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EDGE-COLOURING OF GRAPHS AND HEREDITARY GRAPH PROPERTIES

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Abstract. Edge-colourings of graphs have been studied for decades. We study edge-colourings with respect to hereditary graph properties. For a graph G, a hereditary graph property $\mathcal{P}$ and $l \geqslant 1$ we define $\chi'_{\mathcal{P},l}(G)$ to be the minimum number of colours needed to properly colour the edges of G, such that any subgraph of G induced by edges coloured by (at most) l colours is in $\mathcal{P}$. We present a necessary and sufficient condition for the existence of $\chi'_{\mathcal{P},l}(G)$. We focus on edge-colourings of graphs with respect to the hereditary properties $\mathcal{O}_k$ and $\mathcal{S}_k$, where $\mathcal{O}_k$ contains all graphs whose components have order at most $k+1$, and $\mathcal{S}_k$ contains all graphs of maximum degree at most k. We determine the value of $\chi'_{\mathcal{S}_k,l}(G)$ for any graph G, $k \geqslant 1$, $l \geqslant 1$, and we present a number of results on $\chi'_{\mathcal{O}_k,l}(G)$.

Keywords: edge-colouring; proper colouring; hereditary graph property

MSC 2010: 05C15, 05C35

1. Introduction

In this paper we study undirected graphs without loops and multiple edges. We denote the vertex set of a graph G by $V(G)$ and the edge set by $E(G)$. The order of a graph G is the number of its vertices and the size of G is the number of edges of G. We denote the complete graph, the cycle and the path of order n by K_n, C_n and P_n, respectively. The star of order $n+1$ and size n is denoted by S_n. The distance between two vertices $u, v \in V(G)$ (two edges $e, f \in E(G)$) is the number of edges in a shortest path between u and v (between e and f). The diameter of a graph is the greatest of the distances between all pairs of vertices in a graph. The degree of a vertex $v \in V(G)$ is the number of edges incident to v. The maximum degree of G, denoted by Δ, is the largest vertex degree of G.

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The chromatic index (the edge chromatic number) $\chi'(G)$ of a graph G is the smallest number of colours necessary to colour each edge of G, such that no two edges incident to the same vertex have the same colour. So it is the number of distinct colours in a minimum proper edge-colouring. The chromatic index has been studied extensively. It was shown by Vizing [12] in 1964 that the chromatic index of any graph G is either Δ or $\Delta + 1$. This famous result is known as Vizing's theorem. If the chromatic index of a graph G is Δ, then G is said to be of class one, otherwise G is said to be of class two. In 1977, Erdős and Wilson [5] showed that almost all graphs are of class one. From the work of König [10] published in 1916 it follows that all bipartite graphs are of class one, thus trees and even cycles are of class one, but odd cycles are of class two. It is also well-known that complete graphs of even order are of class one, while complete graphs of odd order are of class two. The general problem of deciding which graphs belong to which class remains unsolved.

Generalized edge-colourings that require edges to be coloured properly are bounded below by the chromatic index. Various generalizations of the concept of edge-colouring have been studied, one example is the strong edge-colouring, introduced by Fouquet and Jolivet [7], which requires all edges adjacent to a given edge to receive different colours. The strong edge-colouring led to the introduction of the k-intersection edge-colouring of a graph by Muthu et al. [11], which is a proper edge-coloring in which the cardinality of the intersection of the two respective sets of colours appearing on edges incident to adjacent vertices u and v is at most k. Then the k-intersection chromatic index of a graph G is the smallest number of colours in a possible k-intersection edge-colouring of G.

Another research problem which has been studied is a proper acyclic edge-colouring of a graph, which is a colouring having no bichromatic (two colourable) cycles. Then the acyclic chromatic index of a graph G is defined to be the minimum number k such that there is an acyclic edge-colouring of G using k colours. The acyclic chromatic index has been studied for various classes of graphs. For example, Basavaraju and Chandran [1] studied complete bipartite graphs, and planar graphs were considered by Fiedorowicz, Hałuszczak and Narayanan [6], and by Hou, Wang and Zhang [9]. For more results on generalized edge-colourings of graphs, see the recent paper of Yang, Chen and Ma [13].

