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GRAUERT'S LINE BUNDLE CONVEXITY, REDUCTION AND RIEMANN DOMAINS

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Abstract. We consider a convexity notion for complex spaces X with respect to a holomorphic line bundle L over X. This definition has been introduced by Grauert and, when L is analytically trivial, we recover the standard holomorphic convexity. In this circle of ideas, we prove the counterpart of the classical Remmert's reduction result for holomorphically convex spaces. In the same vein, we show that if $H^0(X,L)$ separates each point of X, then X can be realized as a Riemann domain over the complex projective space $\mathbb{P}^n$, where n is the complex dimension of X and L is the pull-back of $\mathcal{O}(1)$.

Keywords: Grauert's line bundle convexity; Riemann domain; holomorphic reduction

MSC 2010: 32F17, 32E05, 32E99

1. INTRODUCTION

Let X be a complex space and L a holomorphic line bundle over X with projection map $\pi\colon L \longrightarrow X$. Following Grauert [5], page 380, we say that X is *L-convex* if each compact set K in X admits a compact neighborhood $K^\star$, such that for every point $x \in X \setminus K^\star$ the following property holds true:

For every vector $v \in L_x := \pi^{-1}(x)$ and every open neighborhood U of the zero section of L, there is a global holomorphic section $\sigma \in \Gamma(X,L)$ such that $\sigma(x) = v$ and $\sigma(K) \subset U$.

Obviously, standard holomorphic convexity is regained as L-convexity when L is holomorphically trivial.

Now, a well-known result due to Remmert [8] (see also the work of Cartan [4], Example 1, page 9) asserts that every holomorphically convex space X admits a proper reduction onto a Stein space Y, namely, there is a proper surjective holomorphic map $\pi\colon X \longrightarrow Y$ onto a Stein space Y such that $\pi_\star(\mathcal{O}_X) = \mathcal{O}_Y$; a fortiori π has connected fibers.

On the other hand, if we have enough holomorphic functions on X, such that every point of X is separated by $\mathcal{O}(X)$, a so called *K-complete* space or *holomorphically spreadable* space, then X is a branched Riemann domain over $\mathbb{C}^n$, that is, there is a holomorphic map $\varphi\colon X \longrightarrow \mathbb{C}^n$ with discrete fibers, where $n = \dim(X)$.

The goal of this paper is to prove, in analogy with the usual results on holomorphic convexity and spreadability, the following theorems (definitions are given in sections Section 2 and Section 3).

Theorem 1.1. *Let X be a complex space which is L-convex with respect to a globally generated holomorphic line bundle L over X.*

Then there is a complex space X_0, a proper holomorphic map $\pi\colon X \longrightarrow X_0$ with connected fibres onto a complex space X_0, and a holomorphic line bundle L_0 on X such that:

- ▷ *X_0 is L_0-convex,*
- ▷ *every point of X_0 is separated by $\Gamma(X_0, L_0)$, and*
- ▷ *the canonically induced map $\Gamma(X_0, L_0) \longrightarrow \Gamma(X, L)$ is bijective;* a fortiori $L = \pi^\star(L_0)$.

The counter part to Grauert's Theorem [6], Satz 12, page 253, asserting that every K-complete space of dimension n is a Riemann domain over $\mathbb{C}^n$, reads as follows.

Theorem 1.2. *Let X be a complex space of dimension n and let L be a holomorphic line bundle over X such that each point of X is separated by $\Gamma(X, L)$.*

Then there is a discrete holomorphic map $\pi\colon X \longrightarrow \mathbb{P}^n$ such that $L = \pi^\star(\mathcal{O}(1))$; hence, if X has pure dimension, then it becomes a Riemann domain over $\mathbb{P}^n$.

We conclude this section with an example due to Grauert [5], pages 379–382, that motivates the study of convexity with respect to holomorphic line bundles, namely, there is a 2-dimensional complex manifold F and a nonempty open subset $\Omega \subset F$ with the following properties:

- ▷ The set Ω is *weakly* 1-*complete*, that is, there is a C^∞-smooth proper plurisubharmonic function $\varphi\colon \Omega \longrightarrow [0, \infty)$;
- ▷ every holomorphic function on Ω is constant;
- ▷ the set Ω is a *domain of meromorphy* in F, that is, there is a meromorphic function on Ω which cannot be extended meromorphically across any boundary point of Ω.

This is done as follows: Take a compact Riemann surface S of genus greater than or equal to 1 and a topologically trivial holomomorphic line bundle F over S such that no positive tensor power of F is analytically trivial. As such, we endow F with a flat hermitian metric $\|\cdot\|$ (see the article of Ueda [11]) and set $\Omega := \{v \in F;\ \|v\| < \varrho\}$ for $\varrho > 0$, and $\varphi(v) = 1/(\varrho - \|v\|)$ (for $\varrho = \infty$, we put $\Omega = F$ and $\varphi(v) = \|v\|$).

Compactify each fiber of F with the point at infinity to obtain a ruled surface M over S, that is, a $\mathbb{P}^1$-bundle over S. Since M is projectively algebraic, there is a positive divisor A on M. Let G be its associated holomorphic line bundle. In order to check Theorem 1.2, Grauert proved that Ω is G^k-convex for some positive integer k, sufficiently large.

Throughout this paper, all complex spaces are reduced and with countable topology. For definitions see Section 2 and Section 4.

2. Grauert's convexity

Let X be a complex space and let $\pi\colon L \longrightarrow X$ be a holomorphic line bundle over X. The notion of convexity with respect to holomorphic line bundles appeared for the first time in a paper by Grauert [5].

Definition 2.1. We say that X is L-convex if each compact set K in X admits a compact neighborhood $K^\star$ such that for every point $x \in X \setminus K^\star$ the following property holds:

For every vector $v \in L_x := \pi^{-1}(x)$ and every open neighborhood U of the zero section of L, there is a global holomorphic section $\sigma \in \Gamma(X, L)$ such that $\sigma(x) = v$ and $\sigma(K) \subset U$.

