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THE SECOND ORDER PROJECTION METHOD IN TIME FOR THE TIME-DEPENDENT NATURAL CONVECTION PROBLEM

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Abstract. We consider the second-order projection schemes for the time-dependent natural convection problem. By the projection method, the natural convection problem is decoupled into two linear subproblems, and each subproblem is solved more easily than the original one. The error analysis is accomplished by interpreting the second-order time discretization of a perturbed system which approximates the time-dependent natural convection problem, and the rigorous error analysis of the projection schemes is presented. Our main results of the second order projection schemes for the time-dependent natural convection problem are that the convergence for the velocity and temperature are strongly second order in time while that for the pressure is strongly first order in time.

Keywords: natural convection problem; projection method; stability; convergence

MSC 2010: 65N15, 65N30, 76D07

1. Introduction

The aim of this paper is to establish the convergence of the second-order projection schemes in time for the time-dependent natural convection problem. One major difficulty for the numerical simulation of the incompressible flows is that the velocity and the pressure are coupled by the incompressibility constraint. However, the projection method is an efficient numerical scheme for the incompressible flows, we can refer to the ground breaking works of Chorin [3] and Temam [18]. The most attractive feature of the projection method is that, at each time step, one only needs to solve a sequence of decoupled elliptic equations for different variables; as a consequence, the computational scale is reduced and a lot of CPU time is saved. For example, we

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can refer to [1], [2], [10], [13], [14], [16], [20] for the Navier-Stokes equations. Due to the efficiency of the projection schemes, we consider the second order projection schemes in this paper for the following time-dependent natural convection problem.

$$
(1.1)\qquad \begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = -k\nu^2\mathbf{j}\theta + \mathbf{f} & \text{in } \Omega\times(0,T],\\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega\times(0,T],\\ \theta_t - \lambda\nu\Delta\theta + \mathbf{u}\cdot\nabla\theta = g & \text{in } \Omega\times(0,T],\\ \mathbf{u} = 0,\ \theta = 0 & \text{on } \Gamma\times(0,T],\\ \mathbf{u}(x,0) = \mathbf{u}_0,\ \theta(x,0) = \theta_0 & \text{on } \Omega\times\{0\}, \end{cases}
$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain assumed to have a Lipschitz continuous boundary Γ. Further, $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$ is the fluid velocity, p is the pressure, θ is the temperature, $\nu > 0$ is the viscosity, k is the Groshoff number, $\lambda = \mathrm{Pr}^{-1}$, Pr is the Prandtl number, $\mathbf{j} = (1,0)^{\mathrm{T}}$ is the vector of gravitational acceleration, $T > 0$ is the final time, $\mathbf{f}$ and g are forcing functions.

The natural convection problem (1.1) is an important system which not only contains the incompressibility and strong nonlinearity, but also includes the coupling between the energy equation and the equations governing the fluid motion. Since this system not only comprises the velocity and pressure but also includes the temperature filed, finding the numerical solutions of problem (1.1) becomes a difficult task. Many authors have worked on this problem. Let us mention for example, the standard Galerkin finite element method (FEM) [9], [15], the projection-based stabilized mixed FEM [4], variational multiscale method [5], [21], [24], and the references therein. Here, we need to point out that all these numerical schemes for problem (1.1) are coupled. It means that we need to find the variables $\mathbf{u}, p$, and θ of (1.1) simultaneously, therefore, a large nonlinear algebra system is formed. Generally speaking, it is expensive to find the numerical solutions of the coupled nonlinear system directly by the standard Galerkin FEM. In order to overcome this difficulty, Zhang and his co-authors considered the decoupled schemes for the natural convection problem in [22], [23], [25], and some meaningful results have been established. Recently, Qian and Zhang in [11], [12] considered the first order and higher order projection schemes for the time-dependent natural convection problem.

By the projection schemes, problem (1.1) is decoupled into two small linear subproblems, and each subproblem is solved more easily than the original one. For instance, the following modified projection schemes are analyzed in [11]:

$$
(1.2)\qquad \begin{cases} \dfrac{\widetilde{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\Delta t} - \nu\Delta\widetilde{\mathbf{u}}^{n+1} + (\mathbf{u}^n\cdot\nabla)\widetilde{\mathbf{u}}^{n+1} + \nabla\phi^n = -k\nu^2\mathbf{j}\theta^{n+1} + \mathbf{f}(t_{n+1}),\\ \widetilde{\mathbf{u}}^{n+1}|_\Gamma = 0,\\ \dfrac{\theta^{n+1} - \theta^n}{\Delta t} - \lambda\nu\Delta\theta^{n+1} + (\mathbf{u}^n\cdot\nabla)\theta^{n+1} = g(t_{n+1}), \end{cases}
$$

and

$$
(1.3) \qquad \begin{cases} \dfrac{\mathbf{u}^{n+1} - \widetilde{\mathbf{u}}^{n+1}}{\Delta t} + \alpha_1 \nabla(\phi^{n+1} - \phi^n) = 0, \\ \nabla \cdot \mathbf{u}^{n+1} = 0, \\ \mathbf{u}^{n+1} \cdot \vec{\mathbf{n}}|_\Gamma = 0, \end{cases}
$$

where $\Delta t > 0$ is the time step, $t_n = n\Delta t$, ϕ^0 is an approximation of p^0, $\vec{\mathbf{n}}$ is the normal vector to Γ and $\alpha_1 \geqslant 1$.

Set

$$
H = \{\mathbf{u} \in L^2(\Omega)^2, \nabla \cdot \mathbf{u} = 0,\ \mathbf{u} \cdot \vec{\mathbf{n}}|_\Gamma = 0\},
$$

and denote by P_H the orthogonal projector from $L^2(\Omega)^2$ onto H, i.e.,

$$
(1.4) \qquad (\mathbf{u} - P_H \mathbf{u}, \mathbf{v}) = 0 \quad \forall\, \mathbf{u} \in Y = L^2(\Omega)^2,\ \mathbf{v} \in H.
$$

We can readily check that (1.3) is equivalent to $\mathbf{u}^{n+1} = P_H \widetilde{\mathbf{u}}^{n+1}$, which explains why we call (1.2)–(1.3) the projection schemes. In [11], we developed the classical projection schemes and the modified projection schemes for the time-dependent natural convection problem (1.1). For the classical schemes, we established the convergence of weakly first order for the velocity and temperature and of weakly order $\frac{1}{2}$ for the pressure. For the modified schemes, we improved the convergence to strongly first order for the velocity and temperature and weakly first order for the pressure.

Then, in [12], we investigated the higher order projection schemes

$$
(1.5) \qquad \begin{cases} \dfrac{\widetilde{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\Delta t} - \nu \Delta \widetilde{\mathbf{u}}^{n+1/2} + (\mathbf{u}^n \cdot \nabla) \widetilde{\mathbf{u}}^{n+1/2} + \nabla \phi^n \\ \quad = -k\nu^2 \mathbf{j} \theta^{n+1/2} + \mathbf{f}(t_{n+1/2}), \\ \widetilde{\mathbf{u}}^{n+1/2}|_\Gamma = 0, \\ \dfrac{\theta^{n+1} - \theta^n}{\Delta t} - \lambda \nu \Delta \theta^{n+1/2} + (\mathbf{u}^n \cdot \nabla) \theta^{n+1/2} = g(t_{n+1/2}), \end{cases}
$$

and

$$
(1.6) \qquad \begin{cases} \dfrac{\mathbf{u}^{n+1} - \widetilde{\mathbf{u}}^{n+1}}{\Delta t} + \alpha_2 \nabla(\phi^{n+1} - \phi^n) = 0, \\ \nabla \cdot \mathbf{u}^{n+1} = 0, \\ \mathbf{u}^{n+1} \cdot \vec{\mathbf{n}}|_\Gamma = 0, \end{cases}
$$

where $t_{n+1/2} = (n + \frac{1}{2})\Delta t$, $\widetilde{\mathbf{u}}^{n+1/2} = \frac{1}{2}(\widetilde{\mathbf{u}}^{n+1} + \mathbf{u}^n)$, $\theta^{n+1/2} = \frac{1}{2}(\theta^{n+1} + \theta^n)$ and $\alpha_2 > \frac{1}{2}$. We also obtained that $\mathbf{u}^{n+1}$ in (1.6) is uniquely defined by the relation $\mathbf{u}^{n+1} = P_H \widetilde{\mathbf{u}}^{n+1}$. For the higher projection schemes (1.5)–(1.6), we established convergence of strongly $\frac{3}{2}$ order for the velocity and temperature and of weakly one order for the pressure.