In this paper we study edge-colourings with respect to graph properties. A graph property is any isomorphism-closed class of graphs. We say that a graph G has the property $\mathcal{P}$ if $G \in \mathcal{P}$. A property $\mathcal{P}$ is hereditary if, whenever G is in $\mathcal{P}$, and H is a subgraph of G, then H is also in $\mathcal{P}$. We denote the set of all graphs by $\mathcal{I}$. Let us mention a few hereditary properties:

$$\mathcal{O} = \{G \in \mathcal{I} \colon\ E(G) = \emptyset\},$$

$$\mathcal{S}_k = \{G \in \mathcal{I} \colon \text{ the maximum degree of } G \text{ is at most } k\},$$
$$\mathcal{O}_k = \{G \in \mathcal{I} \colon \text{ each component of } G \text{ has order at most } k+1\},$$
$$\mathcal{I}_k = \{G \in \mathcal{I} \colon G \text{ does not contain } K_{k+2} \text{ as a subgraph}\}.$$

One of the first papers dealing with this topic is the work of Hałuszczak and Vateha [8]. Czap and Mihók [3] considered edge-colourings in which the edges coloured by the same colour induce a subgraph of some hereditary property $\mathcal{P}$. Another research problem on non-proper edge-colourings and hereditary properties is described and studied in [4].

We introduce the following very general problem. For a hereditary graph property $\mathcal{P}$, positive integer l and a graph G we define $\chi'_{\mathcal{P},l}(G)$ to be the minimum number of colours needed to properly colour the edges of G, such that any subgraph of G induced by edges coloured by (at most) l colours is in $\mathcal{P}$. In the next sections we present a number of results on $\chi'_{\mathcal{P},l}(G)$.

2. Preliminary results

Since the colourings which we study are proper, $\chi'_{\mathcal{P},l}(G)$ is bounded below by the chromatic index $\chi'(G)$ of G.

Lemma 2.1. *For any hereditary graph property $\mathcal{P}$, positive integer l and graph G, if $\chi'_{\mathcal{P},l}(G)$ exists, then $\chi'(G) \leqslant \chi'_{\mathcal{P},l}(G)$.*

Let us define the set of minimal forbidden subgraphs of $\mathcal{P}$. We use this concept to prove a necessary and sufficient condition for the existence of $\chi'_{\mathcal{P},l}(G)$.

Definition 2.2. The set of minimal forbidden subgraphs of a hereditary graph property $\mathcal{P}$ is defined as

$$\mathbf{F}(\mathcal{P}) = \{G \in \mathcal{I} \colon G \text{ is not in } \mathcal{P}, \text{ but every proper subgraph of } G \text{ is in } \mathcal{P}\}.$$

It is known (see [2]) that

$$\mathbf{F}(\mathcal{O}) = \{K_2\},$$
$$\mathbf{F}(\mathcal{S}_k) = \{S_{k+1}\},$$
$$\mathbf{F}(\mathcal{O}_k) = \{T_{k+2} \in \mathcal{I} \colon T_{k+2} \text{ is any tree of order } k+2\},$$
$$\mathbf{F}(\mathcal{I}_k) = \{K_{k+2}\}.$$

The following lemma which was presented by Borowiecki et al. in [2] is a direct consequence of the definition of $\mathbf{F}(\mathcal{P})$.

Lemma 2.3. *Let $\mathcal{P}$ be a hereditary graph property. Then a graph G is in $\mathcal{P}$ if and only if no subgraph of G is in $\mathbf{F}(\mathcal{P})$.*

Note that if G is not in $\mathcal{P}$, then by Lemma 2.3 there exists (at least one) subgraph of G that is in $\mathbf{F}(\mathcal{P})$. Therefore there exists a smallest (with respect to size) subgraph G' of G, such that G' is in $\mathbf{F}(\mathcal{P})$.

Now we present a necessary and sufficient condition for the existence of $\chi'_{\mathcal{P},l}(G)$.

Lemma 2.4. *Let $\mathcal{P}$ be any hereditary graph property, let l be any positive integer and let G be any graph. Let us denote by G' a subgraph of G of the smallest size m', such that G' is in $\mathbf{F}(\mathcal{P})$. Then $\chi'_{\mathcal{P},l}(G)$ exists if and only if $l < m'$ or G is in $\mathcal{P}$.*

P r o o f. Suppose that G is in $\mathcal{P}$. Since $\mathcal{P}$ is hereditary, every subgraph of G is in $\mathcal{P}$. We can colour the edges of G properly using $\chi'(G)$ colours. Then $\chi'_{\mathcal{P},l}(G)$ exists and $\chi'_{\mathcal{P},l}(G) = \chi'(G)$.

