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ON THE k-POLYGONAL NUMBERS AND THE MEAN VALUE OF DEDEKIND SUMS

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Abstract. For any positive integer $k \geqslant 3$, it is easy to prove that the k-polygonal numbers are $a_n(k) = (2n+n(n-1)(k-2))/2$. The main purpose of this paper is, using the properties of Gauss sums and Dedekind sums, the mean square value theorem of Dirichlet L-functions and the analytic methods, to study the computational problem of one kind mean value of Dedekind sums $S(a_n(k)\overline{a}_m(k), p)$ for k-polygonal numbers with $1 \leqslant m, n \leqslant p-1$, and give an interesting computational formula for it.

Keywords: Dedekind sums; mean value; computational problem; k-polygonal number; analytic method

MSC 2010: 11L05, 11L10

1. Introduction

It is well known that Pythagoreans linked numbers with geometry. Pythagoras introduced the idea of polygonal numbers: triangular numbers, square numbers, pentagonal numbers, etc. The reason for this geometrical nomenclature is clear when the numbers are represented by dots arranged in the form of triangles, squares, pentagons, etc., as shown in following figure.

For any positive integer $k \geqslant 3$, it is easy to prove that the k-polygonal numbers are $a_n(k) = (2n + n(n-1)(k-2))/2$. For example, $a_1(3) = 1$, $a_2(3) = 3$, $a_3(3) = 6$, $a_4(3) = 10$, $a_5(3) = 15$, $a_6(3) = 21$, $a_7(3) = 28$, $\ldots$ are the triangular numbers. $a_1(5) = 1$, $a_2(5) = 5$, $a_3(5) = 12$, $a_4(5) = 22$, $a_5(5) = 35$, $a_6(5) = 51$, $a_7(5) = 70$, $\ldots$ are the pentagonal numbers.

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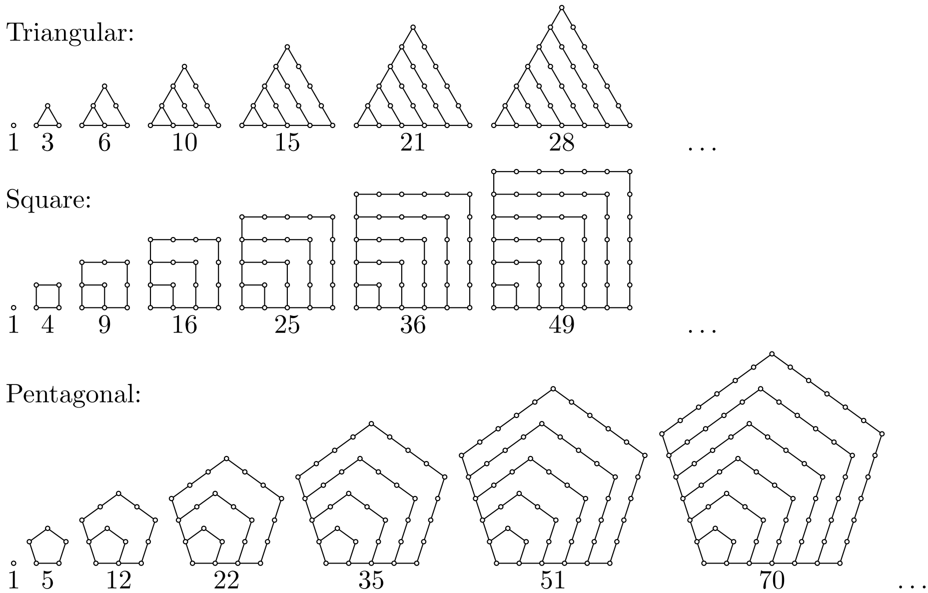

In this paper, we use the analytic methods and the properties of Gauss sums to study the mean value properties of Dedekind sums on the sequence $\{a_n(k)\}$, and give an exact computational formula for it. For convenience, we first introduce the definition of Dedekind sums:

Let q be a natural number and h an integer prime to q. The classical Dedekind sums

$$S(h,q)=\sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer,}\end{cases}$$

describe the behaviour of the logarithm of the eta-function (see [7], [8]) under modular transformations. Many authors have studied the arithmetical properties of $S(h,q)$, and obtained many interesting results, some of which can be found in [2]–[11]. Perhaps the most famous property of Dedekind sums is the reciprocity formula (see references [2], [3], [6] and [7]):

$$S(h,k)+S(k,h)=\frac{h^2+k^2+1}{12hk}-\frac{1}{4} \tag{1}$$

for all $(h,k)=1$, $h>0$ and $k>0$.

An interesting three term version of (1) was also discovered by Rademacher and Grosswald [8].