Remark 2.1. If L is analytically trivial, then X is holomorphically convex precisely when X is L-convex.

A (*singular*) *hermitian metric* h on L is given in any trivialization $\theta\colon\ L|_\Omega \longrightarrow \Omega\times\mathbb{C}$ by

$$\|v\|_h = |t|\mathrm{e}^{\varphi(x)}, \quad x \in \Omega,\ t \in \mathbb{C},\ v = \theta^{-1}(x,t),$$

where $\varphi \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ is called the weight function with respect to the trivialization θ.

Observe that, in general, given $\varepsilon > 0$, the set $\{v \in L;\ \|v\|_h < \varepsilon\}$ is open if the weight functions are upper semi-continuous. In this case the condition in Definition 2.1 becomes:

Let $x \in X \setminus K^\star$. Then, for any vector $v \in L_x$ and $\varepsilon > 0$, there exists a global holomorphic section σ of L over X such that $\sigma(x) = v$ and $\|\sigma(y)\|_h < \varepsilon$ for all $y \in K$.

Now, coming back to L-convexity, because one always can endow L with a continuous (even C^∞) hermitian metric h, Grauert's definition is characterized by the following lemma.

Lemma 2.1. *The space X is L-*convex *if, and only if, for every compact set $K \subset X$ its hull $\widetilde{K}$ with respect to $\Gamma(X,L)$ is relatively compact in X, where*

$$\widetilde{K} := \Big\{ x \in X\,;\ \exists\, C_x > 0\ \forall\, \sigma \in \Gamma(X,L)\ \|\sigma(x)\|_h \leqslant C_x \max_{z\in K} \|\sigma(z)\|_h \Big\}.$$

Observe that the hull $\widetilde{K}$ contains K and does not depend on h; moreover, $\widetilde{K}$ need not be closed in X.

If L is trivial, that is $L = X \times \mathbb{C}$, then $\widetilde{K}$ equals the ordinary holomorphically convex hull $\widehat{K}$, where

$$\widehat{K} := \Big\{ x \in X\,;\ \forall\, f \in \mathcal{O}(X), |f(x)| \leqslant \max_{y\in K} |f(y)| \Big\}.$$

(To see this, note that as $\mathcal{O}(X)$ is a $\mathbb{C}$-algebra, taking powers f^n with $n \in \mathbb{N}$ and then extracting the n-th root, it follows that, in Lemma 2.1, we may choose $C_x = 1$.) This motivates the following definition.

Definition 2.2. We say that X is (L,h)-convex if for every compact subset K of X its hull $\widehat{K}_h$ is compact in X, where

$$\widehat{K}_h := \{ x \in X\,;\ \forall\, \sigma \in \Gamma(X,L)\ \|\sigma(x)\|_h \leqslant \max_{y\in K} \|\sigma(y)\| \}.$$

Remark. Note that if X is L-convex, then for any choice of a continuous hermitian metric h on L, X becomes (L,h)-convex. The converse does not hold (see the subsequent Example 2.3).

Lemma 2.2. *Let X be a Stein space. Then X is L-convex with respect to any holomorphic line bundle L over X.*

Proof. By standard arguments one checks readily that if $T \subset X$ is a holomorphically convex subset of X and $x_0 \in X$, then $T \cup \{x_0\}$ is holomorphically convex.

Now, to check the L-convexity of X, we let K be a compact subset of X. Let $x \in X \setminus \widehat{K}$ and let U be an open neighborhood of the zero section of L. Let $v \in L_x$, $v \neq 0$. Since $\widehat{K} \cup \{x\}$ is holomorphically convex, the restriction map

$$H^0(X,L) \longrightarrow H^0(\widehat{K} \cup \{x\}, L)$$

has dense range. Let $\sigma_1 \in \Gamma(X,L)$ be such that $\sigma_1(\widehat{K}) \subset U/2$ and $|\sigma_1(x)/v - 1| < \frac{1}{2}$. This implies $|\lambda| < 2$, where $\lambda := v/\sigma_1(x)$, and then $\sigma := \lambda\sigma_1$ is such that $\sigma(\widehat{K}) \subset U$ and $\sigma(x) = v$. $\square$

Remark. The conclusion remains valid for 1-convex spaces instead of Stein spaces. (A complex space is called 1-*convex* if it is a proper modification of a Stein space in a finite number of points.)

Example 2.1. Let $\varphi\colon\ X \longrightarrow [0,\infty)$ be an upper semi-continuous function, where X is a complex space without nonconstant holomorphic functions.

Let $L = X \times \mathbb{C}$, $\|v\|_h := |t|\mathrm{e}^{\varphi(x)}$, where $v = (x,t) \in L$. Then, for a compact subset K of X, one has

$$\widehat{K}_h = \Big\{ x \in X\,;\ \varphi(x) \leqslant \sup_{z\in K} \varphi(z) \Big\}.$$

Thus, if φ is continuous, then X is (L,h)-convex if, and only if, φ is proper.

Example 2.2. Here we recall the *intermediate q-holomorphic convexity* defined by Barlet and Silva [3]. Suppose that there are global holomorphic sections $s_0, s_1, \ldots, s_q$ in L without common zeros. They induce a holomorphic map $\pi\colon\ X \longrightarrow \mathbb{P}^q$ and on $L = \pi^\star(\mathcal{O}(1))$, the pull-back of the standard Fubini-Study metric on $\mathbb{P}^q$ induces a hermitian metric $h_\star$ on L; alternatively this can be obtained by setting

$$\|v\|_{h_\star} = \frac{|v|}{\|s(x)\|}, \qquad \text{if}\ \ v \in L_x,$$

where

$$\|s(x)\|^2 = \sum_{j=0}^{q} |s_j(x)|.$$

Notice that the weight functions of $h_\star$ are constant along the fibers of π.

The authors call X *q-holomorphically convex* if X is $(L,h_\star)$-convex. If this is the case, the fibers of π are holomorphically convex.

