Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 66 (2016), No. 1, 57–62

Persistent URL: http://dml.cz/dmlcz/144876

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Czechoslovak Mathematical Journal, 66 (141) (2016), 57–62

A PURE SMOOTHNESS CONDITION FOR RADÓ'S THEOREM FOR α-ANALYTIC FUNCTIONS

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(Received November 11, 2014)

Abstract. Let $\Omega \subset \mathbb{C}^n$ be a bounded, simply connected $\mathbb{C}$-convex domain. Let $\alpha \in \mathbb{Z}_+^n$ and let f be a function on Ω which is separately $C^{2\alpha_j-1}$-smooth with respect to z_j (by which we mean jointly $C^{2\alpha_j-1}$-smooth with respect to $\operatorname{Re} z_j$, $\operatorname{Im} z_j$). If f is α-analytic on $\Omega \setminus f^{-1}(0)$, then f is α-analytic on Ω. The result is well-known for the case $\alpha_i = 1$, $1 \leqslant i \leqslant n$, even when f a priori is only known to be continuous.

Keywords: α-analytic function; polyanalytic function; zero set; Radó's theorem

MSC 2010: 35G05, 30C15, 32A99, 32U15

1. Introduction

Radó's theorem states that a continuous function on an open subset of $\mathbb{C}^n$ that is holomorphic off its zero set extends to a holomorphic function on the given open set. For the one-dimensional result see Radó [14], and for a generalization to several variables, see e.g. Cartan [6].

Definition 1.1. Let $\Omega \subset \mathbb{C}^n$ be an open subset and let $(z_1, \ldots, z_n)$ denote the holomorphic coordinates for $\mathbb{C}^n$. A function f, on Ω, is said to be *separately C^k-smooth with respect to the z_j-variable*, if for any fixed $(c_1, \ldots, c_{n-1}) \in \mathbb{C}^{n-1}$, such that the function

$$z_j \mapsto f(c_1, \ldots, c_{j-1}, z_j, c_j, \ldots, c_{n-1}),$$

is well-defined as z_j varies in some nonempty open set, the latter function is jointly C^k-smooth with respect to $\operatorname{Re} z_j, \operatorname{Im} z_j$. For $\alpha \in \mathbb{Z}_+^n$ we say that f is separately α-smooth if f is separately C^{α_j}-smooth with respect to z_j for each $1 \leqslant j \leqslant n$.

Avanissian and Traore [2], [1] introduced the following definition of polyanalytic functions of order α in several variables.

Definition 1.2 (Avanissian and Traore [2]). Let $\Omega \subset \mathbb{C}^n$ be a domain and let $z = x + \mathrm{i}y$ denote the holomorphic coordinates in $\mathbb{C}^n$. A function $f \in C^\infty(\Omega, \mathbb{C})$ is called *polyanalytic of order* α if there exists a multi-index $\alpha \in \mathbb{Z}_+^n$ such that in a neighborhood of every point of Ω, $(\partial/\partial \overline{z}_j)^{\alpha_j} f(z) = 0$, $1 \leqslant j \leqslant n$. If the integer α_j, $1 \leqslant j \leqslant n$, is minimal, then f is said to be *polyanalytic of exact order* α.

In this paper we shall prove a version of Radó's theorem for polyanalytic functions of order $q \geqslant 1$, that are C^{2q-1}-smooth (note that the case $q = 1$ does *not* reduce to the usual Radó's theorem for holomorphic functions, because we require that the starting function be C^1-smooth, not merely continuous). Our main result is Theorem 2.3 which is the induced result in several variables.

Our proof will rely upon a result from potential theory.

A known result from potential theory. It is known that a C^1-smooth function g on a domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, which is harmonic on $\Omega \setminus g^{-1}(0)$ is automatically harmonic on Ω, see Král [11] (see also Král [10]). Note that this result is not true if we only assume that g is continuous: take for example $g(x, y) = x$ for $x > 0$ and $g(x, y) = 0$ otherwise.

Definition 1.3. Let $\Omega \subset \mathbb{C}$ be an open subset. A function f on Ω is called *polyharmonic of order* q if $f \in C^{2q}(\Omega)$ and $\Delta^q f = 0$ on Ω, where Δ denotes the Laplace operator.

It is known (see e.g. Tarkhanov [15], page 94) that a function u satisfies $\Delta^m u = 0$ if and only if there are harmonic functions u_j, $1 \leqslant j < m$, such that $u(x) = \sum_{j=1}^{m-1} |x|^{2j} u_j(x)$.