The main purpose of this paper is to consider the computational problem of the mean value of Dedekind sums

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} S(a_n(k)\,\overline{a}_m(k), p), \tag{2}$$

where $a_m(k)\,\overline{a}_m(k) \equiv 1 \bmod p$, providing $\overline{a}_n(k) = 0$, if $p \mid a_n(k)$. Here $k \geqslant 3$ is a fixed integer, and p is an odd prime.

We will use the analytic method and the properties of Gauss sums to give an exact computational formula for (2). That is, we shall prove the following:

Theorem. *Let p be an odd prime. Then for any fixed integer $k \geqslant 3$ we have the computational formula*

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} S(a_n(k)\,\overline{a}_m(k), p)$$
$$= \begin{cases} 0, & \text{if } p \mid k-2; \\ 0, & \text{if } p \mid k-4 \text{ and } p \equiv 1 \bmod 4; \\ (p-1)h_p^2, & \text{if } p \mid k-4 \text{ and } p \equiv 3 \bmod 4; \\ \dfrac{(p-1)(p-2)}{12} - \dfrac{1-(-1)^{(p-1)/2}}{2}h_p^2, & \text{otherwise,} \end{cases}$$

where h_p denotes the class number of the quadratic field $\mathbb{Q}(\sqrt{-p})$.

For a general integer $q > 3$, whether there exists a computational formula similar to that of our theorem is an unsolved problem that we will further study.

2. Several lemmas

To complete the proof of our theorem, we need to prove several lemmas. Hereinafter, we shall use some properties of Gauss sums and Dirichlet L-functions, all of which can be found in reference [1], so they will not be repeated here.

Lemma 1. *Let $q > 2$ be an integer. Then for any integer a with $(a, q) = 1$ we have the identity*

$$S(a,q) = \frac{1}{\pi^2 q}\sum_{d\mid q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(a)|L(1,\chi)|^2,$$

where $L(1,\chi)$ denotes the Dirichlet L-function corresponding to the character $\chi \bmod d$.

Proof. See Lemma 2 of [10]. □

Lemma 2. *Let p be an odd prime, χ any primitive character χ* mod p*. Then for any fixed positive integer $k \geqslant 3$ we have the identity*

$$\sum_{n=1}^{p-1} \chi(a_n(k)) = \begin{cases} \dfrac{\overline{\chi}(2)\,\tau(\chi)}{\tau(\overline{\chi})}\,\overline{\chi}(k-2)(p-1), & \textit{if } \chi = \chi_2 \text{ and } p \mid k-4; \\ \dfrac{\overline{\chi}(2)\,\tau(\chi)\,\tau(\overline{\chi}^2)}{\tau(\overline{\chi})}\,\overline{\chi}(k-2)\chi^2(4-k), & \textit{otherwise,} \end{cases}$$

where $\tau(\chi) = \sum\limits_{a=1}^{p-1} \chi(a)e(a/p)$ denotes the classical Gauss sums, χ_2 denotes the Legendre symbol and $e(y) = \mathrm{e}^{2\pi \mathrm{i} y}$.

P r o o f. For any primitive character χ mod p, from the definition of $a_n(k)$ and the properties of Gauss sums we have

$$\begin{aligned} \sum_{n=1}^{p-1} \chi(a_n(k)) &= \sum_{n=1}^{p-1} \chi\Big(\frac{1}{2}(2n + n(n-1)(k-2))\Big) \\ &= \overline{\chi}(2) \sum_{n=1}^{p-1} \chi(n)\chi(2 + (n-1)(k-2)) \\ &= \frac{\overline{\chi}(2)}{\tau(\overline{\chi})} \sum_{n=1}^{p-1} \chi(n) \sum_{b=1}^{p-1} \overline{\chi}(b) e\Big(\frac{b(2+(n-1)(k-2))}{p}\Big) \\ &= \frac{\overline{\chi}(2)}{\tau(\overline{\chi})} \sum_{b=1}^{p-1} \overline{\chi}(b) e\Big(\frac{b(4-k)}{p}\Big) \sum_{n=1}^{p-1} \chi(n) e\Big(\frac{b(k-2)n}{p}\Big) \\ &= \frac{\overline{\chi}(2)\,\tau(\chi)}{\tau(\overline{\chi})} \sum_{b=1}^{p-1} \overline{\chi}(b)\overline{\chi}(b(k-2)) e\Big(\frac{b(4-k)}{p}\Big). \end{aligned} \tag{3}$$

If $\chi = \chi_2$ is the Legendre symbol and $p \mid k-4$, then from (3) we know that

$$\sum_{n=1}^{p-1} \chi(a_n(k)) = \frac{\overline{\chi}(2)\,\tau(\chi)}{\tau(\overline{\chi})}\,\overline{\chi}(k-2)(p-1). \tag{4}$$

