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TORSION UNITS FOR SOME ALMOST SIMPLE GROUPS

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Abstract. We investigate the Zassenhaus conjecture regarding rational conjugacy of torsion units in integral group rings for certain automorphism groups of simple groups. Recently, many new restrictions on partial augmentations for torsion units of integral group rings have improved the effectiveness of the Luther-Passi method for verifying the Zassenhaus conjecture for certain groups. We prove that the Zassenhaus conjecture is true for the automorphism group of the simple group PSL$(2, 11)$. Additionally we prove that the Prime graph question is true for the automorphism group of the simple group PSL$(2, 13)$.

Keywords: Zassenhaus conjecture; torsion unit; partial augmentation; integral group ring

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1. Introduction and main results

Let $U(\mathbb{Z}G)$ be the unit group of the integral group ring of a finite group G. It is well known that

$$U(\mathbb{Z}G) = \{\pm 1\} \times V(\mathbb{Z}G),$$

where $V(\mathbb{Z}G)$ is the group of units of augmentation one. Throughout this article, G is always a finite group and torsion units will always represent torsion units in $V(\mathbb{Z}G) \setminus \{1\}$. A very important conjecture in the theory of integral group rings is:

Conjecture 1.1. *If G is a finite group, then for each torsion unit $u \in V(\mathbb{Z}G)$ there exists $g \in G$, such that $|u| = |g|$ where $|u|$ and $|g|$ are the orders of u and g, respectively.*

A stronger version of this conjecture was formulated by Hans Zassenhaus in [37], which states

Conjecture 1.2. *A torsion unit in $V(\mathbb{Z}G)$ is rationally conjugate to a group element if it is conjugate to an element of G by a unit of the rational group algebra $\mathbb{Q}G$.*

This conjecture was confirmed for some classes of solvable groups in [24], nilpotent groups in [36], [31] and cyclic-by-abelian groups in [18]. The Luthar-Passi method (which was introduced in [29]) is the main investigative tool for simple groups G in relation to the Zassenhaus conjecture for $\mathbb{Z}G$. It was confirmed true for all groups up to order 71, A_5, S_5, central extensions of S_5 and other simple finite groups in [26], [29], [30], [4], [5]. Partial results were given for A_6 in [34] and the remaining cases were dealt with in [21]. Higher order alternating groups were also considered in [33], [32]. It was also proved for $\mathrm{PSL}(2,p)$ when $p = \{7, 11, 13\}$ in [22], $\mathrm{PSL}(2,p)$ when $p = \{8, 17\}$ in [20] and $\mathrm{PSL}(2,p)$ when $p = \{19, 23\}$ in [2]. Further results regarding $\mathrm{PSL}(2,p)$ can be found in [25].

Let H be a group with a torsion part $t(H)$ (i.e. the set of elements of H of finite order) of finite exponent and let $\#H$ be the set of primes dividing the order of elements from the set $t(H)$. The prime graph of H (denoted by $\pi(H)$) is a graph with vertices labeled by primes from $\#H$, such that vertices p and q are adjacent if and only if there is an element of order pq in the group H. The following was composed as a problem in [27], Problem 37:

Question 1.1 (Prime graph question)**.** If G is a finite group, then $\pi(G) = \pi(V(\mathbb{Z}G))$.

This question was upheld for Frobenius and Solvable groups in [28] and was also confirmed for some Sporadic Simple groups in [17], [11], [6], [14], [15], [8], [9], [3], [10], [7], [13], [12]. We use the Luthar-Passi method to obtain our results. Our results are the following:

Theorem 1.1. *The Zassenhaus conjecture is true for the integral group ring of the automorphism group of the group* $\mathrm{PSL}(2, 11)$.

Theorem 1.2. *Let G be the automorphism group of* $\mathrm{PSL}(2, 13)$ *and let u be a torsion unit of $V(\mathbb{Z}G)$. The following conditions hold:*

(i) *If $|u| \in \{2, 3, 13\}$, then u is rationally conjugate to some $g \in G$.*
(ii) *There are no elements of order 21, 26, 39 and 91 in $V(\mathbb{Z}G)$.*
(iii) *If $|u| = 4$, then $\nu_{rx} = 0$ for all $rx \notin \{\nu_{2a}, \nu_{2b}, \nu_{4a}\}$ and*

$$(\nu_{2a}, \nu_{2b}, \nu_{4a}) \in \{(2, 0, -1), (0, 0, 1), (-2, 0, 3), (3, 1, -3), (1, 1, -1), (-1, 1, 1), (-3, 1, 3)\}.$$

(iv) *If $|u| = 6$, then $\nu_{rx} = 0$ for all $rx \notin \{\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{6a}\}$ and*

$$(\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{6a}) \in \{(0, -2, 0, 3), (0, -2, 3, 0), (0, 0, 0, 1), (0, 0, 3, -2), (0, 2, 0, -1), (1, -1, 0, 1), (1, 1, 0, -1)\}.$$

(v) *If* $|u| = 7$, *then* $\nu_{rx} = 0$ *for all* $rx \notin \{\nu_{7a}, \nu_{7b}, \nu_{7c}\}$ *and*

$$\begin{aligned}(\nu_{7a}, \nu_{7b}, \nu_{7c}) \in \{&(2,-3,2),(2,-2,1),(1,-2,2),(2,-1,0),(1,-1,1),(0,-1,2),\\ &(2,0,-1),(1,0,0),(0,0,1),(-1,0,2),(2,1,-2),(1,1,-1),\\ &(0,1,0),(-1,1,1),(-2,1,2),(2,2,-3),(1,2,-2),(0,2,-1),\\ &(-1,2,0),(-2,2,1),(-3,2,2)\}.\end{aligned}$$

Consequently, we obtain the following result:

Corollary 1.1. *The Prime graph question is true for the integral group ring of the automorphism group of the group* PSL(2, 13).

