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A COMPACTNESS RESULT FOR POLYHARMONIC MAPS IN THE CRITICAL DIMENSION

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Abstract. For $n = 2m \geqslant 4$, let $\Omega \in \mathbb{R}^n$ be a bounded smooth domain and $\mathcal{N} \subset \mathbb{R}^L$ a compact smooth Riemannian manifold without boundary. Suppose that $\{u_k\} \in W^{m,2}(\Omega, \mathcal{N})$ is a sequence of weak solutions in the critical dimension to the perturbed m-polyharmonic maps

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} E_m(\Pi(u + t\xi)) = 0$$

with $\Phi_k \to 0$ in $(W^{m,2}(\Omega, \mathcal{N}))^*$ and $u_k \rightharpoonup u$ weakly in $W^{m,2}(\Omega, \mathcal{N})$. Then u is an m-polyharmonic map. In particular, the space of m-polyharmonic maps is sequentially compact for the weak-$W^{m,2}$ topology.

Keywords: polyharmonic map; compactness; Coulomb moving frame; Palais-Smale sequence; removable singularity

MSC 2010: 35J35, 35J48, 58J05

1. Introduction

Let $n \geqslant 4$ and let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Assume that $\mathcal{N}$ is a smooth closed N-dimensional Riemannian manifold isometrically embedded in the L-dimensional Euclidean space $\mathbb{R}^L$. Recall the Sobolev space $W^{l,p}(\Omega, \mathcal{N})$, $1 \leqslant l < \infty$ and $1 \leqslant p < \infty$, is defined by

$$W^{l,p}(\Omega, \mathcal{N}) = \{v \in W^{l,p}(\Omega, \mathbb{R}^L)\colon\ v(x) \in \mathcal{N} \text{ a.e. } x \in \Omega\}$$

and equipped with the topology inherited from the topology of the linear Sobolev space $W^{l,p}(\Omega, \mathbb{R}^L)$.

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For $m \in \mathbb{N}$, we consider the m-harmonic energy functional

$$E_m(v) := \int_\Omega |\nabla^m v|^2 = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\Delta^{m/2} v|^2 & \text{if } m \text{ is even,} \\ \dfrac{1}{2}\displaystyle\int_\Omega |\nabla \Delta^{(m-1)/2} v|^2 & \text{if } m \text{ is odd} \end{cases} \tag{1.1}$$

for any $v \in W^{m,2}(\Omega, \mathcal{N})$. In the following context, we set $n = 2m$ and polyharmonic map is the critical points of energy functional (1.1) with $u \in W^{m,2}(\Omega, \mathcal{N})$ in the weak form. More precisely, we have

Definition 1.1. A map $u \in W^{m,2}(\Omega, \mathcal{N})$ is called weakly m-polyharmonic if u is a critical point of the m-polyharmonic energy functional (1.1) with respect to compactly supported variations on $\mathcal{N}$. That is, if for all $\xi \in C_0^\infty(\Omega, \mathbb{R}^L)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E_m(\Pi(u + t\xi)) = 0 \tag{1.2}$$

where Π denotes the nearest point projection onto $\mathcal{N}$.

Note that we consider the tubular neighborhood V_δ of $\mathcal{N}$ in $\mathbb{R}^L$ for $\delta > 0$ sufficiently small, and the smooth nearest point projection $\Pi_{\mathcal{N}}\colon V_\delta \to \mathcal{N}$. For $u(x) \in \mathcal{N}$, let $P(u) := \nabla\Pi(u)$ be the orthonormal projection onto the tangent space $T_u\mathcal{N}$. The orthonormal projection onto the normal space will be denoted by $P^\perp$. Then we readily have $\Delta^m u \perp T_u\mathcal{N}$ in the sense of distributions, i.e. $P(u)\Delta^m u = 0$, which shows the geometric form of the Euler-Lagrange equation for weakly m-polyharmonic maps (cf. [1], [5]).

Generally speaking, a bubbling phenomenon, which appears in various critical problems for nonlinear systems, especially in the investigation of harmonic, biharmonic and p-harmonic maps, leads to defects of the strong convergence. Typically, some part of the energy functional is lost in the limit passage so that the sequence does not have to converge strongly. Therefore, one is usually forced to use subtle tools coming from Lions' concentration compactness theory [10], [11] and harmonic analysis as Sacks-Uhlenbeck do in their pioneering paper [14]. In addition, there is an important idea, originally introduced by Hélein [7] for harmonic maps due to the target manifolds $\mathcal{N}$ without symmetries, to use Coulomb moving frames such that they satisfy an extra system of differential equations. Here, it is necessary to combine the idea of Coulomb moving frames following Gastel and Scheven [4], [5] for m-polyharmonic maps to establish our compactness result. Before stating our main conclusion, let us recall the concept of Palais-Smale sequence of the energy functionals $E_m(u)$ in the Sobolev space $W^{m,2}(\Omega, \mathcal{N})$.

Definition 1.2. A sequence of maps $\{u_k\} \subset W^{m,2}(\Omega, \mathcal{N})$ is called a Palais-Smale sequence of the energy functionals $E_m(u_k)$ on admissible sets $W^{m,2}(\Omega, \mathcal{N})$, if the following two conditions hold:

(a) $u_k \rightharpoonup u$ weakly in $W^{m,2}(\Omega, \mathcal{N})$,

(b) $E_n'(u_k) \to 0$ in $(W^{m,2}(\Omega, \mathcal{N}))^*$,

where $(W^{m,2}(\Omega, \mathcal{N}))^*$ is the dual of $W^{m,2}(\Omega, \mathcal{N})$.