If G is not in $\mathcal{P}$, then G' exists and we assume that $l < m'$. Let m be the number of edges of G. We colour the edges of G by m different colours (any two edges are coloured by different colours). Then any subgraph H of G induced by edges coloured by at most l colours has size at most l, which is smaller than m'. Since the size of H is less than m' and G' is a subgraph of G in $\mathbf{F}(\mathcal{P})$, which has the smallest size, H is not in $\mathbf{F}(\mathcal{P})$ and no subgraph of H is in $\mathbf{F}(\mathcal{P})$. Therefore by Lemma 2.3, H is in $\mathcal{P}$. Since for this colouring, any subgraph of G induced by edges coloured by at most l colours is in G, we have $\chi'_{\mathcal{P},l}(G) \leqslant m$, so $\chi'_{\mathcal{P},l}(G)$ exists.

Now assume that $\chi'_{\mathcal{P},l}(G)$ exists. So it is possible to colour the edges of G by $\chi'_{\mathcal{P},l}(G)$ colours, such that any subgraph of G induced by edges coloured by at most l colours is in $\mathcal{P}$. Suppose to the contrary that G is not in $\mathcal{P}$ and $l \geqslant m'$. Since G' has size $m' \leqslant l$, edges of G' can be coloured by at most l colours, which implies that G' is in $\mathcal{P}$, contradicting the fact that G' is in $\mathbf{F}(\mathcal{P})$. The proof is complete. □

The first part of the proof of Lemma 2.4 yields the following corollary:

Corollary 2.5. *For any hereditary graph property $\mathcal{P}$ and $l \geqslant 1$, if G is in $\mathcal{P}$, then*

$$\chi'_{\mathcal{P},l}(G) = \chi'(G).$$

We focus on edge-colourings of graphs with respect to the properties $\mathcal{S}_k$ and $\mathcal{O}_k$. Since $\mathbf{F}(\mathcal{S}_k) = \{S_{k+1}\}$, $\mathbf{F}(\mathcal{O}_k) = \{T_{k+2} \in \mathcal{I}\colon\ T_{k+2}$ is any tree of order $k+2\}$ and the graphs S_{k+1} and T_{k+2} have size $k+1$, for the properties $\mathcal{S}_k$ and $\mathcal{O}_k$ we have $m' = k+1$ in Lemma 2.4. Hence we obtain the following corollary:

Corollary 2.6. *For any graph G and $l \geqslant 1$, if $\mathcal{P}$ is the property $\mathcal{S}_k$ or $\mathcal{O}_k$, then $\chi'_{\mathcal{P},l}(G)$ exists if and only if G is in $\mathcal{P}$ or $l \leqslant k$.*

In the next two sections we study the value of $\chi'_{\mathcal{O}_k,l}(G)$ for trees, paths and cycles. By the previous corollary it suffices to consider the case $l \leqslant k$; otherwise $\chi'_{\mathcal{O}_k,l}(G)$ does not exist or if G is in $\mathcal{O}_k$, then by Corollary 2.5 the problem is trivial.

We show that it is not complicated to find the value of $\chi'_{\mathcal{P},l}(G)$ if $\mathcal{P}$ is the property $\mathcal{S}_k$.

Theorem 2.7. *Let $k \geqslant 1$, $l \geqslant 1$ and let G be any graph. If $\chi'_{\mathcal{S}_k,l}(G)$ exists, then*

$$\chi'_{\mathcal{S}_k,l}(G) = \chi'(G).$$

Proof. Suppose that $\chi'_{\mathcal{S}_k,l}(G)$ exists. Then by Corollary 2.6, G is in $\mathcal{S}_k$ or $l \leqslant k$. If G is in $\mathcal{S}_k$, then by Corollary 2.5 we have $\chi'_{\mathcal{S}_k,l}(G) = \chi'(G)$.

Therefore in the rest of this proof we can assume that G is not in $\mathcal{S}_k$ and $l \leqslant k$. Since our colouring must be proper, we need at least $\chi'(G)$ colours to colour the edges of G. Therefore $\chi'_{\mathcal{S}_k,l}(G) \geqslant \chi'(G)$.

Let us colour the edges of G by $\chi'(G)$ colours properly. Since all edges incident to any vertex of G are coloured by different colours, any subgraph H of G induced by edges coloured by at most l colours contains only vertices of degree at most l. Since $l \leqslant k$, H is in $\mathcal{S}_k$ and consequently $\chi'_{\mathcal{S}_k,l}(G) = \chi'(G)$. □

It is well-known that the chromatic index of any graph G is either Δ or $\Delta + 1$. Since $\chi'(G) = \Delta$ for all bipartite graphs G and $\chi'(C_n) = \Delta + 1$ for the cycles C_n if n is odd, we obtain the following corollaries:

Corollary 2.8. *Let $1 \leqslant l \leqslant k$. Then $\chi'_{\mathcal{S}_k,l}(T) = \Delta$ for any tree T.*

Corollary 2.9. *Let $1 \leqslant l \leqslant k$ and $n \geqslant 3$. Then*

$$\chi'_{\mathcal{S}_k,l}(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

3. Edge-colouring of paths and cycles with respect to the property $\mathcal{O}_k$

In this section we study the value of $\chi'_{\mathcal{O}_k,l}(G)$ if G is the path P_n, $n \geqslant 1$, or the cycles C_n, $n \geqslant 3$.