Let $u = \sum a_g g$ be a torsion unit of $V(\mathbb{Z}G)$. Then the sum $\sum_{g \in X^G} a_g$ satisfies $\sum_{g \in X^G} a_g \in \mathbb{Z}$ which is the partial augmentation (denoted by $\varepsilon_C(u)$) of u with respect to its conjugacy classes X^G in G. Let $\nu_i = \varepsilon_{C_i}(u)$ be the i-th partial augmentation of u. It was proved that $\nu_1 = 0$ and $\nu_j = 0$ if the conjugacy class C_j consists of a central element by Higman and Berman [1]. Therefore $\nu_2 + \nu_3 + \ldots + \nu_l = 1$ where l denotes the number of non-central conjugacy classes of G.

Proposition 1.1 ([19])**.** *Let u be a torsion unit of $V(\mathbb{Z}G)$. The order of u divides the exponent of G.*

The following propositions provide relationships between the partial augmentations and the order of a torsion unit.

Proposition 1.2 ([23], Proposition 3.1)**.** *Let u be a torsion unit of $V(\mathbb{Z}G)$. Let C be a conjugacy class of G. If p is a prime dividing the order of a representative of C but not the order of u then the partial augmentation satifies $\varepsilon_C(u) = 0$.*

Proposition 1.3 ([22], Proposition 2.2)**.** *Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$.*

(i) *If u has order p^n, then $\varepsilon_x(u) = 0$ for every x of G whose p-part is of order strictly greater than p^n.*
(ii) *If x is an element of G whose p-part for some prime, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.*

Proposition 1.4 ([29]). *Let u be a torsion unit of $V(\mathbb{Z}G)$ of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C_{i_d} with partial augmentation $\varepsilon_{C_{i_d}}(u^d) \neq 0$.*

For any character χ of G and any torsion unit u of $V(\mathbb{Z}G)$, clearly $\chi(u) = \sum_{i=2}^{l} \nu_i \chi(h_i)$ where h_i is a representative of a non-central conjugacy class C_i.

Proposition 1.5 ([29] and [22], Theorem 1). *Let p be equal to zero or a prime divisor of $|G|$. Suppose that u is an element of $V(\mathbb{Z}G)$ of order k. Let z be a primitive k-th root of unity. Then for every integer l and any character χ of G, the number*

$$\mu_l(u,\chi,p) = \frac{1}{k}\sum_{d|k} \mathrm{Tr}_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi(u^d)z^{-dl}\}$$

is a nonnegative integer.

We will use the notation $\mu_l(u,\chi,*)$ when $p = 0$. The LAGUNA package [16] for the GAP system [35] is a very useful tool when calculating $\mu_l(u,\chi,p)$.

2. Proof of Theorem 1

Let $G = \mathrm{Aut}(\mathrm{PSL}(2,11))$. Clearly $|G| = 1320 = 2^3 \cdot 3 \cdot 5 \cdot 11$ and $\exp(G) = 660 = 2^2 \cdot 3 \cdot 5 \cdot 11$. Initially for any torsion unit of $V(\mathbb{Z}G)$ of order k we have that

$$\nu_{2a} + \nu_{3a} + \nu_{5a} + \nu_{5b} + \nu_{6a} + \nu_{11a} + \nu_{2b} + \nu_{4a} + \nu_{10a} + \nu_{10b} + \nu_{12a} + \nu_{12b} = 1.$$

By Proposition 1.1, we need only to consider torsion units of $V(\mathbb{Z}G)$ of order 2, 3, 4, 5, 6, 10, 11, 12, 15, 20, 22, 33 and 55. We will now consider each case separately.

Case 1: Let $u \in V(\mathbb{Z}G)$ where $|u| = 2$. Using Propositions 1.2 and 1.3, $\nu_{2a} + \nu_{2b} = 1$. Applying Proposition 1.5, we obtain:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{2}(\gamma + 1) \geqslant 0; \quad \mu_1(u,\chi_2,*) = \tfrac{1}{2}(-\gamma + 1) \geqslant 0$$

where $\gamma = \nu_{2a} - \nu_{2b}$. Clearly, $\gamma \in \{1,-1\}$. It follows that the only possible integer solutions for (ν_{2a}, ν_{2b}) are $(0,1)$ and $(1,0)$. Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 1.4.

Case 2: Let $u \in V(\mathbb{Z}G)$ where $|u| = 3$. By Proposition 1.2, $\nu_{kx} = 0$ for all

$$kx \in \{2a, 5a, 5b, 6a, 11a, 2b, 4a, 10a, 10b, 12a, 12b\}.$$

Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 1.4.

Case 3: Let $u \in V(\mathbb{Z}G)$ where $|u| = 4$. Using Propositions 1.2 and 1.3, $\nu_{2a} + \nu_{2b} + \nu_{4a} = 1$. Clearly, $\chi(u^2) = \chi(2a)$. Applying Proposition 1.5, we obtain:

$$\mu_0(u, \chi_2, *) = \tfrac{1}{4}(\gamma_1 + 2) \geqslant 0; \quad \mu_2(u, \chi_2, *) = \tfrac{1}{4}(-\gamma_1 + 2) \geqslant 0;$$
$$\mu_0(u, \chi_4, *) = \tfrac{1}{4}(-\gamma_2 + 8) \geqslant 0; \quad \mu_2(u, \chi_4, *) = \tfrac{1}{4}(\gamma_2 + 8) \geqslant 0;$$
$$\mu_0(u, \chi_3, 11) = \tfrac{1}{4}(\gamma_3 + 2) \geqslant 0; \quad \mu_0(u, \chi_4, 11) = \tfrac{1}{4}(\gamma_4 + 2) \geqslant 0$$

where $\gamma_1 = 2\nu_{2a} - 2\nu_{2b} - 2\nu_{4a}$, $\gamma_2 = 4\nu_{2a} - 4\nu_{4a}$, $\gamma_3 = -2\nu_{2a} - 2\nu_{2b} + 2\nu_{4a}$ and $\gamma_4 = -2\nu_{2a} + 2\nu_{2b} - 2\nu_{4a}$. It follows that the only possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{4a})$ are $(0, 1, 0)$, $(1, 0, 0)$ and $(0, 0, 1)$. Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 1.4.

