, then we have $\alpha, \beta \in \mathbb{R}$. If $\beta \geq \alpha$, then the pair of nonnegative functions $(u - \alpha h, v - \alpha k)$ is a superharmonic on $\omega$. Hence $u - \alpha h$ is a nonnegative $L_1$-superharmonic which is zero at $x_0$, and therefore we have $u = \alpha h$, which is a contradiction with the fact that $\liminf_{x\to z} u(x) \geq 0$ for $z \in \partial\omega$. If $\alpha \geq \beta$, then the l.s.c. function $v/k$ attains its minimum at a point $x_1 \in \omega$. The above argument applied to the superharmonic pair $(u - \beta h, v - \beta k)$ leads to a contradiction. Hence $(u, v) \geq 0$. □

## 5. Potentials

**Definition 5.1.** A nonnegative superharmonic pair $(p, q)$ in $\Omega$ is said to be a potential if, for any nonnegative $L_1$-harmonic function $h$ on $D$ and any nonnegative

$L_2$-harmonic function $k$ on $D$ such that $(V_1h + V_1G_1^{\mu_1}k, V_2G_2^{\mu_2}h + V_2k) \le (p, q)$, we have $h = k = 0$ on $D$.

**Proposition 5.2.** *Let $(p, q)$ be a potential on $D$ and $(h, k)$ a nonnegative harmonic pair on $D$ such that $(h, k) \le (p, q)$. Then $h = k = 0$ on $D$.*

The proposition follows easily from Definition 5.1 and Theorem 4.10.

**Proposition 5.3.** *Let $(p, q)$ be a potential on $D$. Then $p$ is a $L_1$-potential and $v$ is a $L_2$-potential on $D$.*

Proof: Let $h$ and $k$ be nonnegative $L_1$-harmonic and nonnegative $L_2$-harmonic respectively on $D$ such that $h \le p$ and $k \le q$. By Theorem 4.8, we have $p = V_1p_1 + V_1G_1^{\mu_1}p_2$ and $q = V_2G_1^{\mu_2}p_1 + V_2p_2$, where $p_i$ is nonnegative $L_i$-superharmonic on $D$ for each $i = 1, 2$. Then we have $h - p_1 \le G_1^{\mu_1}V_2G_2^{\mu_2}p_1 + V_1G_1^{\mu_1}p_2$. Since the function $h - p_1$ is $L_1$-subharmonic on $D$ and the function $G_1^{\mu_1}V_2G_2^{\mu_2}p_1 + V_1G_1^{\mu_1}p_2$ is a $L_1$-potential, we necessarily have $h - p_1 \le 0$, that is, $h \le p_1$. The same argument applied to $k$ and $q$ gives that $k \le p_2$. Hence we have $(V_1h + V_1G_1^{\mu_1 k}, V_2G_2^{\mu_2}k + V_2k) \le (p, q)$ and therefore $h = k = 0$. It follows then that $p$ is a $L_1$-potential and $q$ is a $L_2$-potential. □

**Proposition 5.4.** *Let $(h, k) = (V_1h_1 + V_1G_1^{\mu_1}h_2, V_2G_2^{\mu_2}h_1 + V_2h_2)$ be a nonnegative harmonic pair and $(u, v) = (V_1t_1 + V_1G_1^{\mu_1}t_2, V_2G_2^{\mu_2}t_1 + V_2t_2)$ a nonnegative superharmonic pair on $D$. If $(h, k) \le (u, v)$ on $D$, then $h_1 \le t_1$ and $h_2 \le t_2$ on $D$.*