In this note we are devoted to the compactness of polyharmonic maps in the critical dimension. This will be shown via a divergence structure of Euler's equations due to geometrical speciality and Coulomb moving frames rather than Lions' concentration compactness argument. In order to further analyse the behaviour of weakly convergent Palais-Smale sequences for the variational functional in the critical dimension, we suppose that u_k is a Palais-Smale sequence to the perturbed polyharmonic map functional, i.e.,

$$-\Delta^m u_k + \Phi_k \perp T_{u_k}\mathcal{N} \tag{1.3}$$

and

$$\{u_k\} \text{ is bounded in } W^{m,2}(\Omega, \mathbb{R}^L), \quad \Phi_k \to 0 \text{ in } (W^{m,2}(\Omega, \mathbb{R}^L))^*. \tag{1.4}$$

It should be pointed out that Euler's equation of polyharmonic maps (1.2) is a higher order elliptic system with critical nonlinearity in the lower order derivatives. Note that E_m is conformally invariant and the conformal group is noncompact, E_m does not satisfy the Palais-Smale condition (cf. [2], [12]). Hence, this is a highly nontrivial question whether any weak limit u of a Palais-Smale sequence of m-polyharmonic maps is still an m-polyharmonic map. To proceed that way, one must be able to analyse the limit behaviour of weakly convergent Palais-Smale sequences for variational functionals. This is a delicate task since the equation is highly nonlinear, and the right-hand side is not continuous with respect to weak convergence. We are now in a position to state our main result.

Theorem 1.3. *Assume that $\{u_k\} \subset W^{m,2}(\Omega, \mathcal{N})$ is a Palais-Smale sequence of energy functionals* (1.1)*, i.e., they satisfy the relation* (1.3) *with $\Phi_k \to 0$ in $(W^{m,2}(\Omega, \mathcal{N}))^*$, and $u_k \rightharpoonup u$ weakly in $W^{m,2}(\Omega, \mathcal{N})$. Then $u \in W^{m,2}(\Omega, \mathcal{N})$ is an m-polyharmonic map.*

We would like to remark that for $n = 2$, Theorem 1.3 has been first proven by Bethuel [2], later simplified via Lions' concentration compactness method due to Freire-Müller-Struwe [3]. Later, Strzelecki-Zatorska [16], Wang [19] and Zheng [20] obtained various corresponding compact results for higher dimensional H-systems,

n-harmonic maps and biharmonic maps which are based on the arguments originating from Freire, Müller, Struwe's work [3]. Recently, Strzelecki (see Proposition 1.2 in [15]) and Goldstein-Strzelecki-Zatorska (see Theorem 1.2 in [6]) derived the compactness of biharmonic maps in dimension four or polyharmonic maps in the critical dimension into spheres via changing the biharmonic or polyharmonic map equations to the corresponding equivalent divergence forms without employing Lions' concentration compactness approach. In particular, it is worth noting that a weak compactness is a direct consequence of the conservation laws for harmonic and biharmonic maps to general manifold, and H-surface due to Rivière and his collaborator's work [13], [8], [9]. Inspired by Uhlenbeck-Rivière decomposition to obtain weak compactness for a class of fourth order systems, we adopt some of their ideas in the process of our main proof.

As an immediate corollary, we get a compactness conclusion for any weakly convergent sequence of weak solutions of m-polyharmonic maps as follows. More precisely, we have

Corollary 1.4. For $n \geqslant 2$, assume that $\{u_k\} \subset W^{m,2}(\Omega, \mathcal{N})$ is a sequence of m-polyharmonic maps converging weakly to u in $W^{m,2}(\Omega, \mathcal{N})$. Then u is an m-polyharmonic map.

This note is organized as follows. In Section 2, we provide some preliminary lemmas by recalling the Coulomb moving frames, and give Euler's equations of the perturbed m-polyharmonic maps. In Section 3, we prove compactness of a Palais-Smale sequence satisfying (1.4) under the smallness condition on the basis of the divergence structure and geometric properties of m-polyharmonic maps, and then derive the main Theorem 1.3 via removable singularity.

2. Some analytic tools

For $\Omega \subset \mathbb{R}^n$ and $u \in W^{m,2}(\Omega, \mathcal{N})$, let us denote by $u^*T\mathcal{N}$ the pull-back bundle of $T\mathcal{N}$ based on u over Ω, and let $\{e_\alpha\}_{\alpha=1}^N$ be a Coulomb moving frame along $u^*T\mathcal{N}$ if $\{e_\alpha\}_{\alpha=1}^N$ forms an orthonormal base of $T_u\mathcal{N}$ which is a tangent space of $\mathcal{N}$ at the point $u(x)$ for a.e. $x \in \Omega$. Then we have the following perturbed polyharmonic map equation via the Coulomb moving frame.