Case 4: Let $u \in V(\mathbb{Z}G)$ where $|u| = 5$. Using Propositions 1.2 and 1.3, $\nu_{5a} + \nu_{5b} = 1$. Applying Proposition 1.5, we obtain

$$\mu_1(u, \chi_7, 11) = \tfrac{1}{5}(\gamma_1 + 7) \geqslant 0; \quad \mu_1(u, \chi_3, 11) = \tfrac{1}{5}(-\gamma_1 + 3) \geqslant 0;$$
$$\mu_2(u, \chi_3, 11) = \tfrac{1}{5}(\gamma_2 + 3) \geqslant 0$$

where $\gamma_1 = 3\nu_{5a} - 2\nu_{5b}$ and $\gamma_2 = 2\nu_{5a} - 3\nu_{5b}$. It follows that the only possible integer solutions for (ν_{5a}, ν_{5b}) are $(0, 1)$ and $(1, 0)$. Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 1.4.

Case 5: Let $u \in V(\mathbb{Z}G)$ where $|u| = 6$. Using Proposition 1.2 and 1.3,

$$\nu_{2a} + \nu_{2b} + \nu_{3a} + \nu_{6a} = 1.$$

Let $\gamma_1 = \nu_{2a} - \nu_{3a} + \nu_{6a}$, $\gamma_2 = 2\nu_{2a} - \nu_{3a} - \nu_{6a}$, $\gamma_3 = \nu_{2a} - 2\nu_{6a} + \nu_{2b}$ and $\gamma_4 = 2\nu_{2a} - 4\nu_{6a} - 2\nu_{2b}$. We will now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2, 3\}$:

▷ $\chi(u^3) = \chi(2a)$ and $\chi(u^2) = \chi(3a)$. Applying Proposition 1.5, we obtain:

$$\mu_0(u, \chi_3, *) = \tfrac{1}{6}(4\gamma_1 + 8) \geqslant 0; \quad \mu_3(u, \chi_3, *) = \tfrac{1}{6}(-4\gamma_1 + 4) \geqslant 0;$$
$$\mu_0(u, \chi_4, *) = \tfrac{1}{6}(-2\gamma_2 + 10) \geqslant 0; \quad \mu_2(u, \chi_4, *) = \tfrac{1}{6}(\gamma_2 + 7) \geqslant 0;$$
$$\mu_0(u, \chi_3, 11) = \tfrac{1}{6}(-2\gamma_3 + 2) \geqslant 0; \quad \mu_2(u, \chi_3, 11) = \tfrac{1}{6}(\gamma_3 + 2) \geqslant 0.$$

Clearly $\gamma_1 \in \{-2, 1\}$ and $\gamma_2 \in \{-7, -1, 5\}$. It follows that the only possible integer solution for $(\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{6a})$ is $(0, 0, 1, 0)$.

▷ $\chi(u^3) = \chi(2b)$ and $\chi(u^2) = \chi(3a)$. Applying Proposition 1.5, we obtain

$$\mu_0(u, \chi_3, *) = \tfrac{1}{6}(4\gamma_1 + 6) \geqslant 0; \quad \mu_3(u, \chi_3, *) = \tfrac{1}{6}(-4\gamma_1 + 6) \geqslant 0;$$
$$\mu_0(u, \chi_4, *) = \tfrac{1}{6}(-\gamma_2 + 12) \geqslant 0; \quad \mu_1(u, \chi_4, *) = \tfrac{1}{6}(-\gamma_2 + 9) \geqslant 0;$$
$$\mu_3(u, \chi_4, *) = \tfrac{1}{6}(2\gamma_2 + 12) \geqslant 0; \quad \mu_0(u, \chi_3, 11) = \tfrac{1}{6}(-2\gamma_3 + 2) \geqslant 0;$$
$$\mu_3(u, \chi_4, 11) = \tfrac{1}{6}(\gamma_4 + 2) \geqslant 0.$$

Clearly $\gamma_1 \in \{0\}$ and $\gamma_2 \in \{-6,-3,0,3,6\}$. It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{6a})$.

Therefore u is rationally conjugated to some element $g \in G$ by Proposition 1.4.

Case 6: Let $u \in V(\mathbb{Z}G)$ where $|u| = 10$. Using Propositions 1.2 and 1.3,

$$\nu_{2a} + \nu_{5a} + \nu_{5b} + \nu_{2b} + \nu_{10a} + \nu_{10b} = 1.$$

Let $\gamma_1 = \nu_{5a} + \nu_{5b} - 4\nu_{2b} + \nu_{10a} + \nu_{10b}$, $\gamma_2 = 2\nu_{5a} - 3\nu_{5b} + 2\nu_{2b} + 2\nu_{10a} - 3\nu_{10b}$, $\gamma_3 = 3\nu_{5a} - 2\nu_{5b} - 2\nu_{2b} + 3\nu_{10a} - 2\nu_{10b}$, $\gamma_4 = -4\nu_{2a} + 2\nu_{5a} + 2\nu_{5b} - 4\nu_{2b} + 6\nu_{10a} + 6\nu_{10b}$ and $\gamma_5 = -4\nu_{2a} + 2\nu_{5a} + 2\nu_{5b} + 4\nu_{2b} - 6\nu_{10a} - 6\nu_{10b}$. We will now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2,5\}$:

▷ $\chi(u^5) = \chi(2a)$ and $\chi(u^2) = \chi(5a)$. Applying Proposition 1.5, we obtain

$$\begin{gathered}
\mu_0(u, \chi_3, *) = \tfrac{1}{10}(8\nu_{2a} + 12) \geqslant 0; \quad \mu_5(u, \chi_3, *) = \tfrac{1}{10}(-8\nu_{2a} + 8) \geqslant 0; \\
\mu_5(u, \chi_{10}, *) = \tfrac{1}{10}(2\gamma_1 + 10) \geqslant 0; \quad \mu_0(u, \chi_{10}, *) = \tfrac{1}{10}(-2\gamma_1 + 10) \geqslant 0; \\
\mu_1(u, \chi_{10}, *) = \tfrac{1}{10}(\gamma_2 + 15) \geqslant 0; \quad \mu_0(u, \chi_3, 11) = \tfrac{1}{10}(\gamma_4 + 4) \geqslant 0; \\
\mu_0(u, \chi_4, 11) = \tfrac{1}{10}(\gamma_5 + 4) \geqslant 0.
\end{gathered}$$

Clearly, $\nu_{2a} \in \{1\}$ and $\gamma_2 \in \{-5,0,5\}$. It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{5a}, \nu_{5b}, \nu_{2b}, \nu_{10a}, \nu_{10b})$.