Proof: The pairs $(u - h, v - k)$, $(u, v)$ are nonnegative and superharmonic on $D$. According to Theorem 4.8, there exist two nonnegative $L_1$-superharmonic functions $t_1,\ t_1'$ on $D$ and two nonnegative $L_2$-superharmonic functions $t_2, t_2'$ on $D$ such that $u = V_1t_1 + V_1G_1^{\mu_1}t_2$, $v = V_2G_2^{\mu_2}t_1 + V_2t_2$, $u - h = V_1t_1' + V_1G_1^{\mu_1}t_2'$ and $v - k = V_2G_2^{\mu_2}t_1' + V_2t_2'$. There exist again by Theorem 4.10 a nonnegative $L_1$-harmonic $h_1$ on $D$ and a nonnegative $L_2$-harmonic function $h_2$ on $D$ such that $h = V_1h_1 + V_1G_1^{\mu_1}h_2$, $k = V_2G_2^{\mu_2}h_1 + V_2h_2$. By the uniqueness in Theorem 4.8 we have necessarily $t_1 = t_1' + h_1$ and $t_2 = t_2' + h_2$, hence $h_1 \le t_1$ and $k_2 \le t_2$. □

**Proposition 5.5.** *For each $i = 1, 2$, let $p_i$ be a $L_i$-potential on $D$. If the pair $(V_1p_1 + V_1G_1^{\mu_2}p_2, V_2G_1^{\mu_2}p_1 + V_2p_2)$ is superharmonic, then it is a potential.*

Proof: Suppose that the pair $(V_1p_1 + V_1G_2^{\mu_2}p_2, V_2G_2^{\mu_2}p_1 + V_2p_2)$ is superharmonic on $D$. Then by Theorem 3.11, for each $i = 1, 2$, there exists a nonnegative $L_i$-harmonic $h_i$ such that $(V_1h_1 + V_1G_1^{\mu_1}h_2, V_2G_2^{\mu_2}h_1 + V_2h_2 \le (V_1p_1 + V_1G_1^{\mu_2}p_2, V_2G_1^{\mu_2}p_1 + V_2p_2)$. According to Proposition 5.3 we have necessarily $h_i \le p_i$, and hence $h_i = 0$, for $i = 1, 2$, and therefore $h = k = 0$ because $p_i$ is a $L_i$-potential. Then $(V_1p_1 + V_1G_1^{\mu_1}p_2, V_2G_2^{\mu_2}p_1 + V_2p_2)$ is a potential. □

**Theorem 5.6** (Riesz decomposition)**.** *Let $(u, v)$ be a nonnegative superharmonic pair on $D$, then for each $i = 1, 2$, there exist a unique nonnegative $L_i$-harmonic function $h_i$ on $D$ and a unique potential $(p, q)$ in $D$ such that $(u, v) = (V_1h_1 + V_1G_1^{\mu_1}k_1, V_2G_2^{\mu_2}h_1 + V_2h_2) + (p, q)$.*

Proof: Let $(u,v)$ be a nonnegative superharmonic pair in $D$. By Theorem 4.8, for each $i=1,2$, there exists a nonnegative $L_i$-superharmonic $t_i$ on $D$ such that $u = V_1t_1 + V_1G_1^{\mu_1}t_2$ and $v = V_2G_2^{\mu_2}t_1 + V_2t_2$. By the Riesz decomposition of nonnegative $L_i$-superharmonic functions, there exists a nonnegative $L_i$-harmonic function $h_i$ in $\Omega$ such that $t_i = h_i + p_i$. Thus we have

$$(u,v) = (V_1h_1 + V_1G_1^{\mu_1}h_2, V_2G_2^{\mu_1}h_1 + V_2h_2) + (V_1p_1 + V_1G_1^{\mu_1}p_2, V_2G_2^{\mu_2}p_1 + V_2p_2).$$

The theorem follows now from Proposition 5.5. □

**Corollary 5.7.** *Let $(p,q)$ be a potential pair on $D$. Then for $i=1,2$, there exists a $L_i$-potential $p_i$ on $D$ such that $p = V_1p_1 + V_1G_1^{\mu_1}p_2$ and $q = V_2G_2^{\mu_2}p_1 + V_2q_2$.*