Lemma 2.1. *For $n \geqslant 2$, suppose that $\{e_\alpha\}_{\alpha=1}^N$ is a Coulomb moving frame along u^*TN. Then $u \in W^{m,2}(\Omega, \mathcal{N})$ satisfies*

$$-\Delta^m u + \Phi \perp T_u\mathcal{N} \quad \textit{a.e. in } \Omega,$$

if and only if for $1 \leqslant \alpha \leqslant N$ *we have*

$$(2.1)\quad \Delta^{m-1}\mathrm{div}\langle \nabla u, e_\alpha\rangle = \langle \Phi, e_\alpha\rangle + \Delta^{m-1}(\langle \nabla u, \nabla e_\alpha\rangle) + \sum_{s=1}^{m-1}\sum_{l=0}^{s}(-1)^{s-l}\binom{s}{l}\big[\Delta^{m-s-1}\mathrm{div}^l(\langle \nabla^{s+1}u, \nabla^{s-l+1}e_\alpha\rangle) + \Delta^{m-s-1}\mathrm{div}^{l+1}(\langle \nabla^s u, \nabla^{s-l+1}e_\alpha\rangle)\big],$$

in the sense of distributions. Here $\Phi \in (W^{m,2}(\Omega, \mathbb{R}^L))^*$.

Proof. Since $\{e_\alpha\}_{\alpha=l}^N$ is a moving frame along u^*TN and $-\Delta^m u + \Phi \perp T_u\mathcal{N}$ a.e. in Ω, we get

$$(2.2)\qquad \langle -\Delta^m u + \Phi, e_\alpha\rangle = 0 \quad \text{for } 1 \leqslant \alpha \leqslant N.$$

A direct computation by induction shows that

$$(2.3)\qquad \Delta^{m-1}\langle \Delta u, e_\alpha\rangle = \langle \Delta^m u, e_\alpha\rangle + \sum_{s=1}^{m-1}\big[\Delta^{m-s-1}\langle \nabla\Delta^s u, \nabla e_\alpha\rangle + \Delta^{m-s-1}\mathrm{div}\langle \Delta^s u, \nabla e_\alpha\rangle\big],$$

which implies

$$(2.4)\qquad \langle \Phi, e_\alpha\rangle = \langle \Delta^m u, e_\alpha\rangle = \Delta^{m-1}\mathrm{div}\langle \nabla u, e_\alpha\rangle - \Delta^{m-1}\langle \nabla u, \nabla e_\alpha\rangle - \sum_{s=1}^{m-1}\big[\Delta^{m-s-1}\langle \nabla\Delta^s u, \nabla e_\alpha\rangle + \Delta^{m-s-1}\mathrm{div}\langle \Delta^s u, \nabla e_\alpha\rangle\big].$$

On the other hand, it follows from Leibniz's rule that

$$(2.5)\qquad \langle \Delta^s u, \nabla e_\alpha\rangle = \sum_{l=0}^{s}(-1)^{s-l}\binom{s}{l}\mathrm{div}^l\langle \nabla^s u, \nabla^{s-l}e_\alpha\rangle.$$

Now we substitute (2.5) into (2.4), and obtain (2.1). □

It is well known that the Coulomb moving frame is an important ingredient for establishing various estimates for harmonic, n-harmonic and biharmonic maps into general target manifolds. Here, the construction of Coulomb moving frames along $W^{m,2}$-maps under the smallness condition on $E_m(u)$ is also inspired by Uhlenbeck's Coulomb gauge construction for Yang-Mills fields [18] and we have to combine it with the higher order estimates (cf. Theorem 5.1 and Lemma 5.3 in [5]). Let us recall the definition and basic properties of Lorentz spaces.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be an open subset. For $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, the Lorentz space $L^{p,q}(\Omega)$ consists of all measurable functions $f\colon \Omega \to \mathbb{R}$ such that

$$\|f\|_{L^{p,q}(\Omega)} = \begin{cases} \left(\displaystyle\int_0^\infty (t^{1/p} f^*(t))^q \frac{\mathrm{d}t}{t}\right)^{1/q} & \text{if } 1 \leqslant q < \infty, \\ \|t^{1/p} f^*(t)\|_{L^\infty(0,\infty)} & \text{if } q = \infty \end{cases}$$

is finite, where $f^* : [0,\infty) \to \mathbb{R}$ denotes the nonincreasing rearrangement of $|f|$:

$$|\{x \in \Omega\colon\ |f(x)| \geqslant s\}| = |\{t \in [0, |\Omega|)\colon\ f^*(t) \geqslant s\}|, \quad s \geqslant 0.$$

It is easy to see that for $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, the Lorentz space $L^{p,q}(\Omega)$ is the dual space of $L^{p/(p-1),q/(q-1)}(\Omega)$. Moreover, $L^{p,p}(\Omega) = L^p(\Omega)$, $L^{p',q'}(\Omega) \subset L^{p,q}(\Omega)$ if $1 < p \leqslant p' < \infty$ and $1 \leqslant q' \leqslant q \leqslant \infty$ and $|\Omega| < \infty$.

In our proof of the main theorem, the following multiplication rule between Lorentz spaces will play a central role, for details see [5], [17].

Proposition 2.3. *Let $1 < a,\ c < \infty$ and $1 \leqslant b,\ d \leqslant \infty$. If $f \in L^{a,b}(\Omega)$, $g \in L^{c,d}(\Omega)$, and $1/a + 1/c = 1/r \leqslant 1$, $1/b + 1/d \geqslant 1/s$, then $fg \in L^{r,s}(\Omega)$, and*

$$\|fg\|_{L^{r,s}(\Omega)} \leqslant C\|f\|_{L^{a,b}(\Omega)}\|g\|_{L^{c,d}(\Omega)}. \tag{2.6}$$

In particular, in the case $1/a + 1/c = 1$, we have $fg \in L^1(\Omega)$, and

$$\|fg\|_{L^1(\Omega)} \leqslant C\|f\|_{L^{a,b}(\Omega)}\|g\|_{L^{c,d}(\Omega)}, \tag{2.7}$$

whenever $1/b + 1/d \geqslant 1$.