▷ $\chi(u^5) = \chi(2a)$ and $\chi(u^2) = \chi(5b)$. Applying Proposition 1.5, we obtain

$$\begin{gathered}
\mu_0(u, \chi_3, *) = \tfrac{1}{10}(8\nu_{2a} + 12) \geqslant 0; \quad \mu_5(u, \chi_3, *) = \tfrac{1}{10}(-8\nu_{2a} + 8) \geqslant 0; \\
\mu_5(u, \chi_{10}, *) = \tfrac{1}{10}(2\gamma_1 + 10) \geqslant 0; \quad \mu_0(u, \chi_{10}, *) = \tfrac{1}{10}(-2\gamma_1 + 10) \geqslant 0; \\
\mu_2(u, \chi_{10}, *) = \tfrac{1}{10}(\gamma_3 + 15) \geqslant 0; \quad \mu_0(u, \chi_3, 11) = \tfrac{1}{10}(\gamma_4 + 4) \geqslant 0; \\
\mu_0(u, \chi_4, 11) = \tfrac{1}{10}(\gamma_5 + 4) \geqslant 0.
\end{gathered}$$

Clearly, $\nu_{2a} \in \{1\}$ and $\gamma_2 \in \{-5,0,5\}$. It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{5a}, \nu_{5b}, \nu_{2b}, \nu_{10a}, \nu_{10b})$.

▷ $\chi(u^5) = \chi(2b)$ and $\chi(u^2) = \chi(5a)$. Applying Proposition 1.5, we obtain

$$\begin{gathered}
\mu_0(u, \chi_3, *) = \tfrac{1}{10}(8\nu_{2a} + 10) \geqslant 0; \quad \mu_5(u, \chi_3, *) = \tfrac{1}{10}(-8\nu_{2a} + 10) \geqslant 0; \\
\mu_5(u, \chi_{10}, *) = \tfrac{1}{10}(2\gamma_1 + 8) \geqslant 0; \quad \mu_0(u, \chi_{10}, *) = \tfrac{1}{10}(-2\gamma_1 + 12) \geqslant 0; \\
\mu_1(u, \chi_{10}, *) = \tfrac{1}{10}(\gamma_2 + 13) \geqslant 0.
\end{gathered}$$

Clearly, $\nu_{2a} \in \{0\}$ and $\gamma_2 \in \{-4,1,6\}$. It follows that the only possible integer solution for $(\nu_{2a}, \nu_{5a}, \nu_{5b}, \nu_{2b}, \nu_{10a}, \nu_{10b})$ is $(0,0,0,0,0,1)$.

▷ $\chi(u^5) = \chi(2b)$ and $\chi(u^2) = \chi(5b)$. Applying Proposition 1.5, we obtain

$$\mu_0(u, \chi_3, *) = \tfrac{1}{10}(8\nu_{2a} + 10) \geqslant 0; \quad \mu_5(u, \chi_3, *) = \tfrac{1}{10}(-8\nu_{2a} + 10) \geqslant 0;$$
$$\mu_5(u, \chi_{10}, *) = \tfrac{1}{10}(2\gamma_1 + 8) \geqslant 0; \quad \mu_0(u, \chi_{10}, *) = \tfrac{1}{10}(-2\gamma_1 + 12) \geqslant 0;$$
$$\mu_2(u, \chi_{10}, *) = \tfrac{1}{10}(\gamma_3 + 17) \geqslant 0.$$

Clearly, $\nu_{2a} \in \{0\}$ and $\gamma_2 \in \{-4, 1, 6\}$. It follows that the only possible integer solution for $(\nu_{2a}, \nu_{5a}, \nu_{5b}, \nu_{2b}, \nu_{10a}, \nu_{10b})$ is $(0, 0, 0, 0, 1, 0)$.

Therefore (in all cases), u is rationally conjugated to some element $g \in G$ by Proposition 1.4.

Case 7: Let $u \in V(\mathbb{Z}G)$ where $|u| = 11$. By Proposition 1.2, $\nu_{kx} = 0$ for all $kx \in \{2a, 3a, 5a, 5b, 6a, 2b, 4a, 10a, 10b, 12a, 12b\}$. Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 1.4.

Case 8: Let $u \in V(\mathbb{Z}G)$ where $|u| = 12$. Using Propositions 1.2 and 1.3,

$$\nu_{2a} + \nu_{3a} + \nu_{6a} + \nu_{2b} + \nu_{4a} + \nu_{12a} + \nu_{12b} = 1.$$

Consider the cases $\chi(u^6) = \chi(2k)$ where $k \in \{a, b\}$. Applying Proposition 1.5 (when $k = a$), we obtain

$$\mu_0(u, \chi_3, *) = \tfrac{1}{12}(8\gamma_1 + 12) \geqslant 0; \quad \mu_6(u, \chi_3, *) = \tfrac{1}{12}(-8\gamma_1 + 12) \geqslant 0;$$
$$\mu_2(u, \chi_4, *) = \tfrac{1}{12}(-2\gamma_2 + 2) \geqslant 0; \quad \mu_6(u, \chi_4, *) = \tfrac{1}{12}(4\gamma_2 + 8) \geqslant 0;$$
$$\mu_1(u, \chi_6, *) = \tfrac{1}{12}(6\gamma_3 + 6) \geqslant 0; \quad \mu_5(u, \chi_6, *) = \tfrac{1}{12}(-6\gamma_3 + 6) \geqslant 0$$

where $\gamma_1 = \nu_{2a} - \nu_{3a} + \nu_{6a}$, $\gamma_2 = 2\nu_{2a} - \nu_{3a} - \nu_{6a} - 2\nu_{4a} + \nu_{12a} + \nu_{12b}$ and $\gamma_3 = \nu_{12a} - \nu_{12b}$. Clearly $\gamma_1 \in \{0\}$, $\gamma_2 \in \{1\}$ and $\gamma_3 \in \{-1, 1\}$. It follows that the only possible integer solutions for $(\nu_{2a}, \nu_{3a}, \nu_{6a}, \nu_{2b}, \nu_{4a}, \nu_{12a}, \nu_{12b})$ are $(0, 0, 0, 0, 0, 0, 1)$ and $(0, 0, 0, 0, 0, 1, 0)$. Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 1.4.