In this paper, instead of the projection schemes (1.2)–(1.3) and (1.5)–(1.6), we consider the following numerical schemes for the problem (1.1):

Let $(\mathbf{u}^0, p^0, \theta^0) = (\mathbf{u}(t_0), p(t_0), \theta(t_0)) \in H_0^1(\Omega)^2 \times (H_0^1(\Omega)/\mathbb{R}) \times H_0^1(\Omega)$, find θ^{n+1}, $\mathbf{u}^{n+1}$ and p^{n+1} satisfying

$$(1.7)\qquad \begin{cases} \dfrac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \nu\Delta\mathbf{u}^{n+1/2} + (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1/2} + \nabla p^n \\ \qquad\qquad = -k\nu^2\mathbf{j}\theta^{n+1/2} + \mathbf{f}(t_{n+1/2}), \\ \mathbf{u}^{n+1}|_\Gamma = 0, \\ \dfrac{\theta^{n+1} - \theta^n}{\Delta t} - \lambda\nu\Delta\theta^{n+1/2} + (\mathbf{u}^n \cdot \nabla)\theta^{n+1/2} = g(t_{n+1/2}) \end{cases}$$

and

$$(1.8)\qquad \nabla \cdot \mathbf{u}^{n+1} - \alpha\Delta t(\Delta p^{n+1} - \Delta p^n) = 0,$$

where $\mathbf{u}^{n+1/2} = \frac{1}{2}(\mathbf{u}^{n+1} + \mathbf{u}^n)$, $\theta^{n+1/2} = \frac{1}{2}(\theta^{n+1} + \theta^n)$ and $\alpha > \frac{1}{4}$, $\mathbb{R}$ is the space of real numbers.

From (1.7) we obtain the numerical solutions $\mathbf{u}^{n+1}$ and θ^{n+1}, but $\mathbf{u}^{n+1}$ may not belong to the divergence-free space in such situation. Then we improve and get $(\mathbf{u}^{n+1}, p^{n+1})$ from (1.8). Since p^n and p^{n+1} are two successive iterative solutions, the difference between Δp^{n+1} and Δp^n tends to zero as n increases, hence we have $\nabla \cdot u^{n+1} = 0$ as n increase.

Compared with the other schemes (1.2)–(1.3) and (1.5)–(1.6), the schemes (1.7)–(1.8) have two advantages: (i) The schemes avoid using the medium $\widetilde{\mathbf{u}}^{n+1}$ to reduce the computation time and storage space; (ii) the flexible constant $\alpha > \frac{1}{4}$ is better than in the other two schemes.

Although the schemes (1.7)–(1.8) do not apply the projection step, we still refer to (1.7)–(1.8) as the projection schemes because of its similarity with (1.5)–(1.6). As mentioned before, at each time step, we only need to solve a Helmholtz equation, a parabolic problem and a Poisson equation. Specially, fast Poisson solvers, if available, can be used. Furthermore, since the velocity, temperature, and pressure in the projection schemes are decoupled from each other, the space discretization for the velocity, the temperature, and the pressure can be chosen independently, and they need not satisfy the Babuška-Brezzi or inf-sup condition.

The article is organized as follows: In Section 2, we recall some notation and present some assumptions which enable us to derive some regularity results required by error analysis. Stabilities of the projection schemes are established in Section 3.

In Section 4, we establish rigorously the convergence: $\mathbf{u}^{n+1}$ and θ^{n+1} are strongly second-order approximations to $\mathbf{u}(t_{n+1})$ and $\theta(t_{n+1})$ in $L^2(\Omega)^2$ and $L^2(\Omega)$, respectively, p^{n+1} is strongly first-order approximation to $p(t_{n+1})$ in $H_0^1(\Omega)/\mathbb{R}$.

2. Preliminaries

In this section, we recall some notation and basic results which are frequently used in the sequel. For the mathematical setting of problem (1.1), we introduce the Hilbert spaces

$$X = H_0^1(\Omega)^2,\ W = H_0^1(\Omega),\ Z = L^2(\Omega),\ M = L_0^2(\Omega) = \Big\{\varphi \in L^2(\Omega)\colon \int_\Omega \varphi \,\mathrm{d}x = 0\Big\}.$$

The standard Sobolev spaces are adopted, for example, $(\cdot,\cdot)$ and $\|\cdot\|_0$ are used to denote the usual inner product and norm in Z or Y. The spaces W and X are equipped with the usual scalar product $(\nabla\cdot,\nabla\cdot)$ and the associated norm $\|\nabla\cdot\|_0$. Let the closed subset V of X be given by

$$V = \{\mathbf{v} \in X,\ \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega\}.$$

In order to simplify the description, we set

$$D(A) = H^2(\Omega)^2 \cap V, \quad E(A) = H^2(\Omega) \cap W.$$

Hereafter, we use $N, \overline{N}, C, C_1, C_2, C_3$ to denote generic positive constants which depend only on Ω, ν, λ. Furthermore, M is a generic positive constant which may additionally depend on $\mathbf{u}^0, \theta^0, \mathbf{f}_\infty, g_\infty$, and α, it may stand for different values at different places, where $\mathbf{f}_\infty = \sup\limits_{t\geqslant 0}|f(t)|,\quad g_\infty = \sup\limits_{t\geqslant 0}|g(t)|$.

If Ω is bounded in some direction then the Poincaré inequality holds:

$$\|\mathbf{v}\|_0 \leqslant C_1\|\nabla\mathbf{v}\|_0 \quad \forall\,\mathbf{v} \in X \text{ or } W. \tag{2.1}$$

The trilinear terms for all $\mathbf{u},\mathbf{v},\mathbf{w} \in X$ and $\theta,\psi \in W$ can be defined as follows

$$\begin{aligned} b(\mathbf{u},\mathbf{v},\mathbf{w}) &= ((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) + \tfrac{1}{2}(\mathbf{v}\operatorname{div}\mathbf{u},\mathbf{w}) = \tfrac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) - \tfrac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v}),\\ \overline{b}(\mathbf{u},\theta,\psi) &= ((\mathbf{u}\cdot\nabla)\theta,\psi) + \tfrac{1}{2}(\theta\operatorname{div}\mathbf{u},\psi) = \tfrac{1}{2}((\mathbf{u}\cdot\nabla)\theta,\psi) - \tfrac{1}{2}((\mathbf{u}\cdot\nabla)\psi,\theta). \end{aligned}$$

It is easy to verify that the trilinear terms $b(\cdot,\cdot,\cdot)$ and $\overline{b}(\cdot,\cdot,\cdot)$ are skew-symmetric with respect to their two arguments,

$$b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0 \quad \forall\,\mathbf{u},\mathbf{v} \in X; \quad \overline{b}(\mathbf{u},\theta,\theta) = 0 \quad \forall\,\mathbf{u} \in X, \theta \in W. \tag{2.2}$$

Furthermore, we recall some properties of the trilinear forms $b(\cdot,\cdot,\cdot)$ and $\overline{b}(\cdot,\cdot,\cdot)$ (see [6], [7], [19]).

Theorem 2.1. *The trilinear forms $b(\cdot,\cdot,\cdot)$ and $\overline{b}(\cdot,\cdot,\cdot)$ satisfy:*

(1) *In view of $H^1(\Omega) \hookrightarrow L^4(\Omega)$, we have*

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \leqslant N\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1 \quad \forall\, \mathbf{u},\mathbf{v},\mathbf{w} \in X,$$
$$|\overline{b}(\mathbf{u},\theta,\psi)| \leqslant \overline{N}\|\mathbf{u}\|_1\|\theta\|_1\|\psi\|_1 \quad \forall\, \mathbf{u} \in X,\ \theta,\psi \in W,$$

where

$$N = \sup_{0\neq \mathbf{u},\mathbf{v},\mathbf{w}\in X} \frac{|b(\mathbf{u},\mathbf{v},\mathbf{w})|}{\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1}, \quad \overline{N} = \sup_{0\neq \mathbf{u}\in X, 0\neq\theta,\psi\in W} \frac{|\overline{b}(\mathbf{u},\theta,\psi)|}{\|\mathbf{u}\|_1\|\theta\|_1\|\psi\|_1}.$$

(2) *The following estimates of trilinear terms $b(\cdot,\cdot,\cdot)$ and $\overline{b}(\cdot,\cdot,\cdot)$ hold:*

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \leqslant C_2\|\mathbf{u}\|_1\|\mathbf{v}\|_2\|\mathbf{w}\|_0, \quad |\overline{b}(\mathbf{u},\theta,\psi)| \leqslant C_3\|\mathbf{u}\|_1\|\theta\|_2\|\psi\|_0,$$

for all $\mathbf{u} \in V$, $\mathbf{v} \in D(A)$, $\mathbf{w} \in X$, $\theta \in E(A)$, $\psi \in W$.