Here, we simply introduce the Sobolev-Lorentz space and some related embedding conclusions. As we know, the Sobolev embedding theorem can be generalized to the scale of Lorentz spaces. If $f \in W^k(\mathbb{R}^n, \mathbb{R})$ with $\nabla^k f \in L^{p,q}(\mathbb{R}^n)$ for some $k \in \mathbb{N}$, $1 < p < n/k$ and $1 \leqslant q \leqslant \infty$, then we have $f \in L^{np/(n-kp),q}(\mathbb{R}^n)$ and

$$\|f\|_{L^{np/(n-kp),q}(\mathbb{R}^n)} \leqslant C\|\nabla^k f\|_{L^{p,q}(\mathbb{R}^n)},$$

which can be found in [5], [17]. If they are defined on the ball, analogous statements hold for a Sobolev-Lorentz's embedding inequality and a Lorentz version of Poincaré's inequality as follows, respectively:

$$\|f\|_{L^{np/(n-kp),q}(\mathbb{B})} \leqslant C\sum_{l=0}^{k} \|\nabla^l f\|_{L^{p,q}(\mathbb{B})}, \tag{2.8}$$

and

$$\|f - P_{k-1}(x)\|_{L^{np/(n-kp),\infty}(\mathbb{B})} \leqslant C\|\nabla^k f\|_{L^{p,\infty}(\mathbb{B})}, \tag{2.9}$$

where $P_{k-1}(x)$ denotes an appropriate $(k-1)$-order average of the Taylor polynomial of f over B.

Without loss of generality it suffices to show that u is an m-polyharmonic map only in $2B \subset \Omega$ by a conformal transformation to the energy functional $E_m(u)$. In the sequel, we denote the ball $B = B_r(x) \subset \mathbb{R}^n$, and $\alpha B = B_{\alpha r}(x)$ for any $\alpha > 0$. In virtue of the construction of the Coulomb moving frames, the following higher order estimates are important to attain our main aim, for details see their proof in Section 5 from Gastel and Scheven's paper [5].

Proposition 2.4. *Let $u \in W^{m,2}(2\mathbb{B}, \mathcal{N})$. If there exists an $\varepsilon_0 > 0$ such that*

$$\|\nabla u\|_{W^{m-1,2}(2\mathbb{B})} \leqslant \varepsilon_0, \tag{2.10}$$

then there exists a Coulomb moving frame $\{e_\alpha\}_{\alpha=1}^N \subset W^{m,2}(\mathbb{B}, T\mathcal{N})$ such that its connection form $A = (A_{\alpha\beta}) := (\langle de_\alpha, e_\beta\rangle)$ satisfies

$$d^*A = 0 \quad \text{in } B; \quad \sum_{j=0}^{m} \|\nabla^j A\|_{L^{2m/j+1,1}(\mathbb{B})} \leqslant C\|\nabla u\|_{W^{m-1,2}(2\mathbb{B})}, \tag{2.11}$$

and

$$\sum_{\alpha=1}^{N} \|\nabla^s e_\alpha\|_{L^{2m/s,p}(\mathbb{B})} \leqslant C \sum_{l=1}^{s} \left(\|\nabla^{l-1} A\|_{L^{2m/l,pk/l}(2\mathbb{B})} + \|\nabla^l u\|_{L^{2m/l,pk/l}(2\mathbb{B})}\right)^{k/l} \tag{2.12}$$

for every $1 \leqslant s \leqslant m$ and $1 \leqslant p \leqslant \infty$. Here, the constant C depends only on m and $\mathcal{N}$.

3. Proof of main result

Lemma 3.1 (ε-weak compactness)**.** *For any $n \geqslant 3$, let there exist an $\varepsilon_0 > 0$ such that if $\{u_k\} \subset W^{m,2}(2\mathbb{B}, \mathcal{N})$ is a Palais-Smale sequence satisfying* (1.3) *and* (1.4) *with Ω replaced by $2\mathbb{B}$, and $u_k \rightharpoonup u$ weakly in $W^{m,2}(2\mathbb{B}, \mathcal{N})$ with the following smallness assumption:*

$$\sum_{l=1}^{m} \|\nabla^l u_k\|_{L^{2m/l}(2\mathbb{B})} \leqslant \varepsilon_0.$$

Then $u \in W^{m,2}(\mathbb{B}, \mathcal{N})$ is a m-polyharmonic map.