When $k = b$, it follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{3a}, \nu_{6a}, \nu_{2b}, \nu_{4a}, \nu_{12a}, \nu_{12b})$ since $\mu_1(u, \chi_2, *) = 1/6$.

Case 9: Let $u \in V(\mathbb{Z}G)$ where $|u| = 15$. Using Propositions 1.2 and 1.3, $\nu_{3a} + \nu_{5a} + \nu_{5b} = 1$. Consider the cases $\chi(u^3) = \chi(5k)$ where $k \in \{a, b\}$. Applying Proposition 1.5, we obtain the following system of inequalities:

$$\mu_3(u, \chi_3, *) = \tfrac{1}{15}(4\nu_{3a} + 6) \geqslant 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{15}(-16\nu_{3a} + 6) \geqslant 0.$$

Clearly, there are no possible integer solutions for $(\nu_{3a}, \nu_{5a}, \nu_{5b})$.

Case 10: Let $u \in V(\mathbb{Z}G)$ where $|u| = 20$. Using Propositions 1.2 and 1.3, $\nu_{2a} + \nu_{5a} + \nu_{5b} + \nu_{2b} + \nu_{4a} + \nu_{10a} + \nu_{10b} = 1$. Consider the cases $\chi(u^{10}) = \chi(2m_1)$ and $\chi(u^5) = \chi(4m_2)$ and $\chi(u^4) = \chi(5m_3)$ and $\chi(u^2) = \chi(10m_4)$, where

$$(m_1, m_2, m_3, m_4) \in \{(a,a,a,a), (a,a,a,b), (a,a,b,a), (a,a,b,b), (b,a,a,a), (b,a,a,b), (b,a,b,a), (b,a,b,b)\}.$$

Now

$$\mu_1(u, \chi_2, *) = -\tfrac{1}{10}$$

when $(m_1, m_2, m_3, m_4) \in \{(a,a,a,a), (a,a,a,b), (a,a,b,a), (a,a,b,b)\}$. Also,

$$\mu_5(u, \chi_2, *) = \tfrac{1}{2}$$

when $(m_1, m_2, m_3, m_4) \in \{(b,a,a,a), (b,a,a,b), (b,a,b,a), (b,a,b,b)\}$. Therefore, there are no possible integer solutions for $(\nu_{2a}, \nu_{5a}, \nu_{5b}, \nu_{2b}, \nu_{4a}, \nu_{10a}, \nu_{10b})$.

Case 11: Let $u \in V(\mathbb{Z}G)$ where $|u| = 22$. Using Propositions 1.2 and 1.3, $\nu_{2a} + \nu_{11a} + \nu_{2b} = 1$. Let $\gamma_1 = \nu_{2a} + \nu_{11a} - \nu_{2b}$, $\gamma_2 = 2\nu_{2a} - \nu_{11a}$ and $\gamma_3 = -2\nu_{2a} - \nu_{11a}$. We will now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2, 11\}$:

▷ $\chi(u^{11}) = \chi(2a)$ and $\chi(u^2) = \chi(11a)$. Applying Proposition 1.5, we obtain

$$\mu_1(u, \chi_2, *) = \tfrac{1}{22}(\gamma_1 - 1) \geqslant 0; \quad \mu_2(u, \chi_2, *) = \tfrac{1}{22}(-\gamma_1 + 1) \geqslant 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{22}(5\gamma_2 + 1) \geqslant 0; \quad \mu_{11}(u, \chi_3, *) = \tfrac{1}{22}(-5\gamma_2 - 1) \geqslant 0.$$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{11a}, \nu_{2b})$.

▷ $\chi(u^{11}) = \chi(2b)$ and $\chi(u^2) = \chi(11a)$. Applying Proposition 1.5, we obtain

$$\mu_1(u, \chi_2, *) = \tfrac{1}{22}(\gamma_1 + 1) \geqslant 0; \quad \mu_2(u, \chi_2, *) = \tfrac{1}{22}(-\gamma_1 - 1) \geqslant 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{22}(\gamma_2) \geqslant 0; \quad \mu_{11}(u, \chi_3, *) = \tfrac{1}{22}(-\gamma_2) \geqslant 0;$$
$$\mu_1(u, \chi_3, *) = \tfrac{1}{22}(\gamma_2 + 11) \geqslant 0; \quad \mu_0(u, \chi_4, *) = \frac{1}{22}(\gamma_3) \geqslant 0;$$
$$\mu_{11}(u, \chi_4, *) = \tfrac{1}{22}(-\gamma_3) \geqslant 0.$$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{11a}, \nu_{2b})$.

Case 12: Let $u \in V(\mathbb{Z}G)$ where $|u| = 33$. Using Propositions 1.2 and 1.3, $\nu_{3a} + \nu_{11a} = 1$. Now $\chi(u^{11}) = \chi(3a)$ and $\chi(u^3) = \chi(11a)$. Applying Proposition 1.5, we obtain

$$\mu_{11}(u, \chi_3, *) = \tfrac{1}{33}(\gamma + 2) \geqslant 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{33}(-2\gamma - 4) \geqslant 0$$

where $\gamma = 20\nu_{3a} + 10\nu_{11a}$. It follows that there are no possible integer solutions for (ν_{3a}, ν_{11a}).