With the above notation, for a given $\mathbf{f} \in L^\infty(0,T;Y)$ with $\mathbf{u}_0 \in D(A)$ and $g \in L^\infty(0,T;Z)$ with $\theta_0 \in E(A)$, based on the backward Euler scheme, the weak form of the projection and linearized time discrete schemes for problems (1.7)–(1.8) read as follows: For all $(\mathbf{v},q,\psi) \in X \times M \times W$, find $(\mathbf{u}^{n+1},p^{n+1},\theta^{n+1})$ such that

$$\tag{2.3} \begin{cases} \Big(\dfrac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t},\mathbf{v}\Big) + \nu(\nabla\mathbf{u}^{n+1/2},\nabla\mathbf{v}) + b(\mathbf{u}^n,\mathbf{u}^{n+1/2},\mathbf{v}) + (\nabla p^n,\mathbf{v}) \\ \qquad = -k\nu^2(\mathbf{j}\theta^{n+1/2},\mathbf{v}) + (\mathbf{f}(t_{n+1/2}),\mathbf{v}), \\ \Big(\dfrac{\theta^{n+1}-\theta^n}{\Delta t},\psi\Big) + \lambda\nu(\nabla\theta^{n+1/2},\nabla\psi) + \overline{b}(\mathbf{u}^n,\theta^{n+1/2},\psi) = (g(t_{n+1/2}),\psi) \end{cases}$$

and

$$\tag{2.4} (\nabla\cdot\mathbf{u}^{n+1},q) - \alpha\Delta t(\Delta p^{n+1}-\Delta p^n,q) = 0.$$

The existence and uniqueness of the numerical solutions of problems (2.3)–(2.4) are ensured by the classical Lax-Milgram theorem. We present some assumptions about the initial data and the regularity of the exact solutions. These assumptions are known to cause nonlocal compatibility conditions on the given data as discussed in [8] in the case of the Navier-Stokes equation. As has been stated in Theorem 1 of [17], we are not concerned with the behavior of the solutions near the initial time but confine ourselves to an ideal case.

(A1) $\mathbf{u}_0 \in H^2(\Omega)^2 \cap V$, $\theta_0 \in H^2(\Omega)$, $\mathbf{f} \in L^\infty(0,T;H)$, $g \in L^\infty(0,T;Z)$, $\sup\limits_{t\in[0,T]}(\|\mathbf{u}(t)\|_1 + \|\theta(t)\|_1) \leqslant M$.

(A2) $\sup_{t\in[0,T]}(\|\mathbf{u}_t(t)\|_1 + \|\theta_t(t)\|_1 + \|p_t(t)\|_1) \leqslant M.$

(A3) $\int_0^T (\|\mathbf{u}_{tt}(t)\|_2^2 + \|\theta_{tt}(t)\|_2^2 + \|\mathbf{u}_{ttt}(t)\|_0^2 + \|\theta_{ttt}(t)\|_0^2 + \|p_{tt}(t)\|_1^2)\,\mathrm{d}t \leqslant M.$

Under the assumption (A1), for all $T > 0$ and $0 < t \leqslant T$, the solutions $(\mathbf{u}, p, \theta)$ of problem (1.1) satisfy (see [16], [19], [22])

$$\sup_{t\in[0,T]}(\|\mathbf{u}(t)\|_2 + \|\mathbf{u}_t(t)\|_0 + \|\theta(t)\|_2 + \|\theta_t(t)\|_0 + \|p(t)\|_1) \leqslant M. \tag{2.5}$$

3. Stabilities of the projection schemes

In this section, we consider the stabilities of the projection numerical schemes under some assumptions presented in Section 2.

Theorem 3.1. *Under the assumptions of* (A1)–(A3), *for* $\alpha > \frac{1}{4}$ *and all* $J = 0, 1, \ldots, [T/\Delta t] - 1$, *the following inequalities for schemes* (2.3)–(2.4) *hold:*

$$\Big(1 - \frac{1}{4\alpha}\Big)\|\mathbf{u}^{J+1}\|_0^2 + \nu\Delta t\sum_{n=0}^{J}\|\mathbf{u}^{n+1/2}\|_1^2 \leqslant s_0^2, \quad \|\theta^{J+1}\|_0^2 + \lambda\nu\Delta t\sum_{n=0}^{J}\|\theta^{n+1/2}\|_1^2 \leqslant s_1^2,$$

where $s_0^2 = \|\mathbf{u}^1\|_0^2 + \frac{1}{2}\alpha\Delta t^2(\|p^1\|_1^2 + \|p^0\|_1^2) + 2C_1^2T\mathbf{f}_\infty^2/\nu + 2C_1^4k^2\nu^2s_1^2/\lambda$, $s_1^2 \leqslant \|\theta^1\|_0^2 + C_1^2Tg_\infty^2/(\lambda\nu)$.

P r o o f. We derive from (1.8) that

$$\nabla\cdot(\mathbf{u}^{n+1} + \mathbf{u}^n) - \alpha\Delta t(\Delta p^{n+1} - \Delta p^{n-1}) = 0. \tag{3.1}$$

Consider the inner product of (1.7) with $2\Delta t\mathbf{u}^{n+1/2}$ and $2\Delta t\theta^{n+1/2}$, respectively. Further, take the inner product of (3.1) with Δtp^n, and sum up these relations. Thanks to (2.2) and the algebraic relations

$$(a-b, 2a) = |a|^2 - |b|^2 + |a-b|^2, \quad (a-b, 2b) = |a|^2 - |b|^2 - |a-b|^2, \\ (a-b, a+b) = |a|^2 - |b|^2, \tag{3.2}$$

we derive that

$$\begin{cases} \|\mathbf{u}^{n+1}\|_0^2 - \|\mathbf{u}^n\|_0^2 + 2\nu\Delta t\|\mathbf{u}^{n+1/2}\|_1^2 + \alpha\Delta t^2(\nabla p^{n+1} - \nabla p^{n-1}, \nabla p^n) \\ \qquad = 2\Delta t(\mathbf{f}(t_{n+1/2}), \mathbf{u}^{n+1/2}) - 2k\nu^2\Delta t(\mathbf{j}\theta^{n+1/2}, \mathbf{u}^{n+1/2}), \\ \|\theta^{n+1}\|_0^2 - \|\theta^n\|_0^2 + 2\lambda\nu\Delta t\|\theta^{n+1/2}\|_1^2 = 2\Delta t(g(t_{n+1/2}), \theta^{n+1/2}). \end{cases} \tag{3.3}$$

Using the inequality (3.2), we find

$$
\begin{aligned}
(3.4)\qquad &(\nabla p^{n+1} - \nabla p^{n-1}, \nabla p^n) \\
&= (\nabla p^{n+1} - \nabla p^n, \nabla p^n) + (\nabla p^n - \nabla p^{n-1}, \nabla p^n) \\
&\leqslant \frac{1}{2}(\|p^{n+1}\|_1^2 - \|p^{n-1}\|_1^2 + \|p^n - p^{n-1}\|_1^2 - \|p^{n+1} - p^n\|_1^2).
\end{aligned}
$$