**Corollary 5.8.** *The set $\mathcal{P}(D)$ of potential pairs on $D$ is a convex cone and a band of $\mathcal{S}(D)$.*

**Theorem 5.9.** *Let $(p,q)$ be a potential pair on $D$. Then there exist two unique nonnegative measures $\mu$ and $\nu$ on $D$ such that*

$$p = \int_D V_1G_1(.,y)d\mu(y) + \int_D V_1G_1^{\mu_1}G_2(.,y)\,d\nu(y)$$

*and*

$$q = \int_D V_2G_2^{\mu_2}G_1(.,y)d\mu(y) + \int_D V_2G_2(.,y)\,d\nu(y).$$

Proof: According to Corollary 5.7, for each $i=1,2$, there exists a unique $L_i$-potential $p_i$ such that $p = V_1p_1 + V_1G_1^{\mu_1}p_2$ and $q = V_2G_2^{\mu_2}p_1 + V_2p_2$. By the Riesz representation Theorem, there exist two unique measures $\mu$ and $\nu$ on $D$ such that $p_1 = \int G_1(x,y)\,d\mu(y)$ and $p_2 = \int G_2(.,y)\,d\nu(y)$. Hence we have by Fubini Theorem,

$$p = \int_\Delta V_1G_1(.,y)\,d\mu(y) + \int_\Delta V_1G_1^{\mu_1}G_2(.,y)\,d\nu(y)$$

and

$$q = \int_\Delta V_2G_2^{\mu_2}G_1(.,y)\,d\mu(y) + \int_\Delta V_1G_2(.,y)\,d\nu(y).$$ □

## 6. Integral representation of nonnegative solutions of (S)

For $i=1,2$, we denote by $\Delta_i$ the Martin boundary of $D$ relative to the operator $L_i$, and by $g_i$ the corresponding Martin kernel on $D\times\Delta_i$. Then, for any $Y\in\Delta_i$, $g_i(.,Y)$ is a nonnegative $L_i$-harmonic function on $D$. We also denote by $\Delta_i^m$ the associated minimal Martin boundary of $D$, that is, the set of minimal points of $\Delta_i$.

**Proposition 6.1.** *Let $\mu$ and $\nu$ be two measures $\geq 0$ on $\Delta_1$ and $\Delta_2$ respectively. If the functions*

$$u = \int_{\Delta_1} V_1g_1(.,Y)\,d\mu(Y) + \int_{\Delta_2} V_1G_1^{\mu_1}g_2(.,Y)\,d\nu(Y)$$

*and*

$$v = \int_{\Delta_1} V_2 G_2^{\mu_2} g_1(.,Y)\,d\mu(Y) + \int_{\Delta_2} V_2 g_2(.,Y)\,d\nu(Y)$$

*are finite and continuous on $D$, then the pair $(u,v)$ is harmonic in $D$.*

Proof: Suppose that $u$ and $v$ are finite and continuous on $D$. It follows then that the functions $h_1 = \int_{\Delta_1} g_1(.,Y)\,d\mu(Y)$ and $h_2 = \int_{\Delta_2} g_2(.,Y)\,d\nu(Y)$ are respectively $L_1$-harmonic and $L_2$-harmonic on $D$ and we have by Fubini's Theorem $u = V_1 h_1 + V_1 G_1^{\mu_1} h_2$ and $v = V_2 G_2^{\mu_2} h_1 + V_2 h_2$. Then the pair $(u,v)$ is harmonic by Proposition 4.1. □

**Lemma 6.2.** *Let $\mu$ and $\nu$ be two finite measures on $\Delta_1$ and $\Delta_2$ respectively, and let $u$ and $v$ the functions defined on $D$ by*

$$u = \int_{\Delta_1} V_1 g_1(.,Y)\,d\mu(Y) + \int_{\Delta_2} V_1 G_1^{\mu_1} g_2(.,Y)\,d\nu(Y)$$

*and*

$$v = \int_{\Delta_1} V_2 G_2^{\mu_2} g_1(.,Y)\,d\mu(Y) + \int_{\Delta_2} V_2 g_2(.,Y)\,d\nu(Y).$$