Case 13: Let $u \in V(\mathbb{Z}G)$ where $|u| = 55$. Using Propositions 1.2 and 1.3, $\nu_{5a} + \nu_{5b} + \nu_{11a} = 1$. Consider the cases $\chi(u^{11}) = \chi(5k)$ where $k \in \{a, b\}$. Applying Proposition 1.5, we obtain

$$\mu_{11}(u, \chi_3, *) = \tfrac{1}{55}(+10\nu_{11a}) \geqslant 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{55}(-40\nu_{11a}) \geqslant 0;$$
$$\mu_1(u, \chi_3, *) = \tfrac{1}{55}(-\nu_{11a} + 11) \geqslant 0.$$

It follows that there are no possible integer solutions for $(\nu_{5a}, \nu_{5b}, \nu_{11a})$ and this completes the proof. □

3. Proof of Theorem 2

Let $G = \mathrm{Aut}(\mathrm{PSL}(2, 13))$. Clearly, $|G| = 2184 = 2^3 \cdot 3 \cdot 7 \cdot 13$ and $\exp(G) = 1092 = 2^2 \cdot 3 \cdot 7 \cdot 13$. Initially for any torsion unit of $V(\mathbb{Z}G)$ of order k we have that

$$\nu_{2a} + \nu_{7a} + \nu_{7b} + \nu_{7c} + \nu_{14a} + \nu_{14b} + \nu_{14c} + \nu_{2b} + \nu_{3a} + \nu_{6a} + \nu_{4a} + \nu_{12a} + \nu_{12b} + \nu_{13a} = 1.$$

In order to prove that the Zassenhaus conjecture holds, it is necessary to consider units of order 2, 3, 4, 6, 7, 12, 13, 14, 21, 26, 28, 39 and 91, by Proposition 1.1. We shall now separately consider units of $V(\mathbb{Z}G)$ of order 2, 3, 4, 6, 7, 13, 21, 26, 39 and 91. Note that we are not considering torsion units of $V(\mathbb{Z}G)$ of order 12, 14 and 28 due to their complicated computations.

Case 1: Let $u \in V(\mathbb{Z}G)$ where $|u| = 2$. Using Propositions 1.2 and 1.3, $\nu_{2a} + \nu_{2b} = 1$. Applying Proposition 1.5, we obtain

$$\mu_1(u, \chi_2, *) = \tfrac{1}{2}(\gamma + 1) \geqslant 0; \quad \mu_0(u, \chi_2, *) = \tfrac{1}{2}(-\gamma + 1) \geqslant 0$$

where $\gamma = \nu_{2a} - \nu_{2b}$. Clearly, $\gamma \in \{1, -1\}$. It follows that the only possible integer solutions for (ν_{2a}, ν_{2b}) are $(0, 1)$ and $(1, 0)$. Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 1.4.

Case 2: Let $u \in V(\mathbb{Z}G)$ where $|u| = 3$. By Proposition 1.2, $\nu_{kx} = 0$ for all $kx \in \{2a, 7a, 7b, 7c, 14a, 14b, 14c, 2b, 6a, 4a, 12a, 12b, 13a\}$. Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 1.4.

Case 3: Let $u \in V(\mathbb{Z}G)$ where $|u| = 4$. Using Propositions 1.2 and 1.3, $\nu_{2a} + \nu_{2b} + \nu_{4a} = 1$. We shall now separately consider the following cases involving $\chi(u^2)$:

▷ $\chi(u^2) = \chi(2a)$. It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{4a})$ since, $\mu_1(u, \chi_2, 0) = 1/2$.

▷ $\chi(u^2) = \chi(2b)$. Applying Proposition 1.5, we obtain

$$\mu_0(u,\chi_2,*) = \tfrac{1}{4}(-2\gamma_1 + 2) \geqslant 0; \quad \mu_2(u,\chi_2,*) = \tfrac{1}{4}(2\gamma_1 + 2) \geqslant 0;$$
$$\mu_0(u,\chi_3,*) = \tfrac{1}{4}(-4\nu_{2a} + 12) \geqslant 0; \quad \mu_2(u,\chi_3,*) = \tfrac{1}{4}(4\nu_{2a} + 12) \geqslant 0$$

where $\gamma_1 = \nu_{2a} - \nu_{2b} + \nu_{4a}$. Clearly $\gamma_1 \in \{-1, 1\}$ and $\nu_{2a} \in \{k;\ -3 \leqslant k \leqslant 3\}$. It follows that the possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{4a})$ are listed in Theorem 1.2.

Case 4: Let $u \in V(\mathbb{Z}G)$ where $|u| = 6$. Using Propositions 1.2 and 1.3, $\nu_{2a} + \nu_{2b} + \nu_{3a} + \nu_{6a} = 1$. Let $\gamma_1 = 2\nu_{2b} - \nu_{3a} - \nu_{6a}$, $\gamma_2 = \nu_{2b} - \nu_{3a} + \nu_{6a}$, $\gamma_3 = -4\nu_{2b} + 2\nu_{3a} + 2\nu_{6a}$ and $\gamma_4 = -4\nu_{2b} - 2\nu_{3a} + 2\nu_{6a}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2, 3\}$:

▷ $\chi(u^3) = \chi(2a)$ and $\chi(u^2) = \chi(3a)$. Applying Proposition 1.5, we obtain

$$\mu_0(u,\chi_{11},*) = \tfrac{1}{6}(2\gamma_1 + 12) \geqslant 0; \quad \mu_1(u,\chi_{11},*) = \tfrac{1}{6}(\gamma_1 + 15) \geqslant 0;$$
$$\mu_3(u,\chi_{13},*) = \tfrac{1}{6}(4\gamma_2 + 18) \geqslant 0; \quad \mu_0(u,\chi_{13},*) = \tfrac{1}{6}(-4\gamma_2 + 18) \geqslant 0;$$
$$\mu_3(u,\chi_{11},*) = \tfrac{1}{6}(\gamma_3 + 12) \geqslant 0.$$

Clearly, $\gamma_1 \in \{-6, -3, 0, 3, 6\}$ and $\gamma_2 \in \{-3, 0, 3\}$. It follows that the only possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{6a})$ are $(1, -1, 0, 1)$ and $(1, 1, 0, -1)$.