For the right-hand side terms of (3.3) we have

$$
\begin{aligned}
(3.5)\qquad &|-2k\nu^2\Delta t(\mathbf{j}\theta^{n+1/2}, \mathbf{u}^{n+1/2})| \leqslant 2k\nu^2\Delta t\|\theta^{n+1/2}\|_0\|\mathbf{u}^{n+1/2}\|_0 \\
&\qquad \leqslant \frac{\nu\Delta t}{2}\|\mathbf{u}^{n+1/2}\|_1^2 + 2C_1^2k^2\nu^3\Delta t\|\theta^{n+1/2}\|_0^2, \\
&|2\Delta t(\mathbf{f}(t_{n+1/2}), \mathbf{u}^{n+1/2})| \leqslant 2\Delta t\|\mathbf{f}(t_{n+1/2})\|_0\|\mathbf{u}^{n+1/2}\|_0 \\
&\qquad \leqslant \frac{\nu\Delta t}{2}\|\mathbf{u}^{n+1/2}\|_1^2 + \frac{2C_1^2\Delta t}{\nu}\|\mathbf{f}(t_{n+1/2})\|_0^2, \\
&|2\Delta t(g(t_{n+1/2}), \theta^{n+1/2})| \leqslant 2\Delta t\|g(t_{n+1/2})\|_0\|\theta^{n+1/2}\|_0 \\
&\qquad \leqslant \lambda\nu\Delta t\|\theta^{n+1/2}\|_1^2 + \frac{C_1^2\Delta t}{\lambda\nu}\|g(t_{n+1/2})\|_0^2.
\end{aligned}
$$

By the above inequalities together with (3.3) and summing (3.3) for n from 1 to J, we get

$$
\begin{aligned}
(3.6)\qquad &\|\mathbf{u}^{J+1}\|_0^2 + \frac{\alpha\Delta t^2}{2}(\|p^{J+1}\|_1^2 + \|p^J\|_1^2) + \nu\Delta t\sum_{n=1}^{J}\|\mathbf{u}^{n+1/2}\|_1^2 \\
&\leqslant \|\mathbf{u}^1\|_0^2 + \frac{\alpha\Delta t^2}{2}(\|p^1\|_1^2 + \|p^0\|_1^2) + \frac{\alpha\Delta t^2}{2}\|p^{J+1} - p^J\|_1^2 \\
&\quad + 2C_1^2k^2\nu^3\Delta t\sum_{n=0}^{J}\|\theta^{n+1/2}\|_0^2 + \frac{2C_1^2\Delta t}{\nu}\sum_{n=1}^{J}\|\mathbf{f}(t_{n+1/2})\|_0^2,
\end{aligned}
$$

$$
(3.7)\qquad \|\theta^{J+1}\|_0^2 + \lambda\nu\Delta t\sum_{n=1}^{J}\|\theta^{n+1/2}\|_1^2 \leqslant \|\theta^1\|_0^2 + \frac{C_1^2\Delta t}{\lambda\nu}\sum_{n=1}^{J}\|g(t_{n+1/2})\|_0^2.
$$

Let $n = J$. Taking the inner product of (1.8) with $p^{J+1} - p^J$, we get

$$
\begin{aligned}
&\alpha\Delta t\|p^{J+1} - p^J\|_1^2 = (\mathbf{u}^{J+1}, \nabla p^{J+1} - \nabla p^J) \\
&\qquad \leqslant \|\mathbf{u}^{J+1}\|_0\|p^{J+1} - p^J\|_1 \leqslant \frac{1}{2\alpha\Delta t}\|\mathbf{u}^{J+1}\|_0^2 + \frac{\alpha\Delta t}{2}\|p^{J+1} - p^J\|_1^2,
\end{aligned}
$$

which gives the inequality

$$
(3.8)\qquad \alpha\Delta t^2\|p^{J+1} - p^J\|_1^2 \leqslant \frac{1}{\alpha}\|\mathbf{u}^{J+1}\|_0^2.
$$

As a consequence, we have

$$\frac{\alpha\Delta t^2}{2}\|p^{J+1}-p^J\|_1^2 \leqslant \frac{1}{4\alpha}\|\mathbf{u}^{J+1}\|_0^2 + \frac{\alpha\Delta t^2}{4}\|p^{J+1}-p^J\|_1^2 \leqslant \frac{1}{4\alpha}\|\mathbf{u}^{J+1}\|_0^2 + \frac{\alpha\Delta t^2}{2}(\|p^{J+1}\|_1^2+\|p^J\|_1^2). \tag{3.9}$$

Substituting (3.7) and (3.9) into (3.6), we deduce that

$$\Big(1-\frac{1}{4\alpha}\Big)\|\mathbf{u}^{J+1}\|_0^2 + \nu\Delta t\sum_{n=1}^{J}\|\mathbf{u}^{n+1/2}\|_1^2 \leqslant \|\mathbf{u}^1\|_0^2 + \frac{\alpha\Delta t^2}{2}(\|p^1\|_1^2+\|p^0\|_1^2) + \frac{2C_1^4k^2\nu^2}{\lambda}\Big(\|\theta^1\|_0^2 + \frac{C_1^2Tg_\infty^2}{\lambda\nu}\Big) + \frac{2C_1^2T\mathbf{f}_\infty^2}{\nu},$$

$$\|\theta^{J+1}\|_0^2 + \lambda\nu\Delta t\sum_{n=1}^{J}\|\theta^{n+1/2}\|_1^2 \leqslant \|\theta^1\|_0^2 + \frac{C_1^2Tg_\infty^2}{\lambda\nu}.$$

□

Theorem 3.2. *From the projection schemes* (1.7)–(1.8) *at* $n=0$, *the following stabilities for numerical solutions* $(\mathbf{u}^1,p^1,\theta^1)$ *hold:*

$$\|\mathbf{u}^1\|_0^2+\nu\Delta t\|\mathbf{u}^{1/2}\|_1^2 \leqslant s_2^2,\ \ \|\theta^1\|_0^2+\lambda\nu\Delta t\|\theta^{1/2}\|_1^2\leqslant s_3^2,\ \ \Delta t^2\|p^1\|_1^2\leqslant s_4^2,$$

where

$$s_2^2 = \|\mathbf{u}^0\|_0^2 + \frac{3C_1^4k^2\nu^2s_3^2}{\lambda} + \frac{3C_1^2\Delta t}{\nu}\|\mathbf{f}(t_{1/2})\|_0^2 + \frac{3C_1^2\Delta t}{\nu}\|p^0\|_1^2,$$
$$s_3^2 = \|\theta^0\|_0^2 + \frac{C_1^2\Delta t}{\lambda\nu}\|g(t_{1/2})\|_0^2,\quad s_4^2 = \Delta t^2\|p^0\|_1^2 + \frac{s_2^2}{\alpha^2}.$$

P r o o f. Taking the inner product of (1.7) at $n=0$ with $2\Delta t\mathbf{u}^{1/2}$ and $2\Delta t\theta^{1/2}$, by using (3.2), we get

$$\begin{cases} \|\mathbf{u}^1\|_0^2 - \|\mathbf{u}^0\|_0^2 + 2\nu\Delta t\|\mathbf{u}^{1/2}\|_1^2 = 2\Delta t(\mathbf{f}(t_{1/2}),\mathbf{u}^{1/2}) \\ \qquad -2k\nu^2\Delta t(\mathbf{j}\theta^{1/2},\mathbf{u}^{1/2}) - 2\Delta t(\nabla p^0,\mathbf{u}^{1/2}), \\ \|\theta^1\|_0^2 - \|\theta^0\|_0^2 + 2\lambda\nu\Delta t\|\theta^{1/2}\|_1^2 = 2\Delta t(g(t_{1/2}),\theta^{1/2}). \end{cases} \tag{3.10}$$

We derive from (3.5) that

$$|-2k\nu^2\Delta t(\mathbf{j}\theta^{1/2},\mathbf{u}^{1/2})| \leqslant 2k\nu^2\Delta t\|\theta^{1/2}\|_0\|\mathbf{u}^{1/2}\|_0 \leqslant \frac{\nu\Delta t}{3}\|\mathbf{u}^{1/2}\|_1^2 + 3C_1^2k^2\nu^3\Delta t\|\theta^{1/2}\|_0^2,$$

$$|2\Delta t(\mathbf{f}(t_{1/2}),\mathbf{u}^{1/2})| \leqslant 2\Delta t\|\mathbf{f}(t_{1/2})\|_0\|\mathbf{u}^{1/2}\|_0 \leqslant \frac{\nu\Delta t}{3}\|\mathbf{u}^{1/2}\|_1^2 + \frac{3C_1^2\Delta t}{\nu}\|\mathbf{f}(t_{1/2})\|_0^2,$$

$$|-2\Delta t(\nabla p^0,\mathbf{u}^{1/2})| \leqslant 2\Delta t\|p^0\|_1\|\mathbf{u}^{1/2}\|_0 \leqslant \frac{\nu\Delta t}{3}\|\mathbf{u}^{1/2}\|_1^2 + \frac{3C_1^2\Delta t}{\nu}\|p^0\|_1^2,$$

$$|2\Delta t(g(t_{1/2}),\theta^{1/2})| \leqslant 2\Delta t\|g(t_{1/2})\|_0\|\theta^{1/2}\|_0 \leqslant \lambda\nu\Delta t\|\theta^{1/2}\|_1^2 + \frac{C_1^2\Delta t}{\lambda\nu}\|g(t_{1/2})\|_0^2.$$