P r o o f. Let $\varepsilon_0 > 0$ be the same constant as in Proposition 2.4. Then it follows that for any $k \geqslant 1$ there is a Coulomb moving frame $\{e^k_\alpha\}_{\alpha=l}^N$ along $u_k^* TN$ such that its connection form $A^k = (A^k_{\alpha\beta}) := (\langle de^k_\alpha, e^k_\beta\rangle)$ satisfies

$$(3.1) \qquad \delta A^k = 0 \quad \text{in } \mathbb{B}; \quad \sum_{j=0}^m \|\nabla^j A^k\|_{L^{2m/(j+1),1}(\mathbb{B})} \leqslant C\|\nabla u_k\|_{W^{m-1,2}(2\mathbb{B})},$$

and

$$(3.2) \qquad \sum_{\alpha=1}^N \|\nabla^s e^k_\alpha\|_{L^{2m/s,p}(\mathbb{B})} \leqslant C\sum_{l=1}^s \big(\|\nabla^{l-1}A^k\|_{L^{2m/l,pk/l}(2\mathbb{B})} + \|\nabla^l u_k\|_{L^{2m/l,pk/l}(2\mathbb{B})}\big)^{k/l}$$

for every $1 \leqslant s \leqslant m$ and $1 \leqslant p \leqslant \infty$.

Therefore, we may assume, after passing to a subsequence, that $e^k_\alpha \rightharpoonup e_\alpha$ weakly in $W^{m,2}(\mathbb{B}, \mathbb{R}^L)$ and strongly in $W^s(\mathbb{B}, \mathbb{R}^L)$ with $0 \leqslant s \leqslant m-1$, and $\nabla^m A^k \rightharpoonup \nabla^m A$ weakly in $L^{2m/(m+1),1}(\mathbb{B})$. Moreover, $\{e_\alpha\}_{\alpha=1}^N$ is a Coulomb moving frame along u^*TN due to the smoothness property of $\mathcal{N}$. Furthermore, on the basis of Proposition 2.4 we conclude that $A := (\langle de_\alpha, e_\beta\rangle)$ satisfies the estimates

$$(3.3) \qquad d^*A = 0 \quad \text{in } \mathbb{B}; \quad \sum_{j=0}^m \|\nabla^j A\|_{L^{2m/(j+1),1}(\mathbb{B})} \leqslant C\|\nabla u\|_{W^{m-1,2}(2\mathbb{B})} \leqslant C\varepsilon_0,$$

and

$$\sum_{\alpha=1}^N \|\nabla^s e_\alpha\|_{L^{2m/s,p}(\mathbb{B})} \leqslant C\sum_{l=1}^s \big(\|\nabla^{l-1}A\|_{L^{2m/l,pk/l}(2\mathbb{B})} + \|\nabla^l u\|_{L^{2m/l,pk/l}(2\mathbb{B})}\big)^{k/l} \leqslant C\varepsilon_0^{k/l}$$

with every $1 \leqslant s \leqslant m$ and $1 \leqslant p \leqslant \infty$; for details see Lemma 5.1 in [5].

Euler's equations of the critical points of m-harmonic energy functional (1.1), yield that the Palais-Smale sequences $\{u_k\}$ of m-polyharmonic maps have the form

$$(3.4)\ \Delta^{m-1}\operatorname{div}\langle\nabla u_k, e^k_\alpha\rangle = \langle\Phi, e^k_\alpha\rangle + \Delta^{m-1}(\langle\nabla u_k, \nabla e^k_\alpha\rangle) + \sum_{s=1}^{m-1}\sum_{l=0}^s (-1)^{s-l}\binom{s}{l} \times \big[\Delta^{m-s-1}\operatorname{div}^l(\langle\nabla^{s+1}u_k, \nabla^{s-l+1}e^k_\alpha\rangle) + \Delta^{m-s-1}\operatorname{div}^{l+1}(\langle\nabla^s u_k, \nabla^{s-l+1}e^k_\alpha\rangle)\big],$$

where u_k and Φ_k are suitable for the convergence assumptions

$$u_k \rightharpoonup u \quad \text{in } W^{m,2}(2\mathbb{B}, \mathbb{R}^L), \quad \Phi_k \to 0 \quad \text{in } (W^{m,2}(2\mathbb{B}, \mathbb{R}^L))^*. \tag{3.5}$$

Observe $\nabla^s e_\alpha^k \to \nabla^s e_\alpha$ strongly in $L^2(\mathbb{B})$ and $\nabla^s u_k \to \nabla^s u$ strongly in $L^2(\mathbb{B}, \mathbb{R}^L)$ for $s = 0, 1, \ldots, m-1$ due to Proposition 2.4 and Rellich's compactness theorem. Then it is obvious that

$$\Delta^{m-1}\operatorname{div}\langle \nabla u_k, e_\alpha^k\rangle \to \Delta^{m-1}\operatorname{div}\langle \nabla u, e_\alpha\rangle \quad \text{in } \mathcal{D}'(\mathbb{B}), \tag{3.6}$$

$$\Delta^{m-1}(\langle \nabla u_k, \nabla e_\alpha^k\rangle) \to \Delta^{m-1}(\langle \nabla u, \nabla e_\alpha\rangle) \quad \text{in } \mathcal{D}'(\mathbb{B}). \tag{3.7}$$

