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SEVERAL NOTES ON THE CIRCUMRADIUS CONDITION

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Abstract. Recently, the so-called circumradius condition (or estimate) was derived, which is a new estimate of the $W^{1,p}$-error of linear Lagrange interpolation on triangles in terms of their circumradius. The published proofs of the estimate are rather technical and do not allow clear, simple insight into the results. In this paper, we give a simple direct proof of the $p = \infty$ case. This allows us to make several observations such as on the optimality of the circumradius estimate. Furthermore, we show how the case of general p is in fact nothing more than a simple scaling of the standard $O(h)$ estimate under the maximum angle condition, even for higher order interpolation. This allows a direct interpretation of the circumradius estimate and condition in the context of the well established theory of the maximum angle condition.

Keywords: finite element method; a priori error estimate; circumradius condition; Lagrange interpolation

MSC 2010: 65N30, 65D05

1. Introduction

In the long history of the finite element method (FEM), much work has been devoted to answer the basic question: when do the approximate solutions converge to the exact solution. In the simplest case, Poisson's problem is treated on a domain Ω and convergence is studied in the energy (semi)norm, i.e., in $H^1(\Omega)$.

We consider a system of conforming triangulations $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ of $\Omega \subset \mathbb{R}^2$, which defines the piecewise linear finite element space $V_h = \{v_h \in C(\overline{\Omega});\ v_h|_K \in P^1(K)$ for all $K \in \mathcal{T}_h\}$, where $P^1(K)$ is the space of linear functions on the triangular element

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$K \in \mathcal{T}_h$. The convergence is then usually measured with respect to the parameters $h_K = \operatorname{diam} K$ and $h = \max\limits_{K \in \mathcal{T}_h} h_K$.

For Poisson's problem, estimates on the finite element error are obtained via Céa's lemma and estimates for Lagrange interpolation on triangles. For $u \in C(\overline{K})$, let $\Pi_K u \in P^1(K)$ be the Lagrange interpolation defined by the vertices of K. From this, we can construct the global interpolant $\Pi_h u \in V_h$ such that $(\Pi_h u)|_K = \Pi_K u$ for all $K \in \mathcal{T}_h$. The error of the finite element method is then estimated by the error of the interpolant.

The first condition for $O(h)$ convergence of $\Pi_h u$ to u in the $H^1(\Omega)$-seminorm is the so-called *minimum angle condition* derived independently in [13], [14].

Lemma 1.1. *Let $\gamma_0 > 0$ and let the minimal angle of $K \in \mathcal{T}_h$ satisfy $\gamma_K \geqslant \gamma_0$. Then there exists a constant $C = C(\gamma_0)$ independent of u and h_K such that*

$$|u - \Pi_K u|_{1,2,K} \leqslant C h_K |u|_{2,2,K}. \tag{1.1}$$

Assuming $\gamma_K \geqslant \gamma_0 > 0$ for all $K \in \mathcal{T}_h$ and all $h \in (0, h_0)$, we obtain

$$|u - \Pi_h u|_{1,2,\Omega} \leqslant C h |u|_{2,2,\Omega}, \tag{1.2}$$

which gives an $O(h)$ error estimate for the FEM in $H^1(\Omega)$.

Later, a generalization of Lemma 1.1 was proved independently in [1], [2], and [6], leading to the *maximum angle condition*:

Lemma 1.2. *Let $\alpha_0 < \pi$ and let the maximal angle of $K \in \mathcal{T}_h$ satisfy $\alpha_K \leqslant \alpha_0$. Then for all $p \in [1, \infty]$ there exists a constant $C_p(\alpha_0)$ independent of u and h_K such that*

$$|u - \Pi_K u|_{1,p,K} \leqslant C_p(\alpha_0) h_K |u|_{2,p,K}. \tag{1.3}$$

Again, assuming that $\alpha_K \leqslant \alpha_0 < \pi$ for all $K \in \mathcal{T}_h$ and all $h \in (0, h_0)$, we obtain a $W^{1,p}(\Omega)$ version of (1.2).

Recently, a generalization of Lemmas 1.1 and 1.2 was given in [8] and [12].