Theorem 3.1. *Let $1 \leqslant l \leqslant k$ and $n \geqslant 1$. Then*

$$\chi'_{\mathcal{O}_k,l}(P_n) = \begin{cases} \chi'(P_n) & \text{if } n \leqslant k+1, \\ l+1 & \text{otherwise.} \end{cases}$$

Proof. If $n \leqslant k+1$, then the path of length n, P_n, is in $\mathcal{O}_k$, and consequently by Corollary 2.5, $\chi'_{\mathcal{O}_k,l}(P_n) = \chi'(P_n)$.

Suppose that $n \geqslant k+2$ (which is at least 3). We have $\chi'_{\mathcal{O}_k,l}(P_n) \geqslant l+1$; otherwise if we colour the edges of P_n by at most l colours, the subgraph of P_n induced by edges coloured by at most l colours is the same graph, P_n, which does not lie in $\mathcal{O}_k$ for $n \geqslant k+2$. Let us denote the edges of P_n by $e_1, e_2, \dots, e_{n-1}$. We colour the edge e_i by the colour $c_{i \bmod (l+1)}$, where $i = 1, 2, \dots, n-1$. Then any subgraph, say H, of P_n induced by edges coloured by l colours consists of paths containing at most l edges, since the edges coloured by one of the $l+1$ colours are not included in H. So every component of H is a path of order at most $l+1$, which is in $\mathcal{O}_l$. Since $l \leqslant k$, we have $\chi'_{\mathcal{O}_k,l}(P_n) = l+1$. □

Theorem 3.2. *Let $1 \leqslant l \leqslant k$ and $n \geqslant 3$.*

(1) *If $n \leqslant k+1$ or $l = 1$, then $\chi'_{\mathcal{O}_k,l}(C_n) = \chi'(C_n)$.*

(2) *If $n \geqslant k+2$ and $l \geqslant 2$, then*

   (i) *$\chi'_{\mathcal{O}_k,l}(C_n) = l+1$ if $0 \leqslant r \leqslant q(k-l)$,*

   (ii) *$l+1 \leqslant \chi'_{\mathcal{O}_k,l}(C_n) \leqslant 2l-k+1+\lceil r/q \rceil$ otherwise, where $n = q(l+1)+r$ and $0 \leqslant r \leqslant l$.*

Proof. (1) If $l = 1$, then in any proper colouring, a subgraph of C_n induced by edges coloured by one colour is a union of K_2's, which is in $\mathcal{O}_1$, therefore $\chi'_{\mathcal{O}_k,1}(C_n) = \chi'(C_n)$.

If $n \leqslant k+1$, then the cycle of order n, C_n is in $\mathcal{O}_k$ and by Corollary 2.5, we have $\chi'_{\mathcal{O}_k,l}(C_n) = \chi'(C_n)$.

(2) Let $n \geqslant k+2$ and $l \geqslant 2$. We have

$$\chi'_{\mathcal{O}_k,l}(C_n) \geqslant l+1;$$

otherwise if $\chi'_{\mathcal{O}_k,l}(C_n) \leqslant l$, then the subgraph of C_n induced by edges coloured with (at most) l colours is the same graph C_n, which is not in $\mathcal{O}_k$ for $n > k+1$.

We have $n = q(l+1) + r$ where $0 \leqslant r \leqslant l$. Since $l \leqslant k$, we have $n \geqslant l+2$, which implies that $q \geqslant 1$. We divide the edges of C_n into q edge disjoint paths $P^1, P^2, \dots, P^q$ (subgraphs of C_n) such that the lengths of any two paths differ by at most 1, the last vertex of P^i is the first vertex of P^{i+1}, $i = 1, 2, \dots, q-1$, and the last vertex of P^q is the first vertex of P^1 (if $q = 1$, we have only one path containing all the edges of C_n). We distinguish 2 cases:

(i) $0 \leqslant r \leqslant q(k-l)$: We have $n = q(l+1)+r \leqslant q(l+1)+q(k-l) = q(k+1)$. From the size of the cycle C_n it follows that $l+1 \leqslant |E(P^i)| \leqslant k+1$ for every $i = 1, 2, \dots, q$. We colour first $l+1$ edges of P^i successively by colours $c_{l+1}, c_l, \dots, c_1$, and if P^i has more than $l+1$ edges, we colour the other edges successively by the colours $c_2, c_1, c_2, c_1, \dots$ (the last edge of P^i is coloured either by c_1 or by c_2). Note that all edges of C_n are coloured and the colouring is proper.