Notice that

$$|\langle \Phi_k, e_\alpha^k\rangle_{\{(W^{m,2})^*, W^{m,2}\}}| \leqslant \|\Phi_k\|_{(W^{m,2}(\mathbb{B}))^*}\|e_\alpha^k\|_{W^{m,2}(\mathbb{B})} \to 0, \tag{3.8}$$

$$\begin{aligned}\Delta^{m-s-1}&\operatorname{div}^l(\langle \nabla^{s+1} u_k, \nabla^{s-l+1} e_\alpha^k\rangle)\\ &\to \Delta^{m-s-1}\operatorname{div}^l(\langle \nabla^{s+1} u, \nabla^{s-l+1} e_\alpha\rangle) \quad \text{in } \mathcal{D}'(\mathbb{B}),\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\Delta^{m-s-1}&\operatorname{div}^{l+1}(\langle \nabla^{s} u_k, \nabla^{s-l+1} e_\alpha^k\rangle)\\ &\to \Delta^{m-s-1}\operatorname{div}^{l+1}(\langle \nabla^{s} u, \nabla^{s-l+1} e_\alpha\rangle) \quad \text{in } \mathcal{D}'(\mathbb{B})\end{aligned} \tag{3.10}$$

for $0 \leqslant s - l \leqslant m-1$ and $1 \leqslant \alpha \leqslant N$, as $k \to \infty$. Therefore, in order to ensure that the limit map u is an m-polyharmonic map in the sense of distributions due to the above various convergence results, it is a key step to prove that for any $1 \leqslant \alpha \leqslant N$ we have

$$\langle \nabla^m u_k, \nabla^m e_\alpha^k\rangle \to \langle \nabla^m u, \nabla^m e_\alpha\rangle \quad \text{in } \mathcal{D}'(\mathbb{B}). \tag{3.11}$$

To deal with the convergence (3.11) in the sense of distributions, let us introduce the orthogonal projections $P(y)\colon \mathbb{R}^L \to T_y\mathcal{N}$ and $P^\perp(y) := \mathrm{Id} - P(y)\colon \mathbb{R}^L \to (T_y\mathcal{N})^\perp$, for which we can refer to [5], [9]. For simplicity, we write P and $P^\perp$ instead of $P(y)$ and $P^\perp(y)$, respectively, then it follows that

$$\begin{aligned}\langle \nabla^m u_k, \nabla^m e_\alpha^k\rangle &= \langle (P\circ u_k)\nabla^m u_k, \nabla^{m-1}((P^\perp\circ u_k)\nabla e_\alpha^k)\rangle\\ &\quad + \langle (P\circ u_k)\nabla^m u_k, \nabla^{m-1}((P\circ u_k)\nabla e_\alpha^k)\rangle\\ &\quad + \langle (P^\perp\circ u_k)\nabla^m u_k, \nabla^m e_\alpha^k\rangle\\ &:= I + II + III.\end{aligned} \tag{3.12}$$

For the estimate of the first term I, by a direct calculation one shows that

$$
\begin{aligned}
I &= \sum_{l=0}^{m-1} \binom{m-1}{l} \langle (P \circ u_k) \nabla^m u_k, \nabla^l (P^\perp \circ u_k) \nabla^{m-l} e_\alpha^k \rangle \\
&= \sum_{l=1}^{m-1} \binom{m-1}{l} \langle (P \circ u_k) \nabla^m u_k, \nabla^l (P^\perp \circ u_k) \nabla^{m-l} e_\alpha^k \rangle,
\end{aligned}
$$

where we employed $\langle (P \circ u_k) \nabla^m u_k, (P^\perp \circ u_k) \nabla^m e_\alpha^k \rangle = 0$ in the second equality above.

Note that, for each $l = 1, \ldots, m-1$, we have

$$
\nabla^l (P^\perp \circ u_k) = \sum_{s=1}^{l} \binom{l}{s} \nabla^{l-s} P^\perp \nabla^s u_k \to \sum_{s=1}^{l} \binom{l}{s} \nabla^{l-s} P^\perp \nabla^s u = \nabla^l (P^\perp \circ u)
$$

strongly in $L^{2m/l,2}(\mathbb{B})$, where we used $\|P^\perp\|_{C^{m-1}} \leqslant C$ and

$$
\nabla^{m-l} e_\alpha^k \to \nabla^{m-l} e_\alpha^k \quad \text{strongly in } L^{2m/(m-l),2}(\mathbb{B}).
$$

Therefore, it follows by the duality property of Lorentz spaces that

$$
\begin{aligned}
I = &\sum_{l=1}^{m-1} \binom{m-1}{l} \langle (P \circ u_k) \nabla^m u_k, \nabla^l (P^\perp \circ u) \nabla^{m-l} e_\alpha \rangle \\
&+ \sum_{l=1}^{m-1} \binom{m-1}{l} \langle (P \circ u_k) \nabla^m u_k, (\nabla^l (P^\perp \circ u_k) \nabla^{m-l} e_\alpha^k - \nabla^l (P^\perp \circ u) \nabla^{m-l} e_\alpha) \rangle \\
\to &\sum_{l=1}^{m-1} \binom{m-1}{l} \langle (P \circ u) \nabla^m u, \nabla^l (P^\perp \circ u) \nabla^{m-l} e_\alpha \rangle \quad \text{in } \mathcal{D}'.
\end{aligned}
$$

To estimate the second term $I\!I$, we use the identity $(P \circ u_k) \nabla e_\alpha^k = \sum_{\beta=1}^{N} A_{\alpha\beta}^k e_\beta^k$, which implies

$$
\begin{aligned}
I\!I &= \sum_{\beta=1}^{N} \sum_{l=0}^{m-1} \binom{m-1}{l} \langle (P \circ u_k) \nabla^m u_k, \nabla^l (A_{\alpha\beta}^k) \nabla^{m-l} e_\beta^k \rangle \\
&= \sum_{\beta=1}^{N} \sum_{l=1}^{m-1} \binom{m-1}{l} \langle (P \circ u_k) \nabla^m u_k, \nabla^l (A_{\alpha\beta}^k) \nabla^{m-l} e_\beta^k \rangle,
\end{aligned}
$$

where the second equality is due to $\delta A_{\alpha\beta}^k = 0$. Noting that $\nabla^l (A_{\alpha\beta}^k) \to \nabla^l (A_{\alpha\beta})$ strongly in $L^{2m/l,2}(\mathbb{B})$ and $\nabla^{m-l} e_\beta^k \to \nabla^{m-l} e_\beta$ strongly in $L^{2m/(m-l),2}(\mathbb{B})$ for