Combining the above inequalities with (3.10), we find that

$$\begin{aligned}\|\mathbf{u}^1\|_0^2 - \|\mathbf{u}^0\|_0^2 + \nu\Delta t\|\mathbf{u}^{1/2}\|_1^2 \leqslant 3C_1^2k^2\nu^3\Delta t\|\theta^{1/2}\|_0^2 + \frac{3C_1^2\Delta t}{\nu}\|\mathbf{f}(t_{1/2})\|_0^2 \\ + \frac{3C_1^2\Delta t}{\nu}\|p^0\|_1^2,\ \|\theta^1\|_0^2 - \|\theta^0\|_0^2 + \lambda\nu\Delta t\|\theta^{1/2}\|_1^2 \\ \leqslant \frac{C_1^2\Delta t}{\lambda\nu}\|g(t_{1/2})\|_0^2.\end{aligned}$$

From (3.8) at $J = 0$ we obtain

$$\Delta t^2\|p^1\|_1^2 - \Delta t^2\|p^0\|_1^2 \leqslant \Delta t^2\|p^1 - p^0\|_1^2 \leqslant \frac{1}{\alpha^2}\|\mathbf{u}^1\|_0^2. \tag{3.11}$$

The proof of Theorem 3.2 is completed. □

4. Error estimates of the projection schemes

This section is devoted to presenting the convergence of the velocity, temperature, and pressure for the projection schemes.

In order to simplify the descriptions, for any function $w(t)$ we denote

$$\widetilde{w}(t_{n+1/2}) = \tfrac{1}{2}(w(t_{n+1}) + w(t_n)),$$

and we also set

$$E_u^n = \mathbf{u}(t_n) - \mathbf{u}^n, \quad E_\theta^n = \theta(t_n) - \theta^n, \quad q^n = p(t_n) - p^n.$$

Let us define the truncation errors R_u^n and R_θ^n by

$$\begin{cases} \dfrac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{\Delta t} - \nu\Delta\widetilde{\mathbf{u}}(t_{n+1/2}) + (\widetilde{\mathbf{u}}(t_{n+1/2})\cdot\nabla)\widetilde{\mathbf{u}}(t_{n+1/2}) + \nabla p(t_n) \\ \qquad = -k\nu^2\mathbf{j}\widetilde{\theta}(t_{n+1/2}) + \mathbf{f}(t_{n+1/2}) + R_u^n, \\ \nabla\cdot\mathbf{u}(t_{n+1}) = 0, \\ \dfrac{\theta(t_{n+1}) - \theta(t_n)}{\Delta t} - \lambda\nu\Delta\widetilde{\theta}(t_{n+1/2}) + (\widetilde{\mathbf{u}}(t_{n+1/2})\cdot\nabla)\widetilde{\theta}(t_{n+1/2}) \\ \qquad = g(t_{n+1/2}) + R_\theta^n. \end{cases} \tag{4.1}$$

Theorem 4.1. *Under the assumptions* (A1)–(A3), *for all* $0 \leqslant J \leqslant [T/\Delta t]-1$ *the following results hold:*

$$\|R_u^n\|_0 \leqslant M\Delta t,\ \|R_\theta^n\|_0 \leqslant M\Delta t,\ \Delta t\sum_{n=0}^{J}\|R_u^n\|_0^2 \leqslant M\Delta t^2,\ \Delta t\sum_{n=0}^{J}\|R_\theta^n\|_0^2 \leqslant M\Delta t^2.$$

Proof. We arrange R_u^n as follows:

$$\begin{aligned}R_u^n &= \Big(\frac{\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)}{\Delta t} - \nu\Delta\widetilde{\mathbf{u}}(t_{n+1/2}) + (\widetilde{\mathbf{u}}(t_{n+1/2})\cdot\nabla)\widetilde{\mathbf{u}}(t_{n+1/2}) + \nabla\widetilde{p}(t_{n+1/2})\\ &\quad + k\nu^2\mathbf{j}\widetilde{\theta}(t_{n+1/2}) - \mathbf{f}(t_{n+1/2})\Big) + (\nabla p(t_n) - \nabla\widetilde{p}(t_{n+1/2}))\\ &= R_{u1}^n + R_{u2}^n.\end{aligned}$$

We have proved the convergence for R_{u1}^n and R_θ^n in [12].

For the term R_{u2}^n, we can easily find

$$R_{u2}^n = \nabla p(t_n) - \nabla\widetilde{p}(t_{n+1/2}) = -\tfrac{1}{2}(\nabla p(t_{n+1}) - \nabla p(t_n)).$$

Then, setting $p(t_{n+1}) - p(t_n) = \Delta t p_t(\varepsilon_n)$, we have

$$\|R_{u2}^n\|_0 = \frac{1}{2}\|p(t_{n+1}) - p(t_n)\|_1 = C\Delta t\|p_t(\varepsilon_n) + \mathcal{O}(\Delta t)\|_1,$$

$$\Delta t\sum_{n=0}^{J}\|R_{u2}^n\|_0^2 = C\Delta t^3\sum_{n=0}^{J}\|p_t(\varepsilon_n) + \mathcal{O}(\Delta t)\|_1 \leqslant M\Delta t^2. \tag{4.2}$$

We completed the proof. □

Theorem 4.2. *Under the assumptions* (A1)–(A3) *and* $\alpha > \frac{1}{4}$, *for every fixed integer* i, *there exists a positive constant* M *such that*

$$\|E_u^i\|_0^2 + \|E_\theta^i\|_0^2 + \Delta t^2\|q^i\|_1^2 \leqslant M\Delta t^4,\quad \|E_u^i\|_1^2 + \|E_\theta^i\|_1^2 \leqslant M\Delta t^3.$$

Proof. From schemes (1.7)–(1.8) and (4.1), we have

$$\begin{cases}\dfrac{E_u^{n+1}-E_u^n}{\Delta t} - \nu\Delta E_u^{n+1/2} + \nabla q^n = -k\nu^2\mathbf{j}E_\theta^{n+1/2} + R_u^n + NLT_1^n,\\ \dfrac{E_\theta^{n+1}-E_\theta^n}{\Delta t} - \lambda\nu\Delta E_\theta^{n+1/2} = R_\theta^n + NLT_2^n,\end{cases} \tag{4.3}$$

and

$$\begin{aligned}(\nabla\cdot E_u^{n+1},\chi) &= \alpha\Delta t(\nabla p(t_{n+1}) - \nabla p(t_n),\nabla\chi)\\ &\quad - \alpha\Delta t(\nabla q^{n+1} - \nabla q^n,\nabla\chi)\quad \forall\,\chi\in H^1(\Omega)/\mathbb{R},\end{aligned} \tag{4.4}$$

where

$$
\begin{aligned}
NLT_1^n &= -B(\widetilde{\mathbf{u}}(t_{n+1/2}), \widetilde{\mathbf{u}}(t_{n+1/2})) + B(\mathbf{u}^n, u^{n+1/2}) \\
&= -\tfrac{1}{2}B(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \widetilde{\mathbf{u}}(t_{n+1/2})) - B(E_u^n, \widetilde{\mathbf{u}}(t_{n+1/2})) - B(\mathbf{u}^n, E_u^{n+1/2}), \\
NLT_2^n &= -\overline{B}(\widetilde{\mathbf{u}}(t_{n+1/2}), \widetilde{\theta}(t_{n+1/2})) + \overline{B}(\mathbf{u}^n, \theta^{n+1/2}) \\
&= -\tfrac{1}{2}\overline{B}(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \widetilde{\theta}(t_{n+1/2})) - \overline{B}(E_u^n, \widetilde{\theta}(t_{n+1/2})) - \overline{B}(\mathbf{u}^n, E_\theta^{n+1/2}).
\end{aligned} \tag{4.5}
$$