▷ $\chi(u^3) = \chi(2b)$ and $\chi(u^2) = \chi(3a)$. Applying Proposition 1.5, we obtain

$$\mu_0(u,\chi_{11},*) = \tfrac{1}{6}(2\gamma_1 + 14) \geqslant 0; \quad \mu_1(u,\chi_{11},*) = \tfrac{1}{6}(\gamma_1 + 13) \geqslant 0;$$
$$\mu_2(u,\chi_{13},*) = \tfrac{1}{6}(2\gamma_2 + 10) \geqslant 0; \quad \mu_0(u,\chi_{13},*) = \tfrac{1}{6}(-4\gamma_2 + 16) \geqslant 0;$$
$$\mu_3(u,\chi_{11},*) = \tfrac{1}{6}(\gamma_3 + 10) \geqslant 0; \quad \mu_0(u,\chi_{14},*) = \tfrac{1}{6}(\gamma_4 + 10) \geqslant 0.$$

Clearly, $\gamma_1 \in \{-7, -4, -1, 2, 5\}$ and $\gamma_2 \in \{-5, -2, 1, 4\}$. It follows that the only possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{6a})$ are $(0, 2, 0, -1)$, $(0, 0, 0, 1)$, $(0, -2, 0, 3)$, $(0, 0, 3, -2)$ and $(0, -2, 3, 0)$.

Case 5: Let $u \in V(\mathbb{Z}G)$ where $|u| = 7$. Using Propositions 1.2 and 1.3, $\nu_{7a} + \nu_{7b} + \nu_{7c} = 1$. Applying Proposition 1.5, we obtain

$$\mu_1(u,\chi_3,*) = \tfrac{1}{7}(\gamma_1 + 12) \geqslant 0; \quad \mu_2(u,\chi_3,*) = \tfrac{1}{7}(\gamma_2 + 12) \geqslant 0;$$
$$\mu_3(u,\chi_3,*) = \tfrac{1}{7}(\gamma_3 + 12) \geqslant 0$$

where $\gamma_1 = -5\nu_{7a} + 2\nu_{7b} + 2\nu_{7c}$, $\gamma_2 = 2\nu_{7a} + 2\nu_{7b} - 5\nu_{7c}$ and $\gamma_3 = 2\nu_{7a} - 5\nu_{7b} + 2\nu_{7c}$. Clearly, $\gamma \in \{1, -1\}$. It follows that the only possible integer solutions for $(\nu_{7a}, \nu_{7b}, \nu_{7c})$ are listed in Theorem 1.2.

Case 6: Let $u \in V(\mathbb{Z}G)$ where $|u| = 13$. By Proposition 1.2, $\nu_{kx} = 0$ for all $kx \in \{2a, 7a, 7b, 7c, 14a, 14b, 14c, 2b, 3a, 6a, 4a, 12a, 12b\}$. Therefore, u is rationally conjugated to some element $g \in G$ by Proposition 1.4.

Case 7: Let $u \in V(\mathbb{Z}G)$ where $|u| = 21$. Using Propositions 1.2 and 1.3, $\nu_{7a} + \nu_{7b} + \nu_{7c} + \nu_{3a} = 1$. Consider the cases $\chi(u^7) = \chi(3a)$ and $\chi(u^3) = m_1\chi(7a) + m_2\chi(7a) + m_3\chi(7a)$. Applying Proposition 1.5, we obtain

$$\mu_0(u, \chi_3, *) = \tfrac{1}{21}(4\gamma_1 + 14) \geqslant 0; \quad \mu_7(u, \chi_3, *) = \tfrac{1}{21}(-2\gamma_1 + k_1) \geqslant 0;$$
$$\mu_6(u, \chi_3, *) = \tfrac{1}{21}(2\gamma_2 + k_2) \geqslant 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{21}(-\gamma_2 + k_3) \geqslant 0;$$
$$\mu_9(u, \chi_3, *) = \tfrac{1}{21}(2\gamma_3 + k_4) \geqslant 0; \quad \mu_2(u, \chi_3, *) = \tfrac{1}{21}(-\gamma_3 + k_5) \geqslant 0;$$
$$\mu_0(u, \chi_9, *) = \tfrac{1}{21}(\gamma_4 + k_6) \geqslant 0$$

where $\gamma_1 = \nu_{7a} + \nu_{7b} + \nu_{7c}$, $\gamma_2 = 2\nu_{7a} + 2\nu_{7b} - 5\nu_{7c}$, $\gamma_3 = 2\nu_{7a} - 5\nu_{7b} + 2\nu_{7c}$ and $\gamma_4 = -12\nu_{7a} - 12\nu_{7b} - 12\nu_{7c} + 12\nu_{3a}$ for all possible m_i, k_j. It follows that there are no possible integer solutions for $(\nu_{7a}, \nu_{7b}, \nu_{7c}, \nu_{3a})$ for all possible m_i, k_j. Note that all possible values for m_i, k_j are listed in Table 1.