The following result appears without proof in Chesnokov [8], page 38, C6, and a proof of the result in the case $n_1 = n$ can be found in Harvey and Polking [9], Theorem 4.3 d.

Theorem 1.4 (Chesnokov [8], and Harvey and Polking [9]). *Let $\Omega \subset \mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ be a domain, let (x, y) denote the Euclidean coordinates, let L be a linear differential operator on Ω that is of order $2m$ with respect to x, and let $l < 2m$. If $Lf = 0$ on $\Omega \setminus A$ for some $f \in C^l(\Omega)$ and A satisfying $H^{n_1-2m+l}(A \cap \{y = 0\}) < \infty$ (here H^α is the α-dimensional (outer) Hausdorff measure), then $Lf = 0$ on Ω.*

Example 1.5. Setting $n_1 = n = 2$, $l = 2m-1$, and $A = f^{-1}(0)$, into Theorem 1.4 will reduce the necessary condition to $H^{n_1-2m+l}(f^{-1}(0)) = H^1(f^{-1}(0)) < \infty$. Hence Theorem 1.4 reduces to stating that if $f \in C^{2m-1}(\Omega)$ (where $\Omega \subset \mathbb{R}^2$ is a bounded domain) is polyharmonic of order m on $\Omega \setminus f^{-1}(0)$ (in the sense that $\Delta^m f = 0$ on $\Omega \setminus f^{-1}(0)$), and if $H^1(f^{-1}(0)) < \infty$, then f is polyharmonic of order m on Ω.

It is well-known that zero sets of (real-valued) harmonic functions are never isolated when $n \geqslant 2$ (see e.g. Axler et al. [3], page 6), and it is also clear that zero sets of polyharmonic functions can be submanifolds of dimension $n-1$. For example, let $\Omega = \{|z| < 1\} \subset \mathbb{C}$ and set $f(z) = z - \overline{z}$. Then $f^{-1}(0) = \Omega \cap \{\operatorname{Im} z = 0\}$ which is a one-dimensional line segment, of finite, one-dimensional Hausdorff measure. If we were to replace Ω by (the unbounded domain) $\{|\operatorname{Im} z| < 1\}$, then $f^{-1}(0)$ would not have finite one-dimensional Hausdorff measure, though f would be a well-defined polyharmonic function of order 2 (in fact it is also 2-analytic) on Ω.

Corollary 1.6 (to Theorem 1.4). *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain. If $f \in C^{2m-1}(\Omega)$ is polyharmonic of order m on $\Omega \setminus f^{-1}(0)$ (in the sense that $\Delta^m f = 0$ on $\Omega \setminus f^{-1}(0)$), then f is polyharmonic of order m on Ω.*

Proof. Let (x, y) denote the Euclidean coordinates for $\mathbb{R}^2$ and assume without loss of generality $0 \in \Omega$ in these coordinates. Setting $n = 2$ and $n_1 = n_2 = 1$, $l = 2m-1$, and $A = f^{-1}(0)$, in Theorem 1.4 will reduce the sufficient condition to $H^{n_1-2m+l}(f^{-1}(0)) = H^0(f^{-1}(0) \cap \{y_1 = 0\}) < \infty$. It is well-known that if u is a harmonic function and if R is an orthogonal matrix then $u(s)$ is harmonic in $s = Rx + p$ (in other words the property of being harmonic is invariant under rigid coordinate changes). Hence if u is polyharmonic of order m, the function $v := \Delta^{m-1} u$ is harmonic and such that

$$0 = \Delta_x^m u(x) = \Delta_x v(x) = \Delta_s v(s) = \Delta_s^m u(s),$$

where s is obtained by rotation and translation with respect to x. We shall need the following.

Definition 1.7 (Balk [5], page 4). Let $U \subset \mathbb{C}$ be a domain and let $p \in E \subset U$. We say that the line $l := \{z \in \mathbb{C}\colon\ z = p + t\mathrm{e}^{\mathrm{i}\theta},\ |t| < \infty,\ t \in \mathbb{R}\}$, p and θ constants, is a *limiting direction of the set E at p* if E contains a sequence of points $z_j = p + t_j \mathrm{e}^{\mathrm{i}\theta_j}$, $t_j \to 0$, $\theta_j \to \theta$, $t_j \neq 0$. The point p is called a *condensation point of order k of E* if there are k different lines through p which are limiting directions of E.

The following uniqueness property is known.