Taking the inner product of (4.3) with $2\Delta t E_u^{n+1/2}$ and $2\Delta t E_\theta^{n+1/2}$, we get

$$
\begin{cases}
\|E_u^{n+1}\|_0^2 - \|E_u^n\|_0^2 + 2\nu\Delta t\|E_u^{n+1/2}\|_1^2 + \Delta t(\nabla q^n, E_u^{n+1} + E_u^n) \\
\quad = 2\Delta t(R_u^n, E_u^{n+1/2}) - 2k\nu^2\Delta t(\mathbf{j}E_\theta^{n+1/2}, E_u^{n+1/2}) + 2\Delta t(NLT_1^n, E_u^{n+1/2}), \\
\|E_\theta^{n+1}\|_0^2 - \|E_\theta^n\|_0^2 + 2\lambda\nu\Delta t\|E_\theta^{n+1/2}\|_1^2 \\
\quad = 2\Delta t(R_\theta^n, E_\theta^{n+1/2}) + 2\Delta t(NLT_2^n, E_\theta^{n+1/2}).
\end{cases} \tag{4.6}
$$

Replacing χ by q^n in (4.4), we obtain

$$
(\nabla \cdot E_u^{n+1}, q^n) = \alpha\Delta t(\nabla p(t_{n+1}) - \nabla p(t_n), \nabla q^n) - \alpha\Delta t(\nabla q^{n+1} - \nabla q^n, \nabla q^n). \tag{4.7}
$$

Since $2E_u^{n+1/2} = E_u^{n+1} + E_u^n$, using (2.2), (3.2) and summing (4.6)–(4.7), we get

$$
\begin{cases}
\|E_u^{n+1}\|_0^2 - \|E_u^n\|_0^2 + 2\nu\Delta t\|E_u^{n+1/2}\|_1^2 \\
\qquad + \dfrac{1}{2}\alpha\Delta t^2(\|q^{n+1}\|_1^2 - \|q^n\|_1^2 - \|q^{n+1} - q^n\|_1^2) \\
\quad = -\Delta t(\nabla q^n, E_u^n) + \alpha\Delta t^2(\nabla p(t_{n+1}) - \nabla p(t_n), \nabla q^n) \\
\qquad - 2k\nu^2\Delta t(\mathbf{j}E_\theta^{n+1/2}, E_u^{n+1/2}) - 2\Delta t b(E_u^n, \widetilde{\mathbf{u}}(t_{n+1/2}), E_u^{n+1/2}) \\
\qquad + 2\Delta t(R_u^n, E_u^{n+1/2}) - \Delta t b(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \widetilde{\mathbf{u}}(t_{n+1/2}), E_u^{n+1/2}), \\
\|E_\theta^{n+1}\|_0^2 - \|E_\theta^n\|_0^2 + 2\lambda\nu\Delta t\|E_\theta^{n+1/2}\|_1^2 \\
\quad = -2\Delta t\overline{b}(E_u^n, \widetilde{\theta}(t_{n+1/2}), E_\theta^{n+1/2}) \\
\qquad + 2\Delta t(R_\theta^n, E_\theta^{n+1/2}) - \Delta t\overline{b}(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \widetilde{\theta}(t_{n+1/2}), E_\theta^{n+1/2}).
\end{cases} \tag{4.8}
$$

On the other hand, setting $\delta = \alpha - \frac{1}{4}$, using $p(t_{n+1}) - p(t_n) = \Delta t p_t(\varepsilon_n)$, and replacing χ by $q^{n+1} - q^n$ in (4.4), we have

$$
\begin{aligned}
\alpha\Delta t\|q^{n+1} - q^n\|_1^2 &= (E_u^{n+1}, \nabla q^{n+1} - \nabla q^n) \\
&\quad + \alpha\Delta t(\nabla p(t_{n+1}) - \nabla p(t_n), \nabla q^{n+1} - \nabla q^n)
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
&\leqslant \Delta t\Big(\frac{\alpha}{2}-\frac{3\delta}{8}\Big)\|q^{n+1}-q^n\|_1^2+\frac{1}{4\Delta t(\alpha/2-3\delta/8)}\|E_u^{n+1}\|_0^2\\
&\quad+\frac{3\delta\Delta t}{8}\|q^{n+1}-q^n\|_1^2+M\Delta t\|p(t_{n+1})-p(t_n)\|_1^2\\
&=\frac{\alpha\Delta t}{2}\|q^{n+1}-q^n\|_1^2+\frac{2}{(1+\delta)\Delta t}\|E_u^{n+1}\|_0^2\\
&\quad+M\Delta t^3\|p_t(\varepsilon_n)\|_1^2.
\end{aligned}
$$

We then derive from (4.9) that

$$
\alpha\Delta t^2\|q^{n+1}-q^n\|_1^2\leqslant\frac{4}{1+\delta}\|E_u^{n+1}\|_0^2+M\Delta t^4\|p_t(\varepsilon_n)\|_1^2.
$$

Hence,

$$
\begin{aligned}
\frac{\alpha\Delta t^2}{2}\|q^{n+1}-q^n\|_1^2&=\frac{(1+\delta/2)\alpha\Delta t^2}{4}\|q^{n+1}-q^n\|_1^2\\
&\quad+\frac{(1-\delta/2)\alpha\Delta t^2}{4}\|q^{n+1}-q^n\|_1^2\\
&\leqslant\frac{1+\delta/2}{1+\delta}\|E_u^{n+1}\|_0^2+M\Delta t^4\|p_t(\varepsilon_n)\|_1^2\\
&\quad+\frac{(1-\delta/2)\alpha\Delta t^2}{2}(\|q^{n+1}\|_1^2+\|q^n\|_1^2).
\end{aligned}\tag{4.10}
$$

Setting $\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)=\Delta t\mathbf{u}_t(\varepsilon_n)$, the terms on the right-hand side of (4.8) can be handled as follows:

$$
\begin{aligned}
&|-\Delta t(E_u^n,\nabla q^n)|\leqslant\Delta t\|E_u^n\|_0\|q^n\|_1\leqslant\|E_u^n\|_0^2+M\Delta t^2\|q^n\|_1^2,\\
&|\alpha\Delta t^2(\nabla(p(t_{n+1})-p(t_n)),\nabla q^n)|\leqslant\alpha\Delta t^2\|p(t_{n+1})-p(t_n)\|_1\|q^n\|_1\\
&\qquad\leqslant M\Delta t^4\|p_t(\varepsilon_n)\|_1^2+M\Delta t^2\|q^n\|_1^2,\\
&|-2k\nu^2\Delta t(\mathbf{j}E_\theta^{n+1/2},E_u^{n+1/2})|\leqslant 2k\nu^2\Delta t\|E_\theta^{n+1/2}\|_0\|E_u^{n+1/2}\|_0\\
&\qquad\leqslant\frac{\delta}{8(1+\delta)}\|E_u^{n+1/2}\|_0^2+M\Delta t^2\|E_\theta^{n+1/2}\|_0^2\\
&\qquad\leqslant\frac{\delta}{16(1+\delta)}\|E_u^{n+1}\|_0^2+\frac{\delta}{16(1+\delta)}\|E_u^n\|_0^2+M\Delta t^2\|E_\theta^{n+1/2}\|_0^2,\\
&|2\Delta t(R_u^n,E_u^{n+1/2})|\leqslant2\Delta t\|R_u^n\|_0\|E_u^{n+1/2}\|_0\leqslant\frac{\delta}{8(1+\delta)}\|E_u^{n+1/2}\|_0^2+M\Delta t^2\|R_u^n\|_0^2\\
&\qquad\leqslant\frac{\delta}{16(1+\delta)}\|E_u^{n+1}\|_0^2+\frac{\delta}{16(1+\delta)}\|E_u^n\|_0^2+M\Delta t^2\|R_u^n\|_0^2,
\end{aligned}
$$