Example 2.3. Let Z be a projective manifold of dimension greater than or equal to 2 and consider a very ample line bundle over Z, denoted by F. Let A be a nonempty analytic subset of Z of codimension greater than or equal to 2. We put $X := Z \setminus A$ and $L := F|_X$. Then X fails to be L-convex; however we may endow L with a smooth hermitian metric h such that X becomes (L,h)-convex.

Indeed, consider finitely many global holomorphic sections, say $s_0, \ldots, s_q$ in $\Gamma(Z,F)$, whose common zero set is A. Put on L the smooth hermitian metric induced by $s_0, \ldots, s_q$ as above. With respect to this hermitian metric, X becomes (L,h)-convex. (As a matter of fact, for this to hold, A may be any proper analytic subset of Z.)

To show that X is not L-convex, one applies Lemma 2.3 from below, because $\Gamma(X,L) \simeq \Gamma(Z,F)$ and the latter has finite dimension over $\mathbb{C}$. Alternatively, we may use the pseudoconcavity[1] of X, which implies that $\Gamma(X,L)$ has finite dimension over $\mathbb{C}$.

Lemma 2.3. *Let X be an irreducible complex space and assume that L is a holomorphic line bundle over X such that the complex vector space $\Gamma(X,L)$ has finite dimension. Then X is L-convex if, and only if, X is compact.*

Proof. Let K be a subset of X with nonempty interior and let $\{\sigma_1,\dots,\sigma_m\}$ be a basis of $\Gamma(X,L)$ over $\mathbb{C}$. Let $\|\cdot\|$ be a continuous hermitian metric on L. Let Λ be the compact subset of $\mathbb{C}^m$ made of $\lambda=(\lambda_1,\dots,\lambda_m)$ with $|\lambda_1|+\dots+|\lambda_m|=1$. We readily see that

$$\mu := \inf_{\lambda\in\Lambda}\max_{y\in K}\|\lambda_1\sigma_1(y)+\dots+\lambda_m\sigma_m(y)\|$$

is strictly positive.

For $x\in X$, we put $C=\max_{\lambda\in\Lambda}\|\lambda_1\sigma_1(x)+\dots+\lambda_m\sigma_m(x)\|$ so that we obtain $\widetilde{K}=X$, which shows that X is compact. □

Corollary 2.1. *Let X be a pseudoconcave irreducible complex space. Then X is L convex with respect to some holomorphic line bundle L over X if, and only if, X is compact.*

Proof. Since the sheaf of germs of holomorphic sections in L is locally free, coherent and torsion free, from [1], the complex vector space $\Gamma(X,L)$ has finite dimension and the proof follows by Lemma 2.3. □

Definiton 2.3. Let $\mathcal{H}$ be a nonempty subset of $\Gamma(X,L)$. We say that

1. the point a of X is separated by $\mathcal{H}$ if there are $\sigma_0,\dots,\sigma_N\in\mathcal{H}$ and an open neighborhood U of a such that $\sigma_0,\sigma_1,\dots,\sigma_N$ does not vanish simultaneously on U and the induced holomorphic map $\chi\colon U\longrightarrow\mathbb{P}^N$ is discrete at a;
2. the family $\mathcal{H}$ gives local coordinates at $a\in X$ if there are finitely many elements $\sigma_0,\dots,\sigma_N$ and U as above such that χ is an immersion at a.

The above conditions can be rewritten by saying that (after shrinking U around a and permuting coordinates, if necessary) σ_0 does not vanish on U and the holomorphic map

$$U\ni x\mapsto\Big(\frac{\sigma_1}{\sigma_0}(x),\dots,\frac{\sigma_N}{\sigma_0}(x)\Big)\in\mathbb{C}^n$$

[1] A complex space X is said to be *pseudoconcave* in the sense of Andreotti (see [1]) if X contains a relatively compact open subset Ω which meets every irreducible component of X and such that each point $x_0\in\partial\Omega$ admits a neighborhood system of open sets $\{U_\nu\}_\nu$ such that, for all ν, x_0 is an interior point of $\widehat{U_\nu\cap\Omega}$ (the holomorphically convex hull of $U_\nu\cap\Omega$ in U_ν with respect of $\mathcal{O}(U_\nu)$).

contains the point a, which is isolated in its fiber, and that

$$\frac{\sigma_i}{\sigma_0} - \frac{\sigma_i}{\sigma_0}(a), \quad i = 1, 2, \ldots, N,$$

generate the space m_a/m_a^2 over $\mathbb{C}$, where m_a is the maximal ideal of $\mathcal{O}_{X,a}$.

It is worth mentioning that, for $L = X \times \mathbb{C}$ so that $\mathcal{H} \subset \mathcal{O}(X)$, we obtain *holomorphic spreadability* (or *K-completeness*) with respect to a subfamily $\mathcal{H}$ of holomorphic functions on X. (This extends the standard terminology for $\mathcal{H} = \mathcal{O}(X)$; in this case we require that for each point $x_0 \in X$, there are finitely many $f_1, \ldots, f_k \in \mathcal{H}$ such that x_0 lies isolated in the fibre $f^{-1}(f(x_0))$, where $f = (f_1, \ldots, f_k)\colon X \longrightarrow \mathbb{C}^k$ (here k may depend on x_0.)

Further, a holomorphic line bundle L is called *globally generated* if, for every point x_0 of X, there is a section $\sigma \in \Gamma(X, L)$ such that $\sigma(x_0) \neq 0$.

Observe that, in this case, given a discrete subset Λ of X, the subset $\mathcal{S}_\Lambda$ of sections $s \in \Gamma(X, L)$ such that $s(x) \neq 0$ for all $x \in \Lambda$ is dense in $\Gamma(X, L)$ (with respect to the topology given by uniform convergence on compact subsets of X). This is a consequence of Baire's theorem if we note: i) $\Gamma(X, L)$ is a Fréchet space and ii) for a point $a \in X$, the subset of $\Gamma(X, L)$ consisting of those sections vanishing at a is closed and has empty interior. (The following observation settles ii) above, namely, for a Fréchet subspace F' of another Fréchet space F, if F' has nonempty interior, then $F' = F$.)