Lemma 1.8 (Balk [4], page 202). *Let $U \subset \mathbb{R}^2$ be a simply connected domain and let u and v be polyharmonic functions on U. Assume that $u = v$ on a subset $E \subset U$ such that E has a condensation point of order ∞. Then $u \equiv v$ on U. Consequently, if a polyharmonic function on U vanishes on E, then it vanishes identically.*

Lemma 1.9. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Assume that $f \in C^{2m-1}(\Omega)$ is polyharmonic of order m on $\Omega \setminus f^{-1}(0)$ (in the sense that $\Delta^m f = 0$ on $\Omega \setminus f^{-1}(0)$).*

Then either f vanishes on Ω, or for each point $p \in f^{-1}(0)$ there is a pair (B, l), where $B \subset \Omega$ is a ball centered at p and l is a straight line through p, such that $l \cap B \cap f^{-1}(0)$ is finite.

Proof. Assume there exists a point $p \in f^{-1}(0)$ such that for every ball $B(p, \varepsilon) \subset \Omega$ (centered at p and of radius $\varepsilon > 0$) and every line $l_\theta := \{(x, y) \in \mathbb{R}^2\colon (x, y) = p + t\mathrm{e}^{\mathrm{i}\theta},\ |t| < \infty,\ t \in \mathbb{R}\}$, the set $l_\theta \cap f^{-1}(0) \cap B(p, \varepsilon)$ contains infinitely many points. Letting $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence of positive real numbers such that $\varepsilon_j \to 0$, we obtain that l_θ is a limiting direction of the set $f^{-1}(0)$ at p. This implies that p is a condensation point of order ∞ of $f^{-1}(0)$. Hence we can apply Lemma 1.8 using $E := f^{-1}(0)$, $U := \Omega$, in order to obtain that $f \equiv 0$. This proves Lemma 1.9. □

Let $p \in f^{-1}(0)$. By Lemma 1.9, there exists a straight line l through p such that $l \cap B \cap f^{-1}(0)$ is finite, which in turn implies that $H^0(l \cap B \cap f^{-1}(0)) < \infty$. Set $s = R[x, y]^{\mathrm{T}} + p$ where R is an orthogonal matrix such that $l = \{R[x, 0]^{\mathrm{T}} + p\colon x \in \mathbb{R}\}$. Then we obtain that $f(s)$ is polyharmonic of order m near $R[0, 0]^{\mathrm{T}} + p$, on $\{R[x, y]^{\mathrm{T}} + p\colon (x, y) \in B\}$. Then f is polyharmonic of order m on an open neighborhood of p in the variables (x, y). Since p was arbitrary in the zero set of f this implies that f is polyharmonic of order m on an open neighborhood of $f^{-1}(0)$ in Ω. This completes the proof of Corollary 1.6. □

Tarkhanov [15], page 42, announced that Chesnokov [7] (which is a dissertation in Russian) generalized Radó's theorem to polyharmonic functions; the announcement of the result of Chesnokov is also made in Pokrovskii [13], page 69, who specifies that this is regarding polyharmonic functions of order k in the class $C^{2k-1}(\Omega)$. *Hence, though we have here presented a separate proof, Corollary* 1.6 *is a known, unpublished, result, due to Chesnokov* [7] *in a dissertation.*

2. Statement and proof of the main theorem

Theorem 2.1 (auxiliary to the main result)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded, simply connected domain. Let $f \in C^{2q-1}(\Omega)$ be a q-analytic function on $\Omega \setminus f^{-1}(0)$, for some $q \geqslant 1$. Then f is q-analytic on Ω.*

Proof. Let $f = u + \mathrm{i}v$ where $u = \operatorname{Re} f$, $v = \operatorname{Im} f$. It is a known result, see Balk [4], page 200, that $\operatorname{Re} f$ is polyharmonic of order q. Now $f^{-1}(0) \subseteq u^{-1}(0)$, whence $u \in C^{2q-1}(\Omega)$ and u is polyharmonic of order q on $\Omega \setminus u^{-1}(0)$. By Radó's theorem for sufficiently smooth polyharmonic functions, given by Corollary 1.6, it follows that u is polyharmonic of order q on all of Ω. Similarly we conclude that v is polyharmonic of order q on Ω. Thus f satisfies

$$0 = \Delta^q f = \overline{D}^q(D^q f), \tag{2.1}$$

meaning that $D^q f$ is q-analytic on Ω. On the other hand, it is known (see e.g. Krantz [12], Lemma 4.6.6, page 197) that if $\overline{D}f$ and Df are L^2, then

$$\|\overline{D}f\|_{L^2} = \|Df\|_{L^2}, \tag{2.2}$$

and by iteration $\|\overline{D}^q f\|_{L^2} = \|D^q f\|_{L^2}$. Furthermore, $\Omega \backslash f^{-1}(0)$ is open and assuming $f \not\equiv 0$ (if $f \equiv 0$ we are done), it is also nonempty, and thus

$$0 = \|\overline{D}^q f\|_{L^2(\Omega \backslash f^{-1}(0))} = \|D^q f\|_{L^2(\Omega \backslash f^{-1}(0))}. \tag{2.3}$$