(m_1, m_2, m_3)	$(k_1, k_2, k_3, k_4, k_5, k_6)$	(m_1, m_2, m_3)	$(k_1, k_2, k_3, k_4, k_5, k_6)$
$(1, 0, 0)$	$(7, 14, 7, 14, 14, 9)$	$(0, 1, 0)$	$(14, 14, 14, 14, 14, 9)$
$(0, 0, 1)$	$(14, 7, 14, 14, 7, 9)$	$(2, -3, 2)$	$(0, 0, 0, 14, 0, 9)$
$(2, -2, 1)$	$(0, 7, 0, 14, 7, 9)$	$(1, -2, 2)$	$(7, 0, 7, 14, 0, 9)$
$(2, -1, 0)$	$(0, 14, 0, 14, 14, 9)$	$(1, -1, 1)$	$(7, 7, 21, 7, 14, 7)$
$(0, -1, 2)$	$(14, 0, 14, 14, 0, 9)$	$(2, 0, -1)$	$(0, 21, 0, 14, 21, 9)$
$(-1, 0, 2)$	$(21, 0, 21, 14, 0, 9)$	$(2, 1, -2)$	$(0, 28, 0, 14, 28, 9)$
$(1, 1, -1)$	$(7, 21, 7, 7, 14, 21)$	$(-1, 1, 1)$	$(21, 7, 7, 21, 14, 7)$
$(-2, 1, 2)$	$(28, 0, 28, 14, 0, 9)$	$(2, 2, -3)$	$(0, 35, 0, 14, 35, 9)$
$(1, 2, -2)$	$(7, 28, 7, 14, 28, 9)$	$(0, 2, 1)$	$(14, 21, 14, 14, 21, 9)$
$(-1, 2, 0)$	$(21, 14, 21, 14, 14, 9)$	$(-2, 2, 1)$	$(28, 7, 28, 14, 7, 9)$
$(-3, 2, 2)$	$(35, 0, 35, 14, 0, 9)$		

Table 1. Possible values for m_i, k_j-units of order 21.

Case 8: Let $u \in V(\mathbb{Z}G)$ where $|u| = 26$. Using Propositions 1.2 and 1.3, $\nu_{2a} + \nu_{2b} + \nu_{13a} = 1$. Let $\gamma_1 = -\nu_{2a} + \nu_{2b} + \nu_{13a}$, $\gamma_2 = 2\nu_{2a} + \nu_{13a}$ and $\gamma_3 = 2\nu_{2a} - \nu_{13a}$. We shall now separately consider the following cases involving $\chi(u^n)$ for $n \in \{2, 13\}$:

▷ $\chi(u^{13}) = \chi(2a)$ and $\chi(u^2) = \chi(13a)$. Applying Proposition 1.5, we obtain

$$\mu_1(u, \chi_2, *) = \tfrac{1}{26}(\gamma_1 + 1) \geqslant 0; \quad \mu_2(u, \chi_2, *) = \tfrac{1}{26}(-\gamma_1 - 1) \geqslant 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{26}(6\gamma_2 - 1) \geqslant 0; \quad \mu_{13}(u, \chi_3, *) = \tfrac{1}{26}(-6\gamma_2 + 1) \geqslant 0.$$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$.

$\triangleright$ $\chi(u^{13}) = \chi(2b)$ and $\chi(u^2) = \chi(13a)$. Applying Proposition 1.5, we obtain

$$\mu_1(u, \chi_2, *) = \tfrac{1}{26}(\gamma_1 - 1) \geqslant 0; \quad \mu_2(u, \chi_2, *) = \tfrac{1}{26}(-\gamma_1 + 1) \geqslant 0;$$
$$\mu_0(u, \chi_6, *) = \tfrac{1}{26}(\gamma_3) \geqslant 0; \quad \mu_{13}(u, \chi_6, *) = \tfrac{1}{26}(-\gamma_3) \geqslant 0;$$
$$\mu_1(u, \chi_3, *) = \tfrac{1}{26}(-\gamma_2 + 13) \geqslant 0.$$

It follows that there are no possible integer solutions for $(\nu_{2a}, \nu_{2b}, \nu_{13a})$.

Case 9: Let $u \in V(\mathbb{Z}G)$ where $|u| = 39$. Using Propositions 1.2 and 1.3, $\nu_{3a} + \nu_{13a} = 1$. Now $\chi(u^{13}) = \chi(3a)$ and $\chi(u^3) = \chi(13a)$. Applying Proposition 1.5, we obtain

$$\mu_{13}(u, \chi_3, *) = \tfrac{1}{39}(+12\nu_{13a}) \geqslant 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{39}(-24\nu_{13a}) \geqslant 0;$$
$$\mu_1(u, \chi_3, *) = \tfrac{1}{39}(-\nu_{13a} + 13) \geqslant 0.$$

It follows that there are no possible integer solutions for (ν_{3a}, ν_{13a}).

Case 10: Let $u \in V(\mathbb{Z}G)$ where $|u| = 91$. Using Propositions 1.2 and 1.3, $\nu_{7a} + \nu_{7b} + \nu_{7c} + \nu_{13a} = 1$. Consider the cases $\chi(u^{13}) = m_1\chi(7a) + m_2\chi(7b) + m_3\chi(7c)$ and $\chi(u^7) = \chi(13a)$ where

$$\begin{aligned}(m_1, m_2, m_3) \in \{&(1,0,0), (0,1,0), (0,0,1), (2,-3,2), (2,-2,1), (1,-2,2),\\ &(2,-1,0), (1,-1,1), (0,-1,2), (2,-1,0), (-1,0,2), (2,1,-2),\\ &(1,1,-1), (-1,1,1), (-2,1,2), (2,2,-3), (1,2,-2), (0,2,-1)\}.\end{aligned}$$

Applying Proposition 1.5, we obtain

$$\mu_0(u, \chi_3, *) = \tfrac{1}{91}(24\gamma + 2) \geqslant 0; \quad \mu_7(u, \chi_3, *) = \tfrac{1}{91}(-2\gamma + 15) \geqslant 0$$

where $\gamma = \nu_{7a} + \nu_{7b} + \nu_{7c} - 3\nu_{13a}$. It follows that there are no possible integer solutions for $(\nu_{7a}, \nu_{7b}, \nu_{7c}, \nu_{13a})$.

We will now consider the prime graph of $G = \mathrm{Aut}(\mathrm{PSL}(2, 13))$. Clearly $[2, 3]$ and $[2, 7]$ are adjacent in $\pi(G)$ and consequently adjacent in $\pi(V(\mathbb{Z}G))$. However, $[2, 13]$, $[3, 7]$, $[3, 13]$ and $[7, 13]$ are not adjacent in $\pi(G)$. Clearly $\pi(G) = \pi(V(\mathbb{Z}G))$, since there are no torsion units of order 21, 26, 39 and 91 in $V(\mathbb{Z}G)$. This completes the proof. $\square$

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