*If the pair $(u,v)$ is harmonic on $D$, then we have*

$$\int_{\Delta} g_1(.,Y)\,d\mu(Y) + \int_{\Delta_2} G_1^{\mu_1} g_2(.,Y)\,d\nu(Y) < +\infty$$

*and*

$$\int_{\Delta_1} G_2^{\mu_2} g_1(.,Y)\,d\mu(Y) + \int_{\Delta_2} g_2(.,Y)\,d\nu(Y) < +\infty.$$

Proof: Indeed we have

$$\int_{\Delta_1} g_1(.,Y)\,d\mu(Y) + \int_{\Delta_2} G_1^{\mu_1} g_2(.,Y)\,d\nu(Y) \le u < +\infty$$

and

$$\int_{\Delta_1} G^{\mu_2} g_1(.,Y) + \int_{\Delta_2} g_2(.,Y)\,d\nu(Y) \le v < +\infty.$$

□

**Theorem 6.3.** *Let $(u,v)$ be a nonnegative harmonic pair on $D$. Then there exist two unique nonnegative finite measures $\mu$ and $\nu$ on $\Delta_1$ and $\Delta_2$ respectively and supported by $\Delta_1^m$ and $\Delta_2^m$ respectively such that*

$$u = \int_{\Delta_1} V_1 g_1(.,Y)\,d\mu(Y) + \int_{\Delta_2} V_1 G_1^{\mu_1} g_2(.,Y)\,d\nu(Y)$$

*and*

$$v = \int_{\Delta_1} V_2 G_2^{\mu_2} g_1(.,Y)\,d\mu(Y) + \int_{\Delta_2} V_2 g_2(.,Y)\,d\nu(Y).$$

Proof: According to Theorem 4.10, for each $i = 1, 2$, there exists a unique nonnegative $L_i$-harmonic function $h_i$ such that $u = V_1 h_1 + V_1 G_1^{\mu_1} h_2$ and $v = V_2 G_2^{\mu_2} h_1 + V_2 h_2$. By the Theorem 2.20, there exist two unique (Radon) measures $\mu$ and $\nu$ on $\Delta_1$ and $\Delta_2$, carried by $\Delta_1^m$ and $\Delta_2^m$ respectively such that $h_1 = \int g_1(x, Y)\, d\mu(Y)$ and $h_2 = \int g_2(., Y)\, d\nu(Y)$. Hence we have by Fubini's Theorem,

$$u = \int_{\Delta_1} V_1 g_1(., Y)\, d\mu(Y) + \int_{\Delta_2} V_1 G_1^{\mu_1} g_2(., Y)\, d\nu(Y)$$

and

$$v = \int_{\Delta_1} V_2 G_2^{\mu_2} g_1(., Y)\, d\mu(Y) + \int_{\Delta_2} V_2 g_2(., Y)\, d\nu(Y).$$

Let $\sigma$ and $\tau$ be two nonnegative (Radon) measures on $\Delta_1$ and $\Delta_2$ and carried by $\Delta_1^m$ and $\Delta_2^m$ respectively, and such that

$$u = \int_{\Delta_1} V_1 g_1(., Y)\, d\sigma(Y) + \int_{\Delta_2} V_1 G_1^{\mu_1} g_2(., Y)\, d\tau(Y)$$

and

$$v = \int_{\Delta_1} V_2 G_2^{\mu_2} g_1(., Y)\, d\sigma(Y) + \int_{\Delta_2} V_2 g_2(., Y)\, d\tau(Y).$$

Then, by Fubini's Theorem,

$$\begin{aligned} (I - G_1^{\mu_1} G_2^{\mu_2}) u &= \int_{\Delta_1} g_1(., Y)\, d\mu(Y) + \int_{\Delta_2} G_2^{\mu_2} g_2(., Y)\, d\nu(Y) \\ &= \int_{\Delta_1} g_1(., Y)\, d\sigma(Y) + \int_{\Delta_2} G_2^{\mu_2} g_2(., Y)\, d\tau(Y). \end{aligned}$$