In particular, we deduce easily that, if X has finite dimension, then there are finitely many sections $\sigma_0, \sigma_1, \ldots, \sigma_k \in \Gamma(X, L)$ without common zeros; hence they induce a holomorphic map $\tau\colon X \longrightarrow \mathbb{P}^k$; besides, one has $L = \tau^\star(\mathcal{O}(1))$. (This can be achieved in a standard way by induction on the complex dimension of X, and is therefore omitted.)

Then, by a proper choice of Λ, we can show that there are "many" global holomorphic sections σ in L whose zero sets $Z(\sigma)$ are rare analytic sets. (For this it is enough to choose Λ to contain at least a point from every positive dimensional irreducible component of X.)

This discussion and the one in [6] improves upon the result due to Grauert [6], Satz 12, page 253.

Theorem 2.1. *Let X be a complex space of dimension n and let $\mathcal{H} \subset \mathcal{O}(X)$ be a family of holomorphic functions such that X is* holomorphically spreadable with respect to $\mathcal{H}$. *If $\mathcal{H}$ is closed in $\mathcal{O}(X)$, then there is a discrete map $\pi\colon X \longrightarrow \mathbb{C}^n$ whose components belong to $\mathcal{H}$.*

3. Proof of Theorem 1.1

Let X be a complex space and let L be a holomorphic line bundle over X. We define an equivalence relation $\mathcal{R}$ on X as the analytic subset of the product space $X \times X$ consisting of all couples $(x, y) \in X \times X$ such that, for every two holomorphic sections σ and τ of L over X,

$$\begin{vmatrix} \sigma(x) & \sigma(y) \\ \tau(x) & \tau(y) \end{vmatrix} = 0.$$

For $x \in X$, we let $\mathcal{R}(x) = \{y \in X\,;\ (x, y) \in \mathcal{R}\}$, that is, $\mathcal{R}(x)$ is the fiber of $\mathcal{R}$ over x.

For a subset T of X we denote by $\mathcal{R}(T)$ the saturated envelope of T, that is,

$$\mathcal{R}(T) = \{y \in X\,;\ (x, y) \in \mathcal{R} \text{ for some } x \in T\}\,.$$

Lemma 3.1. *If L is globally generated, then $\mathcal{R}(x)$ equals the intersection of the zero sets of those holomorphic sections of L which vanish at x.*

Proof. First, let $y \in \mathcal{R}(x)$ and consider $\sigma \in \Gamma(X, L)$ such that $\sigma(x) = 0$. Let $\tau \in \Gamma(X, L)$ with $\tau(x) \neq 0$. From $(x, y) \in \mathcal{R}$, it follows that $\tau(x)\sigma(x) = 0$, hence $\sigma(y) = 0$.

For the reverse inclusion, let $\sigma_1, \sigma_2 \in \Gamma(X, L)$ and let τ be as above. There are $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\sigma_i - \lambda_i\tau$ vanish at x for $i = 1, 2$; hence they vanish at y, too. This immediately shows that

$$\begin{vmatrix} \sigma(x) & \sigma(y) \\ \tau(x) & \tau(y) \end{vmatrix} = 0,$$

and the proof is completed. □

The analytic equivalence relation $\mathcal{R}$ is said to be *proper* if, for every compact subset K of X, the saturated set $\mathcal{R}(K)$ is compact. In that case, we may consider the quotient of X by the induced equivalence analytic relation given by the connected components of $\mathcal{R}(x)$, as x runs throughout X.

Lemma 3.2. *If X is L-convex, then $\mathcal{R}$ is a proper analytic equivalence relation.*

Proof. Indeed, let $K \subset X$ be a compact set. Clearly $\mathcal{R}(K)$ is closed. We will show that $\mathcal{R}(K) \subset \widetilde{K}$. Assume, in order to reach a contradiction, that this is not true; hence there is $y \in X \setminus \widetilde{K}$ and some $x \in K$ such that $y \in \mathcal{R}(x)$.

In order to proceed, observe that there is $\tau \in \Gamma(X, L)$ with $\tau(x) \neq 0$. (Otherwise, assume that $\tau(x) = 0$ for every $\tau \in \Gamma(X, L)$. Then let $v \in L_y$, $v \neq 0$, and put

$\Omega = L \setminus \{v\}$. By the hypothesis, there is a section $\theta \in \Gamma(X, L)$ such that $\theta(y) = v$. But, as $y \in \mathcal{R}(x)$ and since $\theta(x) = 0$ by assumption, it follows that $\theta(y) = 0$, whence $v = 0$, which is a contradiction! In particular, this shows that the base locus of L is compact.)

Fix a continuous hermitian metric h on L. Let $\Omega = \{v \in L;\ \|v\|_h < 1\}$ and take $w \in L_y$ such that $\|w\|_h > \max\{1, \|\tau(y)\|_h/\|\tau(x)\|_h\}$. Then there is a holomorphic section $\sigma \in \Gamma(X, L)$ such that $\sigma(y) = w$ and $\|\sigma(z)\|_h < 1$ for $z \in K$, in particular $\|\sigma(x)\|_h < 1$.

Let $\lambda \in \mathbb{C}$ such that $\sigma(x) = \lambda\tau(x)$. Thus $\sigma - \lambda\tau$ vanishes at x, hence it vanishes at y, too. Therefore $\sigma(y) = \lambda\tau(y)$. Thus $\|w\|_h = |\lambda|\|\tau(y)\|_h$, but $|\lambda| = \|\sigma(x)\|_h/\|\tau(x)\|_h$, so that we obtain $\|w\|_h \leqslant \|\tau(y)\|_h/\|\tau(x)\|_h$, which contradicts the above choice of w. □

Remark 3.1. The conclusion of Lemma 3.2 remains true if we use (L, h)-convexity instead of L-convexity.