Lemma 1.3 (Circumradius estimate). *Let $R_K \leqslant 1$ be the circumradius of K. Then for all $p \in [1, \infty]$ there exists a constant C_p independent of u and K such that*

$$|u - \Pi_K u|_{1,p,K} \leqslant C_p R_K |u|_{2,p,K}. \tag{1.4}$$

Assuming the *circumradius condition*

$$\lim_{h\to 0} \max_{K\in\mathcal{T}_h} R_K \to 0, \tag{1.5}$$

we obtain convergence (not $O(h)$ convergence) of the finite element method similarly as in (1.2). We shall refer to Lemma 1.3 as the *circumradius estimate*, although in [8] both (1.4) and (1.5) are ambiguously called the circumradius condition.

In this paper we show close links between the circumradius estimate and the maximum angle condition. The first such observation is the following. Since α_K is the largest angle in K, its opposing side has length h_K. By the law of sines,

$$2R_K = \frac{h_K}{\sin \alpha_K}. \tag{1.6}$$

If we substitute this expression into (1.4), we get an $O(h)$ estimate if and only if the denominator $\sin \alpha_K$ is uniformly bounded away from zero for all $K \in \mathcal{T}_h$, which is exactly the maximum angle condition. Therefore, as far as $O(h)$ convergence is concerned, Lemmas 1.2 and 1.3 are equivalent.

The history of the circumradius estimate is interesting in itself. For $p = 2$ it was first proved by Rand in his Ph.D. thesis [12], however this result was not published in a journal. The case $p = 2$ was independently shown by Kobayashi in [7] where it is claimed that the constant in (1.4) can be taken as $C_2 = 1$. However, the proof relies heavily on numerical computations and is therefore hard to verify. Finally, the case of general p was proven by Kobayashi and Tsuchiya in [8] using the technique of [1]. However, we must point out that the circumradius estimate was essentially proved already in [10] by Křížek, although the result is not stated—this will be explained in detail at the end of Section 3. Finally, in [9], circumradius-type estimates were derived for higher order Lagrange interpolation.

The proofs of the circumradius estimate presented in [8], [9], and [12] are rather lengthy and technical. It is therefore hard to obtain good insight on why Lemma 1.3 should hold, whether or not the estimate is optimal, etc. The purpose of this paper is twofold: first, to clarify these issues by presenting an as simple as possible straightforward proof of the special case $p = \infty$; and second, to show that the circumradius estimate is essentially an optimal scaling of the $O(h)$ estimate under the maximum angle condition, thus connecting the estimate to well established results.

The structure of the paper is as follows. In Section 2, the case $p=\infty$ is proved. This is especially simple and self-contained, using only simple geometric and analytical tools, hence a good understanding of the circumradius estimate can be gained. In Section 3, the case of $p \in [1,\infty)$ is treated. We show how the result can be obtained by a simple scaling argument from the maximum angle condition. Section 4 is devoted to higher order Lagrange interpolation and again we show that the results proved in [9] can be obtained by scaling from $O(h)$ estimates valid under the maximum angle condition. Finally, in Section 5, we show that the factor R_K in (1.4) cannot be improved for general u and is optimal for $\alpha_K \to \pi$.

1.1. Basic notation. We will use the standard Sobolev spaces $W^{k,p}(G)$, $k \in \mathbb{N}_0$, $p \in [1,\infty]$ on a domain $G \subset \mathbb{R}^2$ equipped with the standard norm $\|\cdot\|_{k,p,G}$ and seminorm $|\cdot|_{k,p,G}$. We will use abbreviated notation for partial derivatives, i.e., $u_{x^iy^j} := \partial^{i+j}u/\partial x^i \partial y^j$. By $\xi_i \in K$ we will denote an auxiliary point defining the Lagrange form of the Taylor remainder or some other similar expression based on the context.

We will use the following notation for the triangle K with vertices ABC, cf. Figure 1. Let the maximal angle α_K be at A, the minimal angle γ_K at C and β_K at B. By a, b, c we denote the sides opposite to α_K, β_K, and γ_K, respectively. Let h_A be the altitude from A to BC and let H be the foot of this altitude. Finally, let x_b and x_c be the lengths of the two segments of BC given by H.