Otherwise, from (3) we have

$$\sum_{n=1}^{p-1} \chi(a_n(k)) = \frac{\overline{\chi}(2)\,\tau(\chi)\,\tau(\overline{\chi}^2)}{\tau(\overline{\chi})}\,\overline{\chi}(k-2)\,\chi^2(4-k). \tag{5}$$

Now combining (4) and (5) we complete the proof of Lemma 2. $\square$

3. Proof of the theorem

In this section, we shall complete the proof of our theorem. First, if p is an odd prime, then from Lemma 1 and the definition of $S(a,p)$ we have

$$S(a,p)=\frac{1}{\pi^2}\frac{p}{p-1}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(a)|L(1,\chi)|^2 \tag{6}$$

and

$$S(1,p)=\sum_{a=1}^{p-1}\left(\frac{a}{p}-\frac{1}{2}\right)^2=\frac{(p-1)(p-2)}{12p}. \tag{7}$$

Combining (6) and (7) we can deduce that

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2=\frac{\pi^2}{12}\frac{(p-1)^2(p-2)}{p^2}. \tag{8}$$

It is clear that if $p \mid k-2$, then from (6) and Lemma 2 we have the identity

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}S(a_n(k)\,\overline{a}_m(k),p)=0. \tag{9}$$

If $p \mid k-4$ and $p\equiv 1 \bmod 4$, then the Legendre symbol χ_2 is not an odd character $\bmod\, p$. So for any odd character $\chi \bmod p$, χ^2 is not the principal character $\bmod\, p$. From (6) and Lemma 2 we have the identity

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}S(a_n(k)\,\overline{a}_m(k),p)=0. \tag{10}$$

If $p \mid k-4$ and $p\equiv 3 \bmod 4$, then the Legendre symbol χ_2 is an odd character $\bmod\, p$. Note that $|\tau(\chi)|=\sqrt{p}$ if χ is not the principal character. From (6) and Lemma 2 we have the identity

$$\begin{aligned}\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}&S(a_n(k)\,\overline{a}_m(k),p)\\ &=\frac{1}{\pi^2}\frac{p}{p-1}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(a_m(k))\overline{\chi}(a_n(k))|L(1,\chi)|^2\\ &=\frac{1}{\pi^2}\frac{p}{p-1}(p-1)^2|L(1,\chi_2)|^2=(p-1)h_p^2,\end{aligned} \tag{11}$$

where we have used the identity $|L(1,\chi_2)| = h_p(\pi/\sqrt{p})$, and h_p denotes the class number of the quadratic field $\mathbb{Q}(\sqrt{-p})$.

If $(p,k-2)=(p,k-4)=1$ and $p\equiv 1 \bmod 4$, then from (6), (8) and Lemma 2 we have the identity

$$
\begin{aligned}
(12)\qquad &\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} S(a_n(k)\,\overline{a}_m(k),p)\\
&=\frac{1}{\pi^2}\frac{p}{p-1}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(a_m(k))\overline{\chi}(a_n(k))|L(1,\chi)|^2\\
&=\frac{1}{\pi^2}\frac{p^2}{p-1}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2=\frac{(p-1)(p-2)}{12}.
\end{aligned}
$$

If $(p,k-2)=(p,k-4)=1$ and $p\equiv 3 \bmod 4$, then note that $\tau(\chi_2^2)=-1$, and from (6), (8) and Lemma 2 we have the identity

$$
\begin{aligned}
(13)\qquad &\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} S(a_n(k)\,\overline{a}_m(k),p)\\
&=\frac{1}{\pi^2}\frac{p}{p-1}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\chi(a_m(k))\overline{\chi}(a_n(k))|L(1,\chi)|^2\\
&=\frac{1}{\pi^2}\frac{p^2}{p-1}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1\\ \chi\neq\chi_2}}|L(1,\chi)|^2+\frac{1}{\pi^2}\frac{p}{p-1}|L(1,\chi_2)|^2\\
&=\frac{1}{\pi^2}\frac{p^2}{p-1}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|L(1,\chi)|^2+\frac{1}{\pi^2}\frac{p-p^2}{p-1}|L(1,\chi_2)|^2\\
&=\frac{(p-1)(p-2)}{12}-h_p^2.
\end{aligned}
$$

Note that $(1-(-1)^{(p-1)/2})/2=1$, if $p\equiv 3 \bmod 4$, and $(1-(-1)^{(p-1)/2})/2=0$, if $p\equiv 1 \bmod 4$. Combining (9)–(13) we immediately complete the proof of our theorem. □

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