End of proof of Theorem 1.1. Thanks to Lemma 3.2, the relation $\mathcal{R}^0$ defined by the connected components of $\mathcal{R}(x)$, $x \in X$, is a proper analytic equivalence relation on X. We shall see that the Hausdorff topological space $X_0 := X/\mathcal{R}^0$ is, in fact, a complex space and the canonical map $\varrho\colon X \longrightarrow X_0$, which is proper and has connected fibres, becomes holomorphic. This follows by [4] if we note the following description of $\mathcal{R}$. Because L is globally generated, we can cover X by open sets of the form $U_\lambda := X \setminus Z(\lambda)$ for $\lambda \in \Gamma(X, L)$ with $\lambda \neq 0$.

Now, fix λ as above and set, for practical purposes, $U = U_\lambda$. We see that $\mathcal{R}$ is given on U by global holomorphic functions, namely,

$$\mathcal{R} \cap (U \times U) = \{(x, y) \in U \times U; f(x) = f(y) \text{ for any } \varphi \in \mathcal{O}^\sharp(U)\},$$

where $\mathcal{O}^\sharp(U) \subset \mathcal{O}(U)$ is formed by taking quotients of $\sigma|_U$ over $\lambda|_U$, and for every compact set $K \subset U$, $\mathcal{R}^0(K) \subset U$.

By a well-known result due to Cartan [4], Main theorem, page 7, it follows that $U/\mathcal{R}^0$ are complex spaces and there are canonical holomorphic mappings $\varrho_U\colon U \longrightarrow U/\mathcal{R}^0$. Moreover, ϱ_U is proper with connected fibers. These data can be glued together to a complex space $X_0 := X/\mathcal{R}^0$ and a holomorphic map $\varrho\colon X \longrightarrow X_0$. (See also the article of Kaup [7], where the case of open equivalence analytic relation is treated.)

Also, the 1-cocycle defining L descends to a 1-cocycle on X_0, thus defining a holomorphic line bundle L_0 over X_0, such that $L = \varrho^\star(L_0)$ and the naturally induced map $\tilde{\varrho}\colon \Gamma(X_0, L_0) \longrightarrow \Gamma(X, L)$ is bijective. This gives readily that each point of X_0 is separated by $\Gamma(X_0, L_0)$ and X_0 is L_0-convex. □

4. Global generation and points separation

This section contains the proof of Theorem 1.2. Recall that a complex space X is said to be a *Riemann domain* over another complex space Y if: i) X is connected and ii) there is a holomorphic map $\pi\colon\ X \longrightarrow Y$ which is open and has discrete fibers. (In the case X is irreducible and Y is locally irreducible and of the same dimension, the openness condition is superfluous. As a matter of fact, the classical setting is for domains over $\mathbb{C}^n$ or $\mathbb{P}^n$.)

We split the proof into two parts (see the subsequent Proposition 4.1 and Proposition 4.2.): first we produce a discrete holomorphic mapping from X into a complex projective space of dimension $k \geqslant n$ and secondly, we decrease, if necessary, the dimension of the target space down to n.

Proposition 4.1. *Let X be a complex space of dimension n and let L be a holomorphic line bundle over X, such that each point of X is separated by $\Gamma(X, L)$.*

Then there are finitely many global holomorphic sections $\sigma_0, \sigma_1, \dots, \sigma_k$ in L which exhibit a holomorphic map $\pi\colon\ X \longrightarrow \mathbb{P}^k$ with discrete fibers. In particular, $L = \pi^\star(\mathcal{O}(1))$.

P r o o f. First, we need the following result.

Claim 1. Let Z be a Stein space and let $A \subset Z$ be a closed analytic subset. Then, for every coherent analytic sheaf $\mathcal{F}$ on Z, the restriction mapping

$$H^0(X, \mathcal{F}) \longrightarrow H^0(X \setminus A, \mathcal{F})$$

has closed image.

Indeed, thanks to Bănică and Stănăşilă [2], Corollaire 6.8, page 101, we know that $H^1_A(Z, \mathcal{F})$ is, with respect to the canonical topology, a Fréchet-Schwartz space. Then, from the long exact cohomology sequence with supports in A, we retain the exact portion

$$H^0(Z, \mathcal{F}) \longrightarrow H^0(Z \setminus A, \mathcal{F}) \longrightarrow H^1_A(Z, \mathcal{F}),$$

from which the desired conclusion follows.

Now, since L is globally generated, there are finitely many globally defined holomorphic sections $\sigma_0, \sigma_1, \dots, \sigma_N$ in L without common zeros and such that each zero set $Z(\sigma_j)$ is a rare analytic set in X.

Let σ be one of the sections $\sigma_0, \sigma_1, \dots, \sigma_N$. Put $A = Z(\sigma)$ and $U = X \setminus A$. Define $\mathcal{Q}(U)$ as the subset of holomorphic functions on U which are obtained taking quotients of restrictions of sections of $\Gamma(X, L)$ to U over $\sigma|_U$.

Claim 2. The set $\mathcal{Q}(U)$ is closed in $\mathcal{O}(U)$.

To check this, let $\{(s_\nu/\sigma)|_U\}_\nu$ be a sequence in $\mathcal{Q}(U)$ that converges uniformly on compact subsets of U to a holomorphic function $f \in \mathcal{O}(U)$. Let $\mathcal{U} = \{U_i\}_i$ be a trivializing covering of X by open (Stein) subsets. Let $\{\xi_{ij}\}_{ij}$ be the defining cocycle of L. Therefore, s_ν is given on U_i by $\{s_\nu^{(i)}\}_\nu \subset \mathcal{O}(U_i)$ such that $s_\nu^{(i)} = \xi_{ij} s_\nu^{(i)}$; similarly σ is given by $\sigma^{(i)} \in \mathcal{O}(U_i)$. Because the sequence $\{(s_\nu^{(i)}/\sigma)|_{U_i \setminus A}\}_\nu$ converges to $f|_{U_i \setminus A}$, it follows from Claim 1 that there are holomorphic functions $s^{(i)} \in \mathcal{O}(U_i)$ such that $f = s^{(i)}\sigma^{(i)}$ on $U_i \setminus A$.