$1 \leqslant l \leqslant m-1$, similarly to the case I we deduce

$$
\begin{aligned}
I\!I &= \sum_{\beta=1}^{N} \sum_{l=1}^{m-1} \binom{m-1}{l} \langle (P \circ u_k) \nabla^m u_k, \nabla^l (A^k_{\alpha\beta}) \nabla^{m-l} e^k_\beta \rangle \\
&\to \sum_{\beta=1}^{N} \sum_{l=1}^{m-1} \binom{m-1}{l} \langle (P \circ u) \nabla^m u, \nabla^l (A_{\alpha\beta}) \nabla^{m-l} e_\beta \rangle \quad \text{in } \mathcal{D}'.
\end{aligned}
$$

Finally, to estimate the third term, in virtue of $(P^\perp \circ u_k)\nabla u_k = 0$ we obtain

$$
0 = \nabla^{m-1}((P^\perp \circ u_k)\nabla u_k) = \sum_{l=1}^{m-1} \binom{m-1}{l} \nabla^l (P^\perp \circ u_k) \nabla^{m-l} u_k + (P^\perp \circ u_k) \nabla^m u_k,
$$

or

$$
I\!I\!I = -\sum_{l=1}^{m-1} \binom{m-1}{l} \langle \nabla^l (P^\perp \circ u_k) \nabla^{m-l} u_k, \nabla^m e^k_\alpha \rangle.
$$

Note that for $1 \leqslant l \leqslant m-1$ we have

$$
\begin{gathered}
\nabla^m e^k_\alpha \rightharpoonup \nabla^m e_\alpha \quad \text{weakly in } L^2(\mathbb{B}), \\
\nabla^{m-l} u_k \to \nabla^{m-l} u_k \quad \text{strongly in } L^{2m/m-l,2}(\mathbb{B})
\end{gathered}
$$

and

$$
\|\nabla^l (P^\perp \circ u_k) - \nabla^l (P^\perp \circ u)\|_{L^{2m/l,2}} \leqslant C \sum_{j=0}^{l} \|\nabla^l u_k - \nabla^l u\|_{L^{2m/l,2}} \to 0 \quad \text{strongly in } \mathbb{B},
$$

which is due to $\|P^\perp\|_{C^{m-1}} \leqslant C$. Therefore,

$$
\begin{aligned}
I\!I\!I &= -\sum_{l=1}^{m-1} \binom{m-1}{l} \langle \nabla^l (P^\perp \circ u_k) \nabla^{m-l} u_k, \nabla^m e^k_\alpha \rangle \\
&\to -\sum_{l=1}^{m-1} \binom{m-1}{l} \langle \nabla^l (P^\perp \circ u) \nabla^{m-l} u, \nabla^m e_\alpha \rangle \quad \text{in } \mathcal{D}'.
\end{aligned}
$$

Now, by substituting the convergence of I, $I\!I$, and $I\!I\!I$ into (3.12), we obtain (3.11).

Putting the above convergence of (3.6), (3.7), (3.8), (3.9), (3.10), and (3.11) together, by (3.4) we have

$$
\begin{aligned}
\Delta^{m-1} \operatorname{div} \langle \nabla u, e_\alpha \rangle = \Delta^{m-1}(\langle \nabla u, \nabla e_\alpha \rangle) &+ \sum_{s=1}^{m-1} \sum_{l=0}^{s} (-1)^{s-l} \binom{s}{l} \\
&\times \big[\Delta^{m-s-1} \operatorname{div}^l (\langle \nabla^{s+1} u, \nabla^{s-l+1} e_\alpha \rangle) \\
&\qquad + \Delta^{m-s-1} \operatorname{div}^{l+1} (\langle \nabla^s u, \nabla^{s-l+1} e_\alpha \rangle) \big],
\end{aligned} \tag{3.13}
$$

which implies that u is an m-polyharmonic map. $\square$

On the basis of Lemma 3.1, the energy of all u_k can concentrate only on a finite "bad set" $\Sigma = \Omega \setminus G$, while the "good set" G consists of those x_0 for which $\liminf\limits_{k\to\infty} \int_{B_r(x_0)} \sum\limits_{l=1}^{m} |\nabla^l u_k|^{2m/l} \leqslant \varepsilon_0^2$ for the given sufficiently small ε_0 and some $r > 0$. Therefore, we derive that the "bad set" Σ is finite due to the boundedness of the total energy. One may have good uniform regularity estimates on G allowing one to pass to the limit on G and leave finitely many singularities in Σ. As a direct consequence of the standard removable singularity argument, we obtain that the weak limit of a sequence of polyharmonic maps is a polyharmonic map. Now, we give the proof of main theorem.