If $V \subset \mathbb{C}$ is a bounded open subset and g is a polyharmonic function of order $k > 1$ on V, then: $(0 = \|g\|_{L^2(V)} \Rightarrow g \equiv 0$ on $V)$. Indeed, $0 = \|g\|_{L^2(V)} = \int_V |g|^2$ implies that the smooth real-valued nonnegative function $g\overline{g}$, vanishes a.e. on V; thus $g\overline{g} \equiv 0$ on V. Hence we have (using $g = D^q f$ and $V = \Omega \setminus f^{-1}(0)$, in the previous argument) that $D^q f = 0$ on $\Omega \backslash f^{-1}(0)$. However we also know that $D^q f$ is q-analytic on Ω, and a q-analytic function which vanishes on an open subset, vanishes on the whole connected component of that subset. Since Ω is connected, this implies that $D^q f = 0$ on Ω, which, as we have pointed out above (i.e. using $|D^q f| \Leftrightarrow D^q f = 0$ and $|D^q f| \Rightarrow |\overline{D}^q f| = 0$), implies that $\overline{D}^q f = 0$ on Ω. This completes the proof. □

Next we shall need the following result on separately polyanalytic functions, due to Avanissian and Traore [1].

Theorem 2.2 ([1], Theorem 1.3, page 264)**.** *Let $\Omega \subset \mathbb{C}^n$ be a domain and let $z = (z_1, \ldots, z_n)$ denote the holomorphic coordinates in $\mathbb{C}^n$ with* $\operatorname{Re} z =: x$, $\operatorname{Im} z = y$. *Let f be a function which, for each j, is polyanalytic of order α_j in the variable $z_j = x_j + \mathrm{i} y_j$ (in such case we shall simply say that f is separately polyanalytic of order α). Then f is jointly smooth with respect to (x, y) on Ω and furthermore is polyanalytic of order $\alpha = (\alpha_1, \ldots, \alpha_n)$ in the sense of Definition* 1.2.

Theorem 2.3 (Main result)**.** *Let $\Omega \subset \mathbb{C}^n$ be a bounded $\mathbb{C}$-convex domain. Let $\alpha \in \mathbb{Z}_+^n$ and let f be a function on Ω which is separately $C^{2\alpha_j - 1}$-smooth with respect to z_j. If f is α-analytic on $\Omega \setminus f^{-1}(0)$, then f is α-analytic on Ω.*

P r o o f. Denote for a fixed $c \in \mathbb{C}^{n-1}$, $\Omega_{c,k} := \{z \in \Omega \colon z_j = c_j,\ j < k,\ z_j = c_{j-1},\ j > k\}$. Since Ω is $\mathbb{C}$-convex, $\Omega_{c,k}$ is simply connected. Consider the function $f_c(z_k) := f(c_1, \ldots, c_{k-1}, z_k, c_k, \ldots, c_{n-1})$. Clearly, f_c is α_k-analytic on $\Omega_{c,k} \setminus f^{-1}(0)$ for any $c \in \mathbb{C}^{n-1}$. Since $f_c^{-1}(0) \subseteq f^{-1}(0)$, Theorem 2.1 applies to f_c meaning that f is *separately* polyanalytic of order α_j in the variable $z_j, 1 \leqslant j \leqslant n$. By Theorem 2.2 the function f must be polyanalytic of order α (in the sense of Definition 1.2) on Ω. This completes the proof. □

We do not know how much it is possible to loosen the smoothness condition on f, but it is clear that continuity alone is not enough. Take for example the function

$$f(z) = \begin{cases} |z|^2 - 1, & |z| \geqslant 1, \\ 1 - |z|^2, & |z| < 1. \end{cases}$$

Then f is continuous and 2-analytic off its zero set $\{|z| = 1\}$, but not 2-analytic.

Acknowledgment. We thank the referee for helpful comments.

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