$$
\begin{aligned}
&|-2\Delta t b(E_u^n, \widetilde{\mathbf{u}}(t_{n+1/2}), E_u^{n+1/2})| \leqslant 2C_2\Delta t\|E_u^n\|_1\|\widetilde{\mathbf{u}}(t_{n+1/2})\|_2\|E_u^{n+1/2}\|_0\\
&\qquad \leqslant \frac{\delta}{8(1+\delta)}\|E_u^{n+1/2}\|_0^2 + M\Delta t^2\|E_u^n\|_1^2,\\
&\qquad \leqslant \frac{\delta}{16(1+\delta)}\|E_u^{n+1}\|_0^2 + \frac{\delta}{16(1+\delta)}\|E_u^n\|_0^2 + M\Delta t^2\|E_u^n\|_1^2,\\
&|-\Delta t b(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \widetilde{\mathbf{u}}(t_{n+1/2}), E_u^{n+1/2})|\\
&\qquad \leqslant C_2\Delta t\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_1\|\widetilde{\mathbf{u}}(t_{n+1/2})\|_2\|E_u^{n+1/2}\|_0\\
&\qquad \leqslant \frac{\delta}{8(1+\delta)}\|E_u^{n+1/2}\|_0^2 + M\Delta t^4\|\mathbf{u}_t(\varepsilon_n)\|_1^2\\
&\qquad \leqslant \frac{\delta}{16(1+\delta)}\|E_u^{n+1}\|_0^2 + \frac{\delta}{16(1+\delta)}\|E_u^n\|_0^2 + M\Delta t^4\|\mathbf{u}_t(\varepsilon_n)\|_1^2,\\
&|2\Delta t(R_\theta^n, E_\theta^{n+1/2})| \leqslant 2\Delta t\|R_\theta^n\|_0\|E_\theta^{n+1/2}\|_0 \leqslant \frac{1}{3}\|E_\theta^{n+1/2}\|_0^2 + M\Delta t^2\|R_\theta^n\|_0^2\\
&\qquad \leqslant \frac{1}{6}\|E_\theta^{n+1}\|_0^2 + \frac{1}{6}\|E_\theta^n\|_0^2 + M\Delta t^2\|R_\theta^n\|_0^2,\\
&|-2\Delta t\overline{b}(E_u^n, \widetilde{\theta}(t_{n+1/2}), E_\theta^{n+1/2})| \leqslant 2C_3\Delta t\|E_u^n\|_1\|\widetilde{\theta}(t_{n+1/2})\|_2\|E_\theta^{n+1/2}\|_0\\
&\qquad \leqslant \frac{1}{3}\|E_\theta^{n+1/2}\|_0^2 + M\Delta t^2\|E_u^n\|_1^2 \leqslant \frac{1}{6}\|E_\theta^{n+1}\|_0^2 + \frac{1}{6}\|E_\theta^n\|_0^2 + M\Delta t^2\|E_u^n\|_1^2,\\
&|-\Delta t\overline{b}(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n), \widetilde{\theta}(t_{n+1/2}), E_\theta^{n+1/2})|\\
&\qquad \leqslant C_3\Delta t\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_1\|\widetilde{\theta}(t_{n+1/2})\|_2\|E_\theta^{n+1/2}\|_0\\
&\qquad \leqslant \frac{1}{3}\|E_\theta^{n+1/2}\|_0^2 + M\Delta t^4\|\mathbf{u}_t(\varepsilon_n)\|_1^2 \leqslant \frac{1}{6}\|E_\theta^{n+1}\|_0^2 + \frac{1}{6}\|E_\theta^n\|_0^2 + M\Delta t^4\|\mathbf{u}_t(\varepsilon_n)\|_1^2.
\end{aligned}
$$

Combining the above inequalities and (4.10) with (4.8), we obtain

$$
\text{(4.11)}\quad \begin{cases}
\dfrac{\delta}{4(1+\delta)}\|E_u^{n+1}\|_0^2 + 2\nu\Delta t\|E_u^{n+1/2}\|_1^2 + \dfrac{\delta\alpha\Delta t^2}{4}\|q^{n+1}\|_1^2\\
\qquad \leqslant M\|E_u^n\|_0^2 + M\Delta t^2\|q^n\|_1^2 + M\Delta t^2\|R_u^n\|_0^2 + M\Delta t^2\|E_u^n\|_1^2\\
\qquad\quad + M\Delta t^4\|p_t(\varepsilon_n)\|_1^2 + M\Delta t^2\|E_\theta^{n+1/2}\|_0^2 + M\Delta t^4\|\mathbf{u}_t(\varepsilon_n)\|_1^2,\\
\dfrac{1}{2}\|E_\theta^{n+1}\|_0^2 + 2\lambda\nu\Delta t\|E_\theta^{n+1/2}\|_1^2\\
\qquad \leqslant M\|E_\theta^n\|_0^2 + M\Delta t^2\|R_\theta^n\|_0^2 + M\Delta t^2\|E_u^n\|_1^2 + M\Delta t^4\|\mathbf{u}_t(\varepsilon_n)\|_1^2.
\end{cases}
$$

First, we present the results of Theorem 4.2 at $i = 1$.

Since $E_u^0 = E_\theta^0 = q^0 = 0$, thanks to Theorem 4.1, problem (4.11) at $n = 0$ can be transformed to

$$
\begin{aligned}
\text{(4.12)}\quad &\frac{1}{2}\|E_\theta^1\|_0^2 + 2\lambda\nu\Delta t\|E_\theta^{1/2}\|_1^2\\
&\qquad \leqslant M\|E_\theta^0\|_0^2 + M\Delta t^2\|R_\theta^0\|_0^2 + M\Delta t^2\|E_u^0\|_1^2 + M\Delta t^4\|\mathbf{u}_t(\varepsilon_0)\|_1^2\\
&\qquad = M\Delta t^2\|R_\theta^0\|_0^2 + M\Delta t^4\|\mathbf{u}_t(\varepsilon_0)\|_1^2 \leqslant M\Delta t^4,
\end{aligned}
$$

$$
\begin{aligned}
(4.13)\quad & \frac{\delta}{4(1+\delta)}\|E_u^1\|_0^2 + 2\nu\Delta t\|E_u^{1/2}\|_1^2 + \frac{\delta\alpha\Delta t^2}{4}\|q^1\|_1^2 \\
& \leqslant M\|E_u^0\|_0^2 + M\Delta t^2\|q^0\|_1^2 + M\Delta t^2\|R_u^0\|_0^2 + M\Delta t^2\|E_u^0\|_1^2 \\
& \quad + M\Delta t^4\|p_t(\varepsilon_0)\|_1^2 + M\Delta t^2\|E_\theta^{1/2}\|_0^2 + M\Delta t^4\|\mathbf{u}_t(\varepsilon_0)\|_1^2 \\
& = M\Delta t^2\|R_u^0\|_0^2 + M\Delta t^4\|p_t(\varepsilon_0)\|_1^2 + M\Delta t^2\|E_\theta^1\|_0^2 + M\Delta t^4\|\mathbf{u}_t(\varepsilon_0)\|_1^2 \\
& \leqslant M\Delta t^4.
\end{aligned}
$$

From (4.12) and the triangle inequality, we get that

$$
\begin{aligned}
&\|E_u^1\|_0^2 + \|E_\theta^1\|_0^2 + \Delta t^2\|q^1\|_1^2 \leqslant M\Delta t^4, \\
&\|E_u^1\|_1^2 \leqslant \|2E_u^{1/2}\|_1^2 + \|E_u^0\|_1^2 \leqslant M\Delta t^3, \\
&\|E_\theta^1\|_1^2 \leqslant \|2E_\theta^{1/2}\|_1^2 + \|E_\theta^0\|_1^2 \leqslant M\Delta t^3.
\end{aligned}
$$

Secondly, assuming that Theorem 4.2 holds for all $i \leqslant m-1$, we have

$$
\begin{aligned}
&\|E_u^{m-1}\|_0^2 + \|E_\theta^{m-1}\|_0^2 + \Delta t^2\|q^{m-1}\|_1^2 \leqslant M\Delta t^4, \\
&\|E_u^{m-1}\|_1^2 \leqslant \|2E_u^{m-3/2}\|_1^2 + \|E_u^{m-2}\|_1^2 \leqslant M\Delta t^3, \\
&\|E_\theta^{m-1}\|_1^2 \leqslant \|2E_\theta^{m-3/2}\|_1^2 + \|E_\theta^{m-2}\|_1^2 \leqslant M\Delta t^3.
\end{aligned}
\tag{4.14}
$$

Finally, we begin to verify that Theorem 4.2 is true at $i = m$.