Consider any subgraph H of C_n induced by edges coloured by at most l colours. Let c_j be the colour which is not used in the colouring of edges of H, $1 \leqslant j \leqslant l+1$. Since the $(l+2-j)$-th edge of every P^i is coloured by c_j and $l+1 \leqslant |E(P^i)| \leqslant k+1$, it follows that H consists of paths of length at most k (and order at most $k+1$). Therefore H is in $\mathcal{O}_k$, which implies that $\chi'_{\mathcal{O}_k,l}(C_n) = l+1$.

(ii) $q(k-l) < r \leqslant l$: Let $r = q(k-l)+r'$. Then $r' > 0$. We have $n = q(l+1)+r = q(l+1)+q(k-l)+r' = q(k+1)+r'$. Note that for any $i = 1, 2, \dots, q$, we have $|E(P^i)|$ is $k+1+\lceil r'/q \rceil$ or $k+1+\lfloor r'/q \rfloor$, where at least one path P^i has size $k+1+\lceil r'/q \rceil$. We colour the first $k-l$ edges of each P^i successively by the colours $c_2, c_1, c_2, c_1, \dots$, and we colour the other $l+1+j$ edges of P^i successively by the colours $c_{l+1+j}, c_{l+j}, \dots, c_1$, where $j = \lceil r'/q \rceil$ or $\lfloor r'/q \rfloor$. The number of colours used for this colouring of edges of C_n is

$$l + 1 + \left\lceil \frac{r'}{q} \right\rceil = l + 1 + \left\lceil \frac{r}{q} - (k-l) \right\rceil = 2l - k + 1 + \left\lceil \frac{r}{q} \right\rceil.$$

We show that any subgraph H' of C_n which is a path of length $k+1$ is coloured by at least $l+1$ colours. Any two edges coloured by a colour c_j, where $j \neq 1$, $j \neq 2$, are at distance at least k in C_n, so there is at most one edge in H' coloured by c_j. On the other hand, C_n contains q paths of length $k-l+2$ as subgraphs, which are coloured only by c_1 and c_2. However, since the length of any P^i, $1 \leqslant i \leqslant q$, is at least $k+1$, it follows that H' (which is of length $k+1$) cannot contain more than $k-l+2$ edges coloured by c_1 or c_2. This means that at least $(k+1) - (k-l+2) = l-1$ other colours must be used to colour H'.

Therefore any subgraph H of C_n induced by edges coloured by at most l colours consists only of paths of length at most k (and order at most $k+1$). Hence H is in $\mathcal{O}_k$, which yields the bound $\chi'_{\mathcal{O}_k,l}(C_n) \leqslant 2l-k+1+\lceil r/q \rceil$. The proof is complete. □

Now we consider the value of $\chi'_{\mathcal{O}_k,l}(C_n)$ for $k = l$. All cases which are not solved completely in the previous theorem are considered and solved in Theorem 3.3.

Theorem 3.3. *Let $k \geqslant 2$ and $n \geqslant k+2$ be such that $n = q(k+1) + r$ where $1 \leqslant r \leqslant k$. Then*

$$\chi'_{\mathcal{O}_k,k}(C_n) = k + 1 + \left\lceil \frac{r}{q} \right\rceil.$$

Proof. From Theorem 3.2, we have $\chi'_{\mathcal{O}_k,k}(C_n) \leqslant k + 1 + \lceil r/q \rceil$. We prove by contradiction that $\chi'_{\mathcal{O}_k,k}(C_n) = k + 1 + \lceil r/q \rceil$.