Without loss of generality, assume that B has coordinates $(0,0)$, C lies at $(h_K,0)$ and A at (x_b,h_A), i.e., the side BC lies on the x-axis. This is possible, since a general triangle K can be transformed into this configuration by $F\colon \mathbb{R}^2 \to \mathbb{R}^2$, a combination of translation and rotation. Such a mapping changes Sobolev seminorms only by a constant factor, specifically, for $u \in W^{k,p}(K)$

$$|u \circ F^{-1}|_{k,p,F(K)} \leqslant C(k,p)|u|_{k,p,K}, \tag{1.7}$$

where $C(k,p)$ is independent of u and K, cf. [3]. A straightforward calculation yields e.g. $C(1,\infty)=\sqrt{2}$ and $C(2,\infty)=2$, which is relevant to the following section.

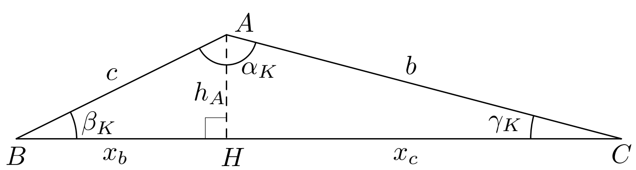

Figure 1. An element $K \in \mathcal{T}_h$.

2. The circumradius estimate for $p = \infty$

Lemma 2.1. *Let K be a triangle with the notation introduced above. Then*

$$\frac{x_b x_c}{h_A} = 2R_K \cos\beta_K \cos\gamma_K. \tag{2.1}$$

P r o o f. By the law of sines,

$$2R_K = \frac{c}{\sin\gamma_K} = \frac{cb}{h_A} = \frac{x_b x_c}{h_A \cos\beta_K \cos\gamma_K},$$

since $x_b = c\cos\beta_K$ and $x_c = b\cos\gamma_K$. Multiplying by $\cos\beta_K\cos\gamma_K$ gives (2.1). □

Theorem 2.1 (Circumradius estimate for $p = \infty$). *Let K be a triangle as in Section 1.1. Let $u \in C^2(K)$. Then*

$$|u - \Pi_K u|_{1,\infty,K} \leqslant (2\sqrt{2} + 2)R_K |u|_{2,\infty,K}. \tag{2.2}$$

P r o o f. Since $\Pi_K w = w$ for all $w \in P^1(K)$, by subtracting a suitable w, we can assume without loss of generality that $u(B) = u(C) = u_y(H) = 0$.

Denote $v := \Pi_K u \in P^1(K)$. Trivially $v_x = (u(C) - u(B))/h_K = 0$ and $v_y = (u(A) - v(H))/h_A = u(A)/h_A$. Since $u_y(H) = 0$, by Taylor's theorem

$$u(A) = u(H) + \tfrac{1}{2}u_{yy}(\xi_1)h_A^2.$$

On the side BC, the function v is simply the 1D linear Lagrange interpolation of u, for which we have the error formula

$$u(H) = u(H) - v(H) = -\tfrac{1}{2}u_{xx}(\xi_2)x_b x_c,$$

cf. [4]. Therefore,

$$|v_y| = \frac{|u(A)|}{h_A} \leqslant \frac{\frac{1}{2}(h_A^2 + x_b x_c)}{h_A}|u|_{2,\infty,K} \leqslant 2R_K|u|_{2,\infty,K},$$

due to Lemma 2.1 and the estimate $h_A \leqslant h_K \leqslant 2R_K$.

Finally, we estimate the left-hand side of (2.2). Since $u(B) = u(C) = 0$, there exists $\widetilde{\boldsymbol{x}}$ on the side BC such that $u_x(\widetilde{\boldsymbol{x}}) = 0$. For $\boldsymbol{x} \in K$

$$|u_x(\boldsymbol{x}) - v_x| = |u_x(\boldsymbol{x})| = |u_x(\boldsymbol{x}) - u_x(\widetilde{\boldsymbol{x}})| = |\nabla u_x(\xi_3)(\boldsymbol{x} - \widetilde{\boldsymbol{x}})| \leqslant \sqrt{2}h_K|u|_{2,\infty,K}.$$