It follows that $s^{(i)} = \xi_{ij} s^{(j)}$ on $U_i \cap U_j \setminus A$. Since A is rare and X reduced, the above equality holds true on $U_i \cap U_j$. Thus $\{s^{(i)}\}_i$ patch to a section $s \in \Gamma(X, L)$ and, consequently, $f \in \mathcal{Q}(U)$.

Finally, the proof of Proposition 4.1 will be concluded thanks to Theorem 2.1 and the above claim. □

Here we give the second key step in the proof of Theorem 1.2.

Proposition 4.2. *Let X be a complex space of dimension n and let L be a holomorphic line bundle on X. Let $\sigma_0, \sigma_1, \ldots, \sigma_m$ be globally defined holomorphic sections of L, which give a discrete holomorphic mapping into some $\mathbb{P}^k$, with $k \geqslant n$.*

Then, for any $\varepsilon > 0$ and any matrix $A = (a_{ij})$ of type $(n+1) \times (k+1)$, there is a matrix $B = (b_{ij})$ of the same type and rank $n+1$ such that $|b_{ij} - a_{ij}| < \varepsilon$ for all i, j, and sections $\sum\limits_{j=0}^{k} b_{ij}\sigma_j$, $0 \leqslant i \leqslant n$, give a discrete holomorphic mapping into $\mathbb{P}^n$.

Before getting involved with the proof, let us mention a few words about the Hausdorff measure (see Shiffman [9]).

Let A be a subset of a metric space X. Let $\delta(A)$ denote the diameter of A, and let

$$\delta^0(A) := \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset, \end{cases}$$

and

$$\delta^p(A) = (\delta(A))^p \quad \text{for } p > 0.$$

For $p \geqslant 0$ and $\varepsilon > 0$ define

$$h^p_\varepsilon(A) := \inf\left\{\sum_{n=1}^{\infty} \delta^p(A_n);\ A \subset \bigcup_{n=1}^{\infty} A_n \text{ and } \delta(A_n) < \varepsilon\right\},$$
$$h^p(A) = \lim_{\varepsilon \to 0} H^p_\varepsilon(A) = \sup_{\varepsilon > 0} H^p_\varepsilon(A).$$

We refer to h^p as the *Hausdorff p-measure*. For any $A \subset X$, $h^0(A)$ equals the number of points of A. These set functions are regular metric outer measures, and hence the Borel sets are h^p-measurable.

If A is h^p-σ-finite, then $h^r(A) = 0$ whenever $r > p$. If M is an n-dimensional Riemannian manifold, then M is h^n-σ-finite, the compact subsets of M have finite Hausdorff n-measure, and hence, the open sets have nonzero Hausdorff n-measure. (For a nonnegative integer k, the usual notion of k-volume in a Riemannian manifold differs from h^k by a universal multiplicative constant c_k, which is the volume of the ball centered at the origin and of radius $1/2$.) It follows that k-dimensional analytic sets have zero Hausdorff $(2k+1)$-measure, since a k-dimensional analytic set is an at most countable union of manifolds of real dimension at most $2k$.

We shall also need a few things concerning the rank of a holomorphic map (see Siu [10]).

Let $\pi\colon X \longrightarrow Y$ be a holomorphic map between reduced complex spaces. Let X^0 be the manifold points of X. By using local embeddings of Y into a suitable Euclidean complex space, we define $\operatorname{rank}(\pi)$ by

$$\operatorname{rank}(\pi) = \sup_{x \in X^0} \operatorname{rank} J_x(\pi),$$

where $J_x(\pi)$ is the complex jacobian matrix of the corresponding holomorphic map $\iota \circ \pi \circ \psi$ at 0; here ψ maps biholomorphically an open neighborhood of $0 \in \mathbb{C}^p$ onto a neighborhood of x and ι is a closed holomomorphic embedding of a neighborhood of $\pi(x)$ into some open subset of $\mathbb{C}^q$. It can be seen that this definition does not depend on the particular choices.

Alternatively, we can give a rank definition à la Remmert as follows. First assume that X is irreducible and set

$$\varrho(\pi) = \dim(X) - \min_{x \in X} \dim_x \pi^{-1}(\pi(x)).$$

In general, if $\{X_i\}_i$ are the irreducible components of X, we put

$$\varrho(\pi) := \sup_i \varrho(\pi|_{X_i}).$$

It can be shown that the two definitions agree, that is, $\operatorname{rank}(\pi) = \varrho(\pi)$. The next result is standard.

Lemma 4.1. *Let $\pi\colon X \longrightarrow Y$ be a holomorphic map between reduced complex spaces of rank q. Then the image of π, $\pi(X)$, is an at most countable union of locally analytic subsets of Y of dimension less than or equal to q.*

We shall also recall a few simple facts from linear algebra. For $p, q \in \mathbb{N}$, denote, as usual, by $M(p,q)$, the complex vector space of type $p \times q$ matrices. Let $\mathrm{St}(p,q)$ be the subset of $M(p,q)$ consisting of the matrices of rank $\min\{p,q\}$. It is an open subset of $M(p,q)$ and is called the Stieffel manifold of type (p,q).

For $A = (a_{ij})_{ij} \in M(p,q)$ put $\|A\| = \max\limits_{i,j} |a_{ij}|$. Observe that if $A \in M(p,q)$ and $B \in M(q,r)$, then $\|AB\| \leqslant q\|A\|\|B\|$.

Then, using block decomposition, one shows that, for $p \leqslant n \leqslant m$, the natural map

$$M(p,n) \times M(n,m) \longrightarrow M(p,m)$$

induced by matrix multiplication is surjective.