Since $\chi'_{\mathcal{O}_k,k}(C_n)$ is an integer and there is no integer in the half-closed interval of real numbers $[k+1+r/q, k+1+\lceil r/q \rceil)$, we can assume that $\chi'_{\mathcal{O}_k,k}(C_n) < k+1+r/q$. Since $n = q(k+1)+r$, there must be a colour, say c, which is used to colour at least $q+1$ edges of C_n (otherwise, if each of the $\chi'_{\mathcal{O}_k,k}(C_n)$ colours is used to colour at most q edges of C_n, then the number of edges $n \leqslant q\chi'_{\mathcal{O}_k,k}(C_n) < q(k+1+r/q) = q(k+1)+r$). Then there must be 2 edges, say e_1, e_2, of distance at most $k-1$ in C_n, which are coloured by c (otherwise, if there are at least k edges between any two edges coloured by c, then $n \geqslant k(q+1)+q+1 = q(k+1)+(k+1)$). This implies that there exists a connected subgraph H of C_n (H is a path), which consists of $k+1$ edges and contains e_1 and e_2. The edges of H are coloured by at most k colours, which means that H is not in $\mathcal{O}_k$. A contradiction. □

4. Edge-colouring of trees with respect to the property $\mathcal{O}_k$

First we present two lemmas, which are used to prove Theorem 4.3.

Lemma 4.1. *Let $1 \leqslant l \leqslant k$. Then for any tree T of order n such that $n \leqslant k+1$,*

$$\chi'_{\mathcal{O}_k,l}(T) = \Delta.$$

Proof. If $n \leqslant k+1$, then any tree T of order n is in $\mathcal{O}_k$ and by Corollary 2.5 we get $\chi'_{\mathcal{O}_k,l}(T) = \chi'(T)$. □

Lemma 4.2. *Let $k \geqslant 1$. Then for any tree T of diameter at most $k+1$ and order n such that $n \geqslant k+2$,*

$$\chi'_{\mathcal{O}_k,k}(T) = n - 1.$$

P r o o f. Let $n \geqslant k+2$. Then T is not in $\mathcal{O}_k$. Since the distance between any two edges in a tree of diameter at most $k+1$ is at most $k-1$, no two different edges e, f of T can be coloured by the same colour. Otherwise any connected subgraph H of T which consists of $k+1$ edges and contains e and f would be of order $k+2$ and coloured by at most k colours (H would not be in $\mathcal{O}_k$). Hence $\chi'_{\mathcal{O}_k,k}(T)$ is equal to the number of edges of T, which is $n-1$. □

In the next two theorems we determine the value of $\chi'_{\mathcal{O}_k,l}(T')$ for all complete Δ-trees T' when $k=l$. Let $\Delta \geqslant 2$ and $h \geqslant 1$. A complete Δ-tree of height h is a tree T', which has exactly one central vertex v, all leaves of T' are at distance h from v, and all internal vertices of T' have degree Δ.

Theorem 4.3. *Let $\Delta \geqslant 2$ and $k \geqslant 1$. Let T' be a complete Δ-tree of height at most $\lceil k/2 \rceil$ and order n. Then*

$$\chi'_{\mathcal{O}_k,k}(T') = \begin{cases} \Delta & \text{if } n \leqslant k+1, \\ n-1 & \text{otherwise.} \end{cases}$$

P r o o f. Since the diameter of T' is at most $k+1$, by Lemma 4.2 we have $\chi'_{\mathcal{O}_k,k}(T') = n-1$ if $n \geqslant k+2$, and by Lemma 4.1 we obtain $\chi'_{\mathcal{O}_k,k}(T') = \Delta$ if $n \leqslant k+1$. □

Now we study complete Δ-trees T' of height at least $\lceil k/2 \rceil + 1$. We call an edge e of T' a descendent of an edge f of T' if and only if the path connecting the central vertex v and e contains f.

Theorem 4.4. *Let $\Delta \geqslant 2$ and $k \geqslant 1$. Let T' be a complete Δ-tree of height at least $\lceil k/2 \rceil + 1$. Then*

$$\chi'_{\mathcal{O}_k,k}(T') = \begin{cases} \Delta \sum\limits_{i=0}^{k/2-1} (\Delta-1)^i + (\Delta-1)^{k/2} & \text{if } k \text{ is even,} \\ \Delta \sum\limits_{i=0}^{(k-1)/2} (\Delta-1)^i & \text{if } k \text{ is odd.} \end{cases}$$

P r o o f. Let T' be a complete Δ-tree of height $h \geqslant \lceil k/2 \rceil + 1$ with central vertex v. An edge is in the i-th level, $i = 1, 2, \ldots, h$, if it is incident to the vertices, such that one of them is at distance $i-1$ from v, and the other vertex is at distance i from v. Note that the number of edges at level at most $\lceil k/2 \rceil$ is

$$\Delta + \Delta(\Delta-1) + \Delta(\Delta-1)^2 + \ldots + \Delta(\Delta-1)^{\lceil k/2 \rceil - 1} = \Delta \sum_{i=0}^{\lceil k/2 \rceil - 1} (\Delta-1)^i = a.$$