Similarly, since $u_y(H) = 0$,

$$\begin{aligned} |u_y(\boldsymbol{x}) - v_y| &\leqslant |u_y(\boldsymbol{x}) - u_y(H)| + |v_y| \leqslant |\nabla u_y(\xi_4)(\boldsymbol{x} - H)| + 2R_K|u|_{2,\infty,K} \\ &\leqslant (\sqrt{2}h_K + 2R_K)|u|_{2,\infty,K}. \end{aligned}$$

Combining the last two estimates gives us (2.2), since $h_K \leqslant 2R_K$. □

R e m a r k 2.1. The factor R_K comes only from the estimation of v_y, all other terms can be estimated using h_K. Hence, (2.2) can be written more finely as

$$|u-\Pi_K u|_{1,\infty,K} \leqslant (\sqrt{2}+\tfrac{1}{2})h_K|u|_{2,\infty,K} + R_K\|u_{xx}\|_{0,\infty,K}.$$

Therefore, if $u_{xx}\approx 0$, we obtain an $O(h_K)$ error estimate instead of $O(R_K)$.

Using Theorem 2.1, we can obtain a 'cheap', slightly weakened variant of Lemma 1.3 for general p.

Theorem 2.2 (Cheap circumradius estimate). *Let $K\subset\mathbb{R}^2$ be an arbitrary triangle. Let $u\in C^2(K)$. Then there exists C_p depending only on $p\in[1,\infty]$ such that*

$$|u-\Pi_K u|_{1,p,K} \leqslant C_p R_K |K|^{1/p}|u|_{2,\infty,K}. \tag{2.3}$$

P r o o f. Let $p\in[1,\infty)$. Then from Theorem 2.1

$$\begin{aligned}|u-v|^p_{1,p,K} &= \|u_x-v_x\|^p_{0,p,K} + \|u_y-v_y\|^p_{0,p,K}\\ &\leqslant 2|K||u-v|^p_{1,\infty,K} \leqslant 2|K|(2\sqrt{2}+2)^p R_K^p|u|^p_{2,\infty,K}.\end{aligned}$$

Taking the p-th root gives (2.3). If K is a general element not aligned with the x-axis, we get an additional factor of $2\sqrt{2}$ in the final estimate obtained by transforming K and the seminorms $|u-v|_{1,\infty,K}$, $|u|_{2,\infty,K}$ via (1.7). □

R e m a r k 2.2. The factor $|K|^{1/p}|u|_{2,\infty,K}$ in (2.3) mimics the behavior of the $W^{2,p}(K)$-seminorm in that it is also additive in the p-th power:

$$|u-\Pi_h u|^p_{1,p,\Omega} = \sum_{K\in\mathcal{T}_h}|u-\Pi_K u|^p_{1,p,K} \leqslant \sum_{K\in\mathcal{T}_h} C_p^p R_K^p|K||u|^p_{2,\infty,K} \leqslant C_p^p R^p|\Omega||u|^p_{2,\infty,\Omega},$$

which is an $O(R)$-estimate in $W^{1,p}(\Omega)$ for the global interpolation error. Furthermore, trivially $|u|_{2,p,K}\leqslant 4^{1/p}|K|^{1/p}|u|_{2,\infty,K}$, i.e., (2.3) is a weaker version of (1.4).

In the proof of Theorem 2.1, we did not need the assumption $R_K\leqslant 1$ from Lemma 1.3. This is true for all the proofs in this paper. We note that the role of this condition is not completely clear in the papers of Kobayashi et al. and the estimate is formulated alternately with and without this assumption even within the same paper. The case of $R_K>1$ may seem irrelevant, since in the end we assume (1.5), however, using the ideas of [5], it can be easily shown that (1.5) is not necessary for convergence or even $O(h)$ convergence of the FEM. Therefore any attempt at formulating a necessary and sufficient condition for FEM convergence will need to take the case of $R_K>1$ into account somehow, cf. also [11].

3. The general circumradius estimate

Here we establish a direct connection between Lemmas 1.2 and 1.3. The main idea is to dilate the triangle K in the y-direction by a suitable factor so that the resulting triangle $\widehat{K}$ has a prescribed maximum angle. One can then use the maximum angle condition on $\widehat{K}$ and transform the resulting estimate back to K. The result of this procedure is the factor R_K instead of h_K in the final estimate. First, we need the following simple scaling result.