Letting $n = m-1$ in (4.11), under the established conditions of (4.14) and $\|E_\theta^{m-1/2}\|_0^2 \leqslant \|E_\theta^m\|_0^2 + \|E_\theta^{m-1}\|_0^2$, we derive

$$
\begin{cases}
\frac{1}{2}\|E_\theta^m\|_0^2 + 2\lambda\nu\Delta t\|E_\theta^{m-1/2}\|_1^2 \leqslant M\|E_\theta^{m-1}\|_0^2 + M\Delta t^2\|R_\theta^{m-1}\|_0^2 \\
\qquad + M\Delta t^2\|E_u^{m-1}\|_1^2 + M\Delta t^4\|\mathbf{u}_t(\varepsilon_{m-1})\|_1^2 \leqslant M\Delta t^4, \\
\frac{\delta}{4(1+\delta)}\|E_u^m\|_0^2 + 2\nu\Delta t\|E_u^{m-1/2}\|_1^2 + \frac{\delta\alpha\Delta t^2}{4}\|q^m\|_1^2 \leqslant M\|E_u^{m-1}\|_0^2 \\
\qquad + M\Delta t^2\|q^{m-1}\|_1^2 + M\Delta t^2\|R_u^{m-1}\|_0^2 + M\Delta t^2\|E_u^{m-1}\|_1^2 \\
\qquad + M\Delta t^4\|p_t(\varepsilon_{m-1})\|_1^2 + M\Delta t^2\|E_\theta^{m-1/2}\|_0^2 + M\Delta t^4\|\mathbf{u}_t(\varepsilon_{m-1})\|_1^2 \\
\qquad \leqslant M\Delta t^4.
\end{cases}
\tag{4.15}
$$

Hence,

$$
\begin{aligned}
&\|E_u^m\|_1^2 \leqslant \|2E_u^{m-1/2}\|_1^2 + \|E_u^{m-1}\|_1^2 \leqslant M\Delta t^3, \\
&\|E_\theta^m\|_1^2 \leqslant \|2E_\theta^{m-1/2}\|_1^2 + \|E_\theta^{m-1}\|_1^2 \leqslant M\Delta t^3.
\end{aligned}
\tag{4.16}
$$

By the mathematical induction and deduction methods, we complete the proof. □

R e m a r k 4.1. Assume the initial data $(\mathbf{u}^0, p^0, \theta^0)$ for the schemes (1.7)–(1.8) satisfies

$$\|\mathbf{u}^0 - \mathbf{u}(t_0)\|_0 + \|\theta^0 - \theta(t_0)\|_0 \leqslant C\Delta t^2, \tag{4.17}$$

$$\|\mathbf{u}^0 - \mathbf{u}(t_0)\|_1 + \|\theta^0 - \theta(t_0)\|_1 \leqslant C\Delta t^{3/2},$$

$$\|p^0 - p(t_0)\|_1 \leqslant C\Delta t. \tag{4.18}$$

Then we can get the convergence of strongly second order in time for the velocity and temperature and of strongly first order in time for the pressure.

R e m a r k 4.2. Assume that we are given initial data $(\mathbf{u}^0, \phi^0, \theta^0)$ which are the corresponding approximations to $(\mathbf{u}(t_0), p(t_0), \theta(t_0))$ for the projection schemes (1.2)–(1.3) and (1.5)–(1.6). If the initial data satisfy (4.17) and

$$\|\phi^0 - p(t_0)\|_1 \leqslant C\Delta t,$$

or the schemes start with $(\mathbf{u}^0, \theta^0) = (\mathbf{u}(t_0), \theta(t_0))$ and $\|\phi^0 - p(t_0)\|_1 \leqslant C\Delta t$, we can also achieve the same results as those in Theorem 4.2.

References

[1] *R. Araya, G. R. Barrenechea, A. H. Poza, F. Valentin*: Convergence analysis of a residual local projection finite element method for the Navier-Stokes equations. SIAM J. Numer. Anal. *50* (2012), 669–699.

[2] *G. Chen, M. Feng, H. Zhou*: Local projection stabilized method on unsteady Navier-Stokes equations with high Reynolds number using equal order interpolation. Appl. Math. Comput. *243* (2014), 465–481.

[3] *A. J. Chorin*: Numerical solution of the Navier-Stokes equations. Math. Comput. *22* (1968), 745–762.

[4] *A. Çıbık, S. Kaya*: A projection-based stabilized finite element method for steady-state natural convection problem. J. Math. Anal. Appl. *381* (2011), 469–484.

[5] *B. Du, H. Su, X. Feng*: Two-level variational multiscale method based on the decoupling approach for the natural convection problem. Int. Commun. Heat. Mass. *61* (2015), 128–139.

[6] *Y. He*: Two-level method based on finite element and Crank-Nicolson extrapolation for the time-dependent Navier-Stokes equations. SIAM J. Numer. Anal. *41* (2003), 1263–1285.

[7] *Y. He*: Stability and error analysis for a spectral Galerkin method for the Navier-Stokes equations with H^2 or H^1 initial data. Numer. Methods Partial Differ. Equations *21* (2005), 875–904.

[8] *J. G. Heywood, R. Rannacher*: Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization. SIAM J. Numer. Anal. *19* (1982), 275–311.

[9] *M. T. Manzari*: An explicit finite element algorithm for convection heat transfer problems. Int. J. Numer. Methods Heat Fluid Flow *9* (1999), 860–877.

[10] *J. H. Pyo*: A classification of the second order projection methods to solve the Navier-Stokes equations. Korean J. Math. *22* (2014), 645–658.
[11] *Y. X. Qian, T. Zhang*: On error estimates of the projection method for the time-dependent natural convection problem: first order scheme. Submitted to Comput. Math. Appl.
[12] *Y. X. Qian, T. Zhang*: On error estimates of a higher projection method for the time-dependent natural convection problem. Submitted to Front. Math. China.
[13] *J. Shen*: On error estimates of projection methods for Navier-Stokes equations: First-order schemes. SIAM J. Numer. Anal. *29* (1992), 57–77.
[14] *J. Shen*: On error estimates of some higher order projection and penalty-projection methods for Navier-Stokes equations. Numer. Math. *62* (1992), 49–73.
[15] *S. Shen*: The finite element analysis for the conduction-convection problems. Math. Numer. Sin. *16* (1994), 170–182. (In Chinese.)
[16] *J. Shen*: On error estimates of the projection methods for the Navier-Stokes equations: Second-order schemes. Math. Comput. *65* (1996), 1039–1065.
[17] *M. Tabata, D. Tagami*: Error estimates of finite element methods for nonstationary thermal convection problems with temperature-dependent coefficients. Numer. Math. *100* (2005), 351–372.
[18] *R. Témam*: Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires. II. Arch. Ration. Mech. Anal. *33* (1969), 377–385. (In French.)
[19] *R. Témam*: Navier-Stokes Equations. Theory and Numerical Analysis. Studies in Mathematics and Its Applications, Vol. 2, North-Holland, Amsterdam, 1984.
[20] *A. W. Vreman*: The projection method for the incompressible Navier-Stokes equations: the pressure near a no-slip wall. J. Comput. Phys. *263* (2014), 353–374.
[21] *Y. Zhang, Y. Hou, J. Zhao*: Error analysis of a fully discrete finite element variational multiscale method for the natural convection problem. Comput. Math. Appl. *68* (2014), 543–567.
[22] *T. Zhang, Z. Tao*: Decoupled scheme for time-dependent natural convection problem II: time semidiscreteness. Math. Probl. Eng. *2014* (2014), Article ID 726249, 23 pages.
[23] *T. Zhang, J. Yuan, Z. Si*: Decoupled two-grid finite element method for the time-dependent natural convection problem I: Spatial discretization. Numer. Methods Partial Differ. Equations *31* (2015), 2135–2168.
[24] *X. Zhang, P. Zhang*: Meshless modeling of natural convection problems in non-rectangular cavity using the variational multiscale element free Galerkin method. Eng. Anal. Bound. Elem. *61* (2015), 287–300.
[25] *T. Zhang, X. Zhao, P. Huang*: Decoupled two level finite element methods for the steady natural convection problem. Numer. Algorithms *68* (2015), 837–866.

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