Any two edges e, f of T' which are of distance at most $k-1$ must be coloured by different colours. Otherwise any connected subgraph H of T' which consists of $k+1$ edges and contains e and f would be of order $k+2$ and coloured by at most k colours (so H would not be in $\mathcal{O}_k$). Since the distance between any two edges of T' which are at level at most $\lceil k/2 \rceil$ is at most $k-1$, they must be coloured by different colours and we obtain

$$\chi'_{\mathcal{O}_k,k}(T') \geqslant a.$$

So we use a colours to colour the edges in the first $\lceil k/2 \rceil$ levels.

For $i \geqslant \lceil k/2 \rceil + 1$ an edge f from the $(i - \lceil k/2 \rceil)$-th level is the parent edge of an edge e from the i-th level if e is a descendent of f. Note that there are exactly $(\Delta-1)^{\lceil k/2 \rceil}$ edges in the i-th level which are descendents of the same edge (called their parent edge) from the $(i - \lceil k/2 \rceil)$-th level, thus the edges in the i-th level can be divided into sets of cardinality $(\Delta-1)^{\lceil k/2 \rceil}$, such that all the edges in one set have the same parent edge.

For even k, let S_1 be any set of $(\Delta-1)^{k/2}$ edges in the $(k/2+1)$-th level such that all edges in S_1 have the same parent edge (which is in the first level). Since the distance between any two edges in S_1 is at most $k-2$ and the distance between any edge in S_1 and an edge which is in a level at most $k/2$ is smaller than k, we must use $(\Delta-1)^{k/2}$ new colours to colour the edges in S_1. This implies that if k is even, then

$$\chi'_{\mathcal{O}_k,k}(T') \geqslant a + (\Delta-1)^{k/2}.$$

It remains to show that we can colour the edge set of T' by a colours if k is odd, and by $a + (\Delta-1)^{k/2}$ colours if k is even, such that any subgraph H of T induced by edges coloured by at most k colours consists of components of order at most $k+1$ (and size k). This means that we must show that any two edges e, f coloured by the same colour are at distance at least k in T' (which means that there must be at least k edges in the path connecting e and f coloured by k different colours).

We are going to colour the edges in the $(\lceil k/2 \rceil + 1)$-th level, then the edges in the $(\lceil k/2 \rceil + 2)$-th level, and so on, till we colour all h levels. We study the cases when k is even or odd separately.

(i) k is even: In the $(k/2+1)$-th level there are Δ sets of edges such that all edges in one set have the same parent edge. We can denote these sets by $S_1, S_2, \dots, S_\Delta$; $|S_i| = (\Delta-1)^{k/2}$, $i = 1, 2, \dots, \Delta$. The same set of $(\Delta-1)^{k/2}$ colours which were used to colour the edges of S_1 can be used to colour the edges of any set S_i, $i = 2, 3, \dots, \Delta$, because the distance between any edge in S_i and any edge in S_j is k $(1 \leqslant i < j \leqslant k)$.

We show that we do not need any new colours to colour the rest of $E(T')$. Let us colour the edges in the i-th level for any $k/2 + 2 \leqslant i \leqslant h$ and assume that all edges in the levels up to $i-1$ have been coloured, and no edges in the levels greater than i have been coloured.

Let S be any set of $(\Delta-1)^{k/2}$ edges in the i-th level such that all edges in S have the same parent edge, say g. Note that among the colored edges any edge which is at distance at most $k/2-2$ from some edge in S is a descendent of g, and except for descendents of g, the edge g is the only coloured edge which is at distance $k/2-1$ from any edge in S. This implies that among the coloured edges, the number of edges at distance at most $k/2-1$ from the edges in S is

$$b = 1 + (\Delta-1) + (\Delta-1)^2 + \ldots + (\Delta-1)^{k/2-1}.$$

Except for these edges, the other coloured edges which are at distance at most $k-1$ from the edges in S can be reached only via the edge g and therefore their number is

$$c = (\Delta-1) + (\Delta-1)^2 + \ldots + (\Delta-1)^{k/2}.$$

Since $b+c=a$, the number of coloured edges at distance at most $k-1$ from the edges in S is a, and hence at most a colours are used to colour these edges. Therefore there are $(\Delta-1)^{k/2}$ colours among the colours used to colour the previous levels of T' available to colour the edges of S. It follows that the edges of T' can be coloured by $a+(\Delta-1)^{k/2}$ colours, such that any two edges coloured by the same colour are at distance at least k.