For $A \in \mathrm{St}(k,k+1)$ the kernel of the induced linear map from $\mathbb{C}^{k+1}$ into $\mathbb{C}^k$, which is surjective, is a complex line; therefore it defines a point $\xi_A \in \mathbb{P}^k$. Consequently, we obtain a holomorphic map

$$\Phi\colon\ \mathrm{St}(k,k+1) \longrightarrow \mathbb{P}^k,$$

which is, in fact, a $\mathrm{Gl}(k)$-principal bundle over $\mathbb{P}^k$. Furthermore, multiplication by A induces also a canonical projection map

$$\varphi_A\colon\ \mathbb{P}^k \setminus \{\xi_A\} \longrightarrow \mathbb{P}^{k-1},$$

which is a $\mathbb{C}^\star$-principal bundle over $\mathbb{P}^{k-1}$.

Now, if $\Sigma \subset \mathbb{P}^k$ is an at most countable union of locally analytic subsets Σ_i of $\mathbb{P}^k$ (that is, each Σ_i is closed analytic in some open subset Ω_i of $\mathbb{P}^k$) of dimension less than or equal to n, with $n < k$, then the subset $\mathcal{A}_1$ of matrices $A \in \mathrm{St}(k,k+1)$ for which $\xi_A \in \Sigma$, has σ-finite Hausdorff $2p$-measure, where $p := k^2 + n$. In particular, $\mathcal{A}_1$ has zero Hausdorff $(2k^2 + 2k - 1)$-measure.

Lemma 4.2. *The set $\mathcal{A}_2$ of matrices $A \in \mathrm{St}(k,k+1)$ for which $\xi_A \notin \Sigma$ and φ_A is not discrete on Σ has σ-finite Hausdorff $2q$-measure, where $q = k^2 + k - 1$. In particular, $\mathcal{A}_2$ has zero Hausdorff $(2k^2 + 2k - 1)$-measure.*

P r o o f. Thanks to the fact that each fiber $\varphi_A^{-1}(\varphi_A(x))$ with $x \in \Sigma$ is one dimensional, $\mathcal{A}_2$ consists of the matrices $A \in \mathrm{St}(k,k+1) \setminus \mathcal{A}_1$ such that there is $x_0 \in \Sigma$ and a nonempty open neighborhood W of x_0 such that $\varphi_A^{-1}(\varphi_A(x_0)) \cap W \subset \Sigma \cap W$.

Therefore, the proof is a straightforward consequence of the following lemma (see also the paper of Grauert [6], pages 249–250). $\square$

Lemma 4.3. *Let Ω be a nonempty domain (open and connected) of $\mathbb{C}^n$ and let Σ be a proper analytic subset of Ω. Then the subset Λ of Ω made of all $z \in \Omega$ for which there is a point $z_0 \in \Sigma$ and $\varepsilon > 0$ (which might depend on z_0), such that $z_0 + tz \in \Sigma$ for all $t \in \mathbb{C}$, $|t| < \varepsilon$, has zero Hausdorff $(2n-1)$-measure.*

P r o o f. We show that Λ is the image of an analytic subset of $\Omega \times \Sigma$ through a holomorphic map of rank less than n. Consider, for this, the subset T of the product space $\Omega \times \Sigma$ consisting of couples (z, a), for which there is $\varepsilon(a) > 0$ such that $a + tz$ lies in Σ for all $t \in \mathbb{C}$ with $|t| < \varepsilon(a)$.

We claim that T is an analytic subset of $\Omega \times \Sigma$ and, if we denote by π the restriction to T of the first projection map from $\Omega \times \Sigma$ onto Ω, then $\operatorname{rank}(\pi) \leqslant n - 1$.

Clearly $\Lambda = \pi(T)$. Hence, it suffices to test the claim from above. While the analyticity part is more or less standard (so, it is left to the interested reader), let us focus on the $\operatorname{rank}(\pi)$ part.

Let $\zeta_0 = (z_0, a_0)$ be a manifold point of T, where $\operatorname{rank}_{\zeta_0}(\pi) = n$; thus, in an open neighborhood U of ζ_0 in $\Omega \times \Sigma$, one has $\operatorname{rank}_{\zeta}(\pi) = n$ for $\zeta \in U \cap T$.

Hence, after shrinking U about ζ_0 if necessary, $\pi|_{U \cap T}$ admits a local holomorphic inverse θ, which is defined on an open neighborhood V of z_0 in Ω onto $U \cap T$. But, θ has the form $\theta(z) = (z, \gamma(z))$, $z \in V$, with $\gamma\colon\ V \longrightarrow \Sigma$ holomomorphic. Now, let $V^{\star}$ be a nonempty open neighborhood of z_0, whose closure M is compact. Then, by compactness argument, there is $\varepsilon > 0$ such that for all $z \in M$ and $t \in \mathbb{C}$ with $|t| < \varepsilon$, one has $(z, z + t\gamma(z)) \in U$.

On the one hand, for each $z \in M$ there is $\varepsilon(z) > 0$ such that $z + t\gamma(z) \in \Sigma$ for $t \in \mathbb{C}$ with $|t| < \varepsilon(z)$. The Identity theorem for irreducible analytic sets then implies that $z + t\gamma(z) \in \Sigma$ for $z \in V^{\star}$ and $t \in \mathbb{C}$ with $|t| < \varepsilon$. On the other hand, there is $t_0 \in \mathbb{C}$ with $|t_0| < \varepsilon$ and such that the jacobian matrix of $V^{\star} \ni z \mapsto z + t_0\gamma(z) \in \Omega$ is not zero, so that, for this t_0, the corresponding map is open showing that Σ contains a nonempty open subset of Ω, which contradicts the hypothesis. □

Remark 4.1. Note that $\mathcal{A}_1 \cup \mathcal{A}_2$ has zero Hausdorff $(2k^2 + 2k - 1)$-measure. In particular, its complement in $\mathrm{St}(k, k+1)$ will be dense.