Lemma 3.1. *Let $a > 0$ and $\widehat{K}$ be obtained from K by the transformation $F\colon K \to \widehat{K}$, $F(x,y) = (x, ay) = (\widehat{x}, \widehat{y})$. For $f\colon K \to \mathbb{R}$ let $\widehat{f}\colon \widehat{K} \to \mathbb{R}$ be defined by $\widehat{f}(\widehat{x},\widehat{y}) = f(F^{-1}(\widehat{x},\widehat{y}))$. Then for $i,j \in \mathbb{N}_0$*

$$\|f_{x^i y^j}\|_{0,p,K} = a^{j-1/p}\|\widehat{f}_{\widehat{x}^i\widehat{y}^j}\|_{0,p,\widehat{K}}. \tag{3.1}$$

P r o o f. By the chain rule and substitution,

$$\begin{aligned}\int_K \big|f_{x^i y^j}(x,y)\big|^p \,\mathrm{d}x\,\mathrm{d}y &= \int_K \big|\widehat{f}_{\widehat{x}^i\widehat{y}^j}(F(x,y))a^j\big|^p \,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{a^{jp}}{\det J_F}\int_{\widehat{K}} \big|\widehat{f}_{\widehat{x}^i\widehat{y}^j}(\widehat{x},\widehat{y})\big|^p \,\mathrm{d}\widehat{x}\,\mathrm{d}\widehat{y},\end{aligned}$$

where $\det J_F = a$ is the Jacobian of F. Taking the p-th root gives (3.1). $\square$

Theorem 3.1 (Circumradius estimate)**.** *Let $u \in W^{2,p}(K)$. There exists C_p depending only on $p \in [1,\infty]$ such that*

$$|u - \Pi_K u|_{1,p,K} \leqslant C_p R_K |u|_{2,p,K}. \tag{3.2}$$

P r o o f. Let $\alpha_0 = \frac{3}{4}\pi$. If $\alpha_K \leqslant \alpha_0$, K satisfies the maximum angle condition, therefore

$$|u - \Pi_K u|_{1,p,K} \leqslant C_p(\alpha_0) h_K |u|_{2,p,K} \leqslant 2C_p(\alpha_0) R_K |u|_{2,p,K},$$

since $h_K \leqslant 2R_K$. Here $C_p(\alpha_0)$ is the constant from Lemma 1.2.

If $\alpha_K > \alpha_0$, we find the unique triangle $\widehat{K}$ with vertices $\widehat{A}, B, C$ such that $\alpha_{\widehat{K}} := \angle B\widehat{A}C = \alpha_0$ and $\widehat{A}, A$ have the same foot H of their altitudes to BC. This is possible, since the set of all vertices $\widehat{A}$ such that $\angle B\widehat{A}C = \alpha_0$ is a circular arc, cf. Figure 2.

Triangles K, $\widehat{K}$ are related as in Lemma 3.1 with $a = h_{\widehat{A}}/h_A$, where $h_{\widehat{A}}$ is the altitude from $\widehat{A}$ to BC. We estimate the factor a. Because $\beta_K \geqslant \gamma_K$, then also

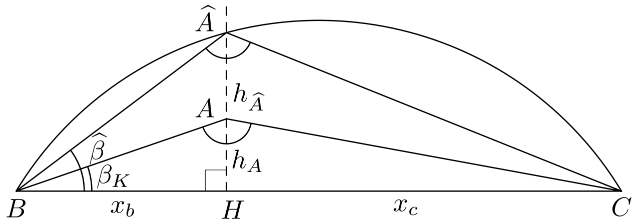

Figure 2. Proof of Theorem 3.1.