(ii) k is odd: We can assume that all edges in the levels up to $i-1$ have been coloured and we present a colouring of edges in the i-th level for any $(k+1)/2+1 \leqslant i \leqslant h$.

Let S be any set of $(\Delta-1)^{(k+1)/2}$ edges in the i-th level such that all edges in S have the same parent edge, say g. Among the coloured edges all edges at distance at most $(k-1)/2$ from the edges in S are descendents of g or the edge g, and their number is

$$b = 1 + (\Delta-1) + (\Delta-1)^2 + \ldots + (\Delta-1)^{(k-1)/2}.$$

Except for these edges, the other coloured edges which are at distance at most $k-1$ from the edges in S can be reached only via the edge g and their number is

$$c = (\Delta-1) + (\Delta-1)^2 + \ldots + (\Delta-1)^{(k-1)/2}.$$

The number of coloured edges at distance at most $k-1$ from the edges in S is $b+c = a-(\Delta-1)^{(k+1)/2}$. Hence $(\Delta-1)^{(k+1)/2}$ edges of S can be colored by $(\Delta-1)^{(k+1)/2}$ colours which have been used to colour previous levels, but none of these colours is used to colour the edges at distance at most $k-1$ from edges in S. So in total a colours suffice to colour the edges of T', such that any two edges coloured by the same colour are at distance at least k. The proof is complete. □

Finally, we present a corollary of Theorem 4.4, which gives the value of $\chi'_{\mathcal{O}_k,k}(T)$ for a wider range of trees T.

Corollary 4.5. *Let $\Delta \geqslant 2$ and $k \geqslant 1$. Let T be any tree which contains a complete Δ-tree T' of height $\lceil (k+1)/2 \rceil$ as a subgraph ($V(T') \subset V(T)$). Then*

$$\chi'_{\mathcal{O}_k,k}(T) = \begin{cases} \Delta \sum\limits_{i=0}^{k/2-1} (\Delta-1)^i + (\Delta-1)^{k/2} & \text{if } k \text{ is even,} \\ \Delta \sum\limits_{i=0}^{(k-1)/2} (\Delta-1)^i & \text{if } k \text{ is odd.} \end{cases}$$

P r o o f. Note that $n \geqslant k+2$, so T is not in $\mathcal{O}_k$. Let T' be a subgraph of T such that T' is a complete Δ-tree of height $\lceil (k+1)/2 \rceil$. If k is even, by Theorem 4.4 we have $\chi'_{\mathcal{O}_k,k}(T') = \Delta \sum\limits_{i=0}^{k/2-1} (\Delta-1)^i + (\Delta-1)^{k/2}$. If k is odd, from the proof of Theorem 4.4 it follows that we need at least $\Delta \sum\limits_{i=0}^{(k-1)/2} (\Delta-1)^i$ colours (which is the number of edges of T') to colour the edges of T'.

Since any finite tree T is a subgraph of a complete Δ-tree of height h for some positive integer h, by Theorem 4.4 we also have

$$\chi'_{\mathcal{O}_k,k}(T) \leqslant \begin{cases} \Delta \sum\limits_{i=0}^{k/2-1} (\Delta-1)^i + (\Delta-1)^{k/2} & \text{if } k \text{ is even,} \\ \Delta \sum\limits_{i=0}^{(k-1)/2} (\Delta-1)^i & \text{if } k \text{ is odd.} \end{cases}$$

The result follows. □

5. Open problems

In Section 3 we presented the value of $\chi'_{\mathcal{O}_k,l}(G)$ for $1 \leqslant l \leqslant k$ if G is any path, and we found the value of $\chi'_{\mathcal{O}_k,k}(G)$ for any $k \geqslant 1$ if G is a cycle. We also gave results on $\chi'_{\mathcal{O}_k,l}(C_n)$. The only open problem in this area is

Problem 5.1. Let $2 \leqslant l < k$ and $n \geqslant k+2$. Find the exact value of $\chi'_{\mathcal{O}_k,l}(C_n)$ for $r > q(k-l)$, where $n = q(l+1) + r$ and $0 \leqslant r \leqslant l$.

In Section 4 we studied the value of $\chi'_{\mathcal{O}_k,l}(T)$ for trees T, mostly in the case when $k = l$. However, for some classes of trees the problem has not been resolved. Therefore we state the following open problem:

Problem 5.2. Find the value of $\chi'_{\mathcal{O}_k,k}(T)$ for any tree T where $k \geqslant 1$.

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