The functions $\int_{\Delta_1} g_1(., Y)\, d\mu(Y)$ and $\int_{\Delta_1} g_1(., Y)\, d\sigma(Y)$ are $L_1$-harmonic and $L_2$-harmonic respectively, and the functions

$$\int_{\Delta_1} G_2^{\mu_2} g_1(., Y)\, d\nu(Y) \quad \text{and} \quad \int_{\Delta_2} G_2^{\mu_2} g_2(., Y)\, d\tau(Y)$$

are a $L_1$-potential and $L_2$ potential respectively, then by the uniqueness of the Riesz decomposition we have $\int_{\Delta} g_1(., Y)\, d\mu(Y) = \int_{\Delta} g_1(., Y)\, d\sigma(y)$, hence $\mu = \sigma$ by the uniqueness of the integral representation of nonnegative $L_1$-harmonic functions by mean of measure on $\Delta_1$ carried by $\Delta_1^m$. In the same way we have $\nu = \tau$. □

**Corollary 6.4.** *Let $(u, v)$ be a nonnegative harmonic pair on $D$ admitting the integral representation*

$$u = \int_{\Delta_1} V_1 g_1(., Y)\, d\mu(Y) + \int_{\Delta_2} V_1 G_1^{\mu_1} g_2(., Y)\, d\nu(Y)$$

*and*

$$v = \int_{\Delta_1} V_2 G_2^{\mu_2} g_1(.,Y)\, d\mu(Y) + \int_{\Delta_2} V_2 g_2(.,Y)\, d\nu(Y)$$

*on $D$, where $\mu$ and $\nu$ are two finite nonnegative measures on $\Delta_1$ and $\Delta_2$ respectively. Then the pair $(h,k) = ((I - G_1^{\mu_1} G_2^{\mu_2})u, \int_{\Delta_2} g_2(.,Y)\, d\nu(Y))$ is a solution of the system $L_1 h = -\mu_1 k$, $L_2 k = 0$.*

Proof: Indeed, we have $(I - G_1^{\mu_1} G_2^{\mu_2})(V_1 g_1(.,Y)) = g_1(.,Y)$ for each $Y \in \Delta_1$ and $(I - G_1^{\mu_1} G_2^{\mu_2}) V_1 G_1^{\mu_1} g_2(.,Y)) = G_1^{\mu_1} g_2(.,Y)$ for each $Y \in \Delta_2$. Hence, by Fubini's Theorem, $(I - G_1^{\mu_1} G_2^{\mu_2})u = \int_{\Delta_1} g_1(.,Y)\, d\mu(Y) + G_1^{\mu_1} \int_{\Delta_2} g_2(.,Y)\, d\nu(Y)$. By applying the operator $L_1$ to the members of this equality, we obtain $L_1(I - G_1^{\mu_1} G_2^{\mu_2})u = -\mu_1 \int_{\Delta_2} g_2(.,Y)\, d\nu(Y)$ because the function $\int_{\Delta_1} g_1(.,Y)\, d\mu(Y)$ is $L_1$-harmonic. Since the function $\int_{\Delta_1} g_1(.,Y)\, d\nu(Y)$ is $L_2$-harmonic, we deduce that the pair $(h,k) = ((I - G_1^{\mu_1} G_2^{\mu_2})u, \int_{\Delta_2} g_2(.,y)\, d\nu(Y))$ is a solution of the system $L_1 h = -\mu k$, $L_2 k = 0$. □

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Aslimani A., El Ghazi I., El Kadiri M.:
Université Mohammed V, Département de Mathématiques, Faculté des Sciences, B.P. 1014, Rabat, Morocco

*E-mail:* slimonier.math@gmail.com
el.imad.imad@gmail.com
elkadiri@fsr.ac.ma

Haddad S.:
CRMEF, Rabat-Akkari, Morocco

*E-mail:* sabahhaddad24@gmail.com

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