E n d o f p r o o f of Proposition 4.2. We do this by induction. Since the case $k = n$ is obvious, let us assume that the statement holds true for dimensions k', $n \leqslant k' < k$.

In order to complete the induction step, let us observe the following. Let $A \in M(n+1, k+1)$ and let $\varepsilon > 0$. There are $B' \in M(n+1, k)$ and $C' \in M(k, k+1)$ such that $A = B'C'$ (see the discussion after Lemma 4.2).

Now, let $C \in \mathrm{St}(k, k+1)$ be such that $C\sigma\colon\ X \longrightarrow \mathbb{P}^{k-1}$ is well-defined, discrete and $\|C - C'\| < \delta$ (δ will be chosen later in the proof). Here and afterwards, by $C\sigma$ we mean the canonical map into $\mathbb{P}^{k-1}$ induced by sections $\sum\limits_{j=0}^{k} c_{ij}\sigma_j$, $0 \leqslant i < k$, where $C = (c_{ij})_{ij}$.

Then, select $B \in \mathrm{St}(n+1,k)$ such that $B(C\sigma)\colon X \longrightarrow \mathbb{P}^n$ is well-defined and discrete, $\|B - B'\| < \delta$. The choices of C and B are possible by Remark 4.1 and the induction step, respectively.

It follows that $\|BC - A\| \leqslant k(\|B - B'\|\|C\|) + \|B'\|\|C - C'\|$, which in turn is $\leqslant k\delta(\delta + \|B'\| + \|C'\|)$ and this can be made less than ε by choosing $\delta > 0$ small enough, whence the proof of the proposition. □

End of proof of Theorem 1.2. This is obvious from propositions Proposition 4.1 and Proposition 4.2. □

In the circle of ideas discussed up to now, in closing this section, we mention, for the sake of completeness, the following proposition.

Proposition 4.3. *Let X be a complex space of dimension n and let L be a holomorphic line bundle over X such that each point of X is separated by $\Gamma(X,L)$. Then there exists a dense Zariski open subset in X on which $\Gamma(X,L)$ gives local coordinates.*

Proof of Proposition 4.3, beginning. We show that the subset A of X of all points at which $\Gamma(X,L)$ does not give local coordinates is closed and analytic; moreover, as $\Gamma(X,L)$ locally separates points in X, A is rare (that is, A has empty interior).

For this, we need the following lemma.

Lemma 4.4. *Let $f\colon Y \longrightarrow Z$ be a holomorphic map of complex spaces. Then the subset Σ_f of Y defined by*

$$\Sigma_f = \{y \in Y\,;\ f \text{ is not an immersion at } y\}$$

is analytic. Moreover, if for every irreducible component Y_i of Y there exists a point $y_i \in Y_i$ which is isolated in $f_i^{-1}(f_i(y_i))$, where $f_i : Y_i \longrightarrow Z$ is the restriction of f to Y_i, then Σ_f is rare. In particular, this holds if f has discrete fibers.

Proof. To see that Σ_f is analytic we proceed as follows. The holomorphic map f induces a sheaf morphisms

$$df\colon\ f^\star(\Omega^1_Z) \longrightarrow \Omega^1_Y$$

of $\mathcal{O}_Y$-modules[2]. Let $\Omega^1_{Y|Z}$ be the cokernel of df. This is a coherent analytic sheaf on Y. The important fact is that f is an immersion at a point $y \in Y$ if and only if

[2] For a complex space W, we let Ω^1_W be the sheaf of germs of holomorphic 1-forms on W, which is a coherent analytic sheaf on W.

the stalk at y of $\Omega^1_{Y|Z}$ vanishes; in other words

$$\Sigma_f = \mathrm{Supp}(\Omega^1_{Y|Z}),$$

whence the analyticity of Σ_f.

For the "moreover" part, it is sufficient to treat the case when Y is irreducible. Because Σ_f is analytic, the assertion follows if we find a nonempty open set U of Y such that $U \cap \Sigma_f$ is rare. To show that it exists, let $a \in Y$ be an isolated point in $f^{-1}(f(a))$. Then there are open neighborhoods U and V of a and $f(a)$, respectively, such that $f(U) \subset V$ and the induced map $f|_U \colon U \longrightarrow V$ is a branched covering. Since $U \cap \Sigma_f = \Sigma_{f|U}$, the lemma follows. □

P r o o f of Proposition 4.3, concluded. First, notice that A is a closed subset of X. In order to show that A is analytic, let $\sigma_0 \in \Gamma(X, L)$, $\sigma_0 \neq 0$. Let $Y = X \setminus Z(\sigma_0)$. Consider $\mathcal{O}^\sharp(Y)$, the set of those holomorphic functions on Y that are quotients of $\sigma|_Y$ over $\sigma_0|_Y$. Let $\Sigma^\star$ denote the intersection of Σ_f, performed over all holomorphic maps $f\colon Y \longrightarrow \mathbb{C}^m$ with $m \in \mathbb{N}$, whose components are in $\mathcal{O}^\sharp(Y)$. Since each Σ_f is analytic due to Lemma 4.4, $\Sigma^\star$ is also analytic.

The proof will be concluded if we check that $A \cap Y = \Sigma^\star$, because X admits an open covering by such sets Y and A is closed. To verify the above equality, observe that the inclusion "$\supset$" is obvious; as for "$\subset$", one uses the straightforward fact that, if a holomorphic map $g\colon Y \longrightarrow \mathbb{C}^m$ gives local coordinates at a point $x_0 \in Y$, then for any $h \in \mathcal{O}^\star(Y)$, the map from Y into $\mathbb{C}^{m+1}$, given by

$$x \mapsto \Big(\frac{1}{h(x)}, h(x)g_1(x), \ldots, h(x)g_m(x)\Big),$$

gives also local coordinates at x_0.

Finally, the fact that A is a rare analytic set is an immediate consequence of Definition 2.3 and Lemma 4.4. □

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