$\widehat{\beta} \geqslant \widehat{\gamma}$, where $\widehat{\beta} := \angle \widehat{A}BC$, $\widehat{\gamma} := \angle \widehat{A}CB$. Since $\alpha_{\widehat{K}} = \frac{3}{4}\pi$, then $\widehat{\beta} \in [\frac{1}{8}\pi, \frac{1}{4}\pi)$ and $\tan\widehat{\beta} \in [\sqrt{2}-1, 1)$. As $h_{\widehat{A}} = x_b \tan\widehat{\beta}$, we obtain

$$(\sqrt{2}-1)x_b \leqslant h_{\widehat{A}} < x_b.$$

Because $a = h_{\widehat{A}}/h_A$,

$$(\sqrt{2}-1)x_b h_A^{-1} \leqslant a < x_b h_A^{-1}. \tag{3.3}$$

On the other hand, $h_A = x_b \tan\beta_K < x_b$, since $\beta_K < \pi - \alpha_K < \frac{1}{4}\pi$. Thus $1 < x_b h_A^{-1}$ and from (3.3) we get the sought estimate for a:

$$(\sqrt{2}-1) \leqslant a < x_b h_A^{-1}. \tag{3.4}$$

Finally, we estimate (3.2). Defining $e := u - \Pi_K u$, by Lemma 3.1

$$\begin{aligned} |e|_{1,p,K}^p &= \|e_x\|_{0,p,K}^p + \|e_y\|_{0,p,K}^p = a^{-1}\|\widehat{e}_{\widehat{x}}\|_{0,p,\widehat{K}}^p + a^{p-1}\|\widehat{e}_{\widehat{y}}\|_{0,p,\widehat{K}}^p \\ &\leqslant (a^{-1} + a^{p-1})|\widehat{e}|_{1,p,\widehat{K}}^p \leqslant (a^{-1} + a^{p-1})C_p(\alpha_0)^p h_{\widehat{K}}^p |\widehat{u}|_{2,p,\widehat{K}}^p, \end{aligned} \tag{3.5}$$

since $\widehat{K}$ satisfies the maximum angle condition and $\widehat{e}$ is simply the Lagrange interpolation error for $\widehat{u}$ on $\widehat{K}$. Transforming back to K gives us

$$\begin{aligned} |\widehat{u}|_{2,p,\widehat{K}}^p &= \|\widehat{u}_{\widehat{x}\widehat{x}}\|_{0,p,\widehat{K}}^p + \|\widehat{u}_{\widehat{x}\widehat{y}}\|_{0,p,\widehat{K}}^p + \|\widehat{u}_{\widehat{y}\widehat{y}}\|_{0,p,\widehat{K}}^p \\ &= a\|u_{xx}\|_{0,p,K}^p + a^{1-p}\|u_{xy}\|_{0,p,K}^p + a^{1-2p}\|u_{yy}\|_{0,p,K}^p \\ &\leqslant (a + a^{1-p} + a^{1-2p})|u|_{2,p,K}^p. \end{aligned}$$

Together with (3.5), this gives

$$|e|_{1,p,K}^p \leqslant (2 + 2a^{-p} + a^{-2p} + a^p)C_p(\alpha_0)^p h_K^p |u|_{2,p,K}^p, \tag{3.6}$$

since $h_{\widehat{K}} = h_K$. Due to (3.4), we have $a^{-p} \leqslant (\sqrt{2}-1)^{-p}$, $a^{-2p} \leqslant (\sqrt{2}-1)^{-2p}$, which are constants depending only on p. As for the last remaining factor a^p in (3.6), we have due to (3.4) and Lemma 2.1

$$a^p h_K^p < (x_b h_A^{-1} 2x_c)^p < (4R_K)^p, \tag{3.7}$$

since $h_K/2 < x_c$. Collecting the estimates for a^{-p} and a^{-2p} along with the trivial inequality $h_K^p \leqslant 2^p R_K^p$ and combining with (3.7), we finally obtain from (3.6)

$$|e|_{1,p,K}^p \leqslant \bigl(2^{p+1} + (\sqrt{2}-1)^{-p}2^{p+1} + (\sqrt{2}-1)^{-2p}2^p + 4\bigr)C_p(\alpha_0)^p R_K^p |u|_{2,p,K}^p.$$

Taking the p-th root gives us (3.2). $\square$

Having established a connection with the maximum angle condition, it is perhaps not surprising that all the ingredients needed to prove Theorem 3.1 are already contained in one of the papers dealing with the maximum angle condition.