Proof of Theorem 1.3. Assume that $\{u_k\}_{k=1}^{\infty} \in W^{m,2}(\Omega, \mathcal{N})$ is a sequence of extrinsic m-polyharmonic maps and $u_k \rightharpoonup u$ weakly in $W^{m,2}(\Omega, \mathcal{N})$. Since u_k is a bounded sequence in $W^{m,2}(\Omega, \mathcal{N})$, we have that $\mu_k := \int_\Omega \sum\limits_{l=1}^{m} |\nabla^l u_k|^{2m/l} \,\mathrm{d}x$ is a family of nonnegative Radon measures with $M = \sup\limits_k \mu_k(\Omega) < \infty$. Therefore, after passing to a subsequence, we may suppose that there is a nonnegative Radon measure μ on Ω such that

$$\mu_k := \int_\Omega \sum_{l=1}^{m} |\nabla^l u_k|^{2m/l} \,\mathrm{d}x \to \mu,$$

as the convergence of Radon measures. Let $\varepsilon_0 > 0$ be the same constant as in Lemma 3.1 and define Σ by

$$\Sigma := \{x \in \Omega \colon \mu(\{x\}) \geqslant \varepsilon_0^2\}. \tag{3.14}$$

Then by a simple covering argument we have that Σ is a finite set. In fact

$$H^0(\Sigma) \leqslant \frac{M}{\varepsilon_0^2}, \quad M = \sup_k \int_\Omega \sum_{l=1}^{m} |\nabla^l u_k|^{2m/l} \,\mathrm{d}x < \infty,$$

where H^0 is a 0-dimensional Hausdorff measure. In fact, let $\Sigma' = \{x_1, \ldots, x_m\} \subset \Sigma$ be any finite subset of Σ, then there is a small $\delta_0 > 0$ such that $\{B_{\delta_0}(x_i)\}_{i=1}^{s}$ are mutually disjoint balls and

$$\liminf_{k\to\infty} \int_{B_{\delta_0}(x_i)} \sum_{l=1}^{m} |\nabla^l u_k|^{2m/l} \,\mathrm{d}x \geqslant \varepsilon_0^2, \quad i = 1, 2, \ldots, s,$$

which implies that there is a natural number $K_s \in \mathbb{N}$ such that for any $k \geqslant K_s$ we have

$$\int_{B_{\delta_0}(x_i)} \sum_{l=1}^{m} |\nabla^l u_k|^{2m/l} \,\mathrm{d}x \geqslant \varepsilon_0^2, \quad i = 1, 2, \ldots, s.$$

Therefore, for any $k \geqslant K_s$ we have

$$
\begin{aligned}
s\varepsilon_0^2 &\leqslant \sum_{i=1}^{s} \int_{B_{\delta_0}(x_i)} \sum_{l=1}^{m} |\nabla^l u_k|^{2m/l}\,\mathrm{d}x = \int_{\bigcup_{i=1}^{s} B_{\delta_0}(x_i)} \sum_{l=1}^{m} |\nabla^l u_k|^{2m/l}\,\mathrm{d}x \\
&\leqslant \int_{\Omega} \sum_{l=1}^{m} |\nabla^l u_k|^{2m/l}\,\mathrm{d}x \leqslant M < \infty.
\end{aligned}
$$

This implies $s \leqslant M\varepsilon_0^{-2}$.

Therefore, for any $x_0 \in \Omega \setminus \Sigma$ there exists an $r_0 > 0$ such that $\mu(B_{2r_0}(x_0)) < \varepsilon_0^2$. On account of

$$
\limsup_{k\to\infty} \int_{B_{r_0}(x_0)} \sum_{l=1}^{m} |\nabla^l u_k|^{2m/l}\,\mathrm{d}x \leqslant \mu(B_{2r_0}(x_0)),
$$

we may assume that there exists $K_0 \geqslant 1$ such that

$$
\int_{B_{r_0}(x_0)} \sum_{l=1}^{m} |\nabla^l u_k|^{2m/l}\,\mathrm{d}x \leqslant \varepsilon_0^2. \tag{3.15}
$$

Therefore, such a property leads to non-compactness of polyharmonic maps in dimension $n = 2m$ only at a finite number of points in Ω. Lemma 3.1 implies that u is an m-polyharmonic map in $B_{r_0}(x_0)$. Since the point $x_0 \in \Omega \setminus \Sigma$ is arbitrary, we conclude that u is an m-polyharmonic map in $\Omega \setminus \Sigma$. It is a standard argument to show that u is an m-polyharmonic map in Ω (see [2], [3] or [1], [5], [16]). The proof of Theorem 1.3 is complete. □

Remark 3.2. We claim that every $W^{m,2}$-weak limit of a sequence of polyharmonic maps u_k into a compact Riemannian manifold $\mathcal{N}$ is again a polyharmonic map. Here we follow a scheme using the method of Coulomb moving frames and removability of isolated singularities. In fact, a polyharmonic map is a higher order elliptic system with critical nonlinearity. Note that E_m is conformal invariant and the conformal group is non-compact, E_m does not satisfy the Palais-Smale condition. Hence, this is a highly nontrivial result in the setting of the general Riemannian manifold $\mathcal{N}$ because there are unbounded functions in $W^{m,2}(\mathbb{R}^n, \mathcal{N})$, $n = 2m$. In particular, for the special case $\mathcal{N} = \mathbb{S}^{k-1}$ this is indeed trivial since it follows from the fact that the polyharmonic map equations can be rewritten in the divergence form.

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