As stated on the introduction, everything needed to prove Theorem 3.1 is already present in [10], although the final result is never formulated, since the paper only deals with $O(h)$ convergence and the maximum angle condition. However, estimates using R_K are used throughout the paper and Theorem 3.1 could have been obtained in the following way. An intermediate step of (2.22) in [10] states

$$|\det B_K| = \frac{f_K g_K h_K}{2R_K}, \tag{3.8}$$

where $\det B_K$ is the Jacobian of the mapping from a reference triangle to K and f_K, g_K, h_K are the lengths of sides of K. An intermediate step of the chain of inequalities following (2.22) in [10] is

$$|v - \Pi_K v|_{1,p,K} \leqslant 32\widehat{C}|\det B_K|^{-1} f_K g_K h_K |v|_{2,p,K}. \tag{3.9}$$

Substituting (3.8) into (3.9) immediately gives (3.2). The reason this is not done in the paper is that (3.8) is further estimated using the maximum angle condition, thus eliminating R_K from the final estimate in order to obtain $O(h)$ convergence.

4. The circumradius estimate for higher order interpolation

Now we turn our attention to higher order Lagrange interpolation. This is treated in detail in [9], where estimates of the error in $W^{m,p}(K)$ for general m are derived. Here we only show how the $W^{1,p}(K)$ results can again be simply obtained by scaling from existing results assuming the maximum angle condition.

On a triangle K, we define the Lagrange interpolation points of degree $k \in \mathbb{N}$ by

$$\Sigma^k(K) = \Bigl\{\Bigl(\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}\Bigr) \in K;\ a_i \in \{0, \ldots, k\},\ a_1 + a_2 + a_3 = k\Bigr\}, \tag{4.1}$$

where a triplet $(\lambda_1, \lambda_2, \lambda_3)$ denotes the barycentric coordinates on K. Let $\Pi_K^k\colon C(\overline{K}) \to P^k(K)$ be the Lagrange interpolation of degree k on the set $\Sigma^k(K)$. Under the maximum angle condition on $\mathcal{T}_h$, we have the following theorem proved in [6] and indicated in [1] for $p = 2$.

Theorem 4.1 (Jamet). *Let $k \geqslant 2$ be an integer and $p \in [1, \infty]$. Let $u \in W^{k+1,p}(K)$. Then there exists a constant $C_{p,k}(\alpha_K)$ depending only on p, k, and the maximum angle α_K of K such that*

$$|u - \Pi^k_K u|_{1,p,K} \leqslant C_{p,k}(\alpha_K) h^k_K |u|_{k+1,p,K}. \tag{4.2}$$

Here we show that the ideas of Section 3 can be used to obtain circumradius-type estimates for higher order Lagrange interpolation from Theorem 4.1.

Theorem 4.2 (Higher order circumradius estimate). *Let $u \in W^{k+1,p}(K)$. There exists $C_{p,k}$ depending only on $p \in [1, \infty]$ and $k \in \mathbb{N}$ such that*

$$|u - \Pi^k_K u|_{1,p,K} \leqslant C_{p,k} R_K h^{k-1}_K |u|_{k+1,p,K}. \tag{4.3}$$

P r o o f. We take a as in Theorem 3.1 and apply Lemma 3.1. As in (3.5) we get

$$|e|^p_{1,p,K} \leqslant (a^{-1} + a^{p-1}) |\widehat{e}|^p_{1,p,\widehat{K}} \leqslant (a^{-1} + a^{p-1}) C_{p,k}(\alpha_0)^p h^{kp}_{\widehat{K}} |\widehat{u}|^p_{k+1,p,\widehat{K}}\,, \tag{4.4}$$

where $C_{p,k}(\alpha_0)$ is the constant from Theorem 4.1. Due to Lemma 3.1,

$$\begin{aligned} |\widehat{u}|^p_{k+1,p,\widehat{K}} &= \sum_{i=0}^{k+1} \|\widehat{u}_{\widehat{x}^{k+1-i}\widehat{y}^i}\|^p_{0,p,\widehat{K}} = \sum_{i=0}^{k+1} a^{1-ip} \|u_{x^{k+1-i}y^i}\|^p_{0,p,K} \\ &\leqslant \left(\sum_{i=0}^{k+1} a^{1-ip}\right) |u|^p_{k+1,p,K}. \end{aligned}$$

Together with (4.4), we obtain

$$|e|^p_{1,p,K} \leqslant \left(\sum_{i=0}^{k+1} a^{-ip} + \sum_{i=0}^{k+1} a^{(1-i)p}\right) C_{p,k}(\alpha_0)^p h^{kp}_K |u|^p_{k+1,p,K}. \tag{4.5}$$

Due to (3.4), we have $a^{-ip} \leqslant (\sqrt{2} - 1)^{-ip}$ for $i = 0, \ldots, k+1$, which are constants. Only a^p, the first term of the second sum, is estimated using (3.7), resulting in the factor $R^p_K h^{(k-1)p}_K$. Taking the p-th root gives (4.3). □

We note that in [6], Theorem 4.1 is stated under the assumption $k > 2/p$ if $p > 1$ and holds for $k \geqslant 2$ if $p = 1$, although this case is not stated explicitly. In Theorem 4.1, we have simplified these assumptions to $k \geqslant 2$, since the remaining cases correspond to linear interpolation and are treated in Section 3.

5. Optimality of the circumradius estimate

In this section, we show that the circumradius estimate gives the correct scaling as $\alpha_K \to \pi$, i.e., that the factor R_K in (1.4) cannot be improved. This observation relies on the fact that for a quadratic function, the estimates in the proof of Theorem 2.1 are sharp. We consider $p \in [1, \infty)$, the case $p = \infty$ follows similarly.

We take $u(x, y) = x^2$ and again assume without loss of generality that the longest side BC of K is aligned with the x-axis. Then

$$|u|_{2,p,K} = \left(\int_K |u_{xx}|^p \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p} = 2|K|^{1/p}. \tag{5.1}$$

Since $u(x, y) = x^2$, we have $u(A) = u(H)$ and $u_y = 0$. As in the proof of Theorem 2.1, $v := \Pi_K u$ is the 1D linear Lagrange interpolation of u on the side BC, hence

$$|u_y - v_y| = |v_y| = \frac{|u(A) - v(H)|}{h_A} = \frac{|u(H) - v(H)|}{h_A} = \frac{\frac{1}{2} x_b x_c |u_{xx}|}{h_A} = \frac{x_b x_c}{h_A}.$$

Therefore,

$$\begin{aligned} |u - v|_{1,p,K} \geqslant \|u_y - v_y\|_{0,p,K} &= \left(\int_K |u_y - v_y|^p \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p} = \frac{x_b x_c}{h_A} |K|^{1/p} \\ &= 2R_K \cos\beta_K \cos\gamma_K |K|^{1/p} = R_K \cos\beta_K \cos\gamma_K |u|_{2,p,K} \end{aligned} \tag{5.2}$$

due to Lemma 2.1 and (5.1). Taking e.g. $\alpha_K \geqslant \frac{2}{3}\pi$, we have $\beta_K, \gamma_K \leqslant \frac{1}{3}\pi$ and (5.2) gives us

$$|u - \Pi_K u|_{1,p,K} \geqslant \tfrac{1}{4} R_K |u|_{2,p,K}. \tag{5.3}$$

Therefore, the factor R_K in (1.4) cannot be improved for general u and α_K large.

We note that as $\alpha_K \to \pi$, the factor $\cos\beta_K \cos\gamma_K \to 1$. This gives support to the claim of [7], where (1.4) is shown with $C_2 = 1$ based on numerical computations.

Of course for special functions, e.g. $u(x, y) = y^2$, the factor R_K can be replaced by h_K via Remark 2.1.

6. Conclusions

We have presented several observations on the circumradius estimate (condition) and its relation to the maximum angle condition. Since the original proofs of [8] and [12] are lengthy and technical, we gave a simple straightforward proof of the $p = \infty$ case. For $p \in [1, \infty)$, we showed that the result can be obtained by simple scaling of the classical $O(h)$ estimate under the maximum angle condition. This holds also in the case of higher order Lagrange interpolation. Finally, we showed that the factor R_K in the circumradius estimate cannot be improved in general.

Future work includes a purely analytic proof (i.e., without the assistance of numerical computations) of the result from [7] that $C_2 = 1$ can be taken